AN EXPLORATION OF THE GROWTH IN MATHEMATICAL UNDERSTANDING OF GRADE 10 LEARNERS

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AN EXPLORATION OF THE GROWTH IN MATHEMATICAL UNDERSTANDING OF GRADE 10 LEARNERS

by

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DECLARATION

I declare that the mini-dissertation hereby submitted to the University of Limpopo, for the degree of master of education in mathematics education has not previously been submitted by me for a degree at this or any other university; that it is my work in design and in execution, and that all material contained herein has been duly acknowledged.

Mokwebu D. J. (Mrs) __________________ __________

Table of Contents

LIST OF FIGURES

LIST OF TABLES

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ABSTRACT

In this study, I presented the exploration of Mpho's growth in mathematical understanding. Mpho is a grade 10 mathematics learner. To fulfil such, a qualitative research method was employed. I explored her growth in understandings in the context of co-ordinate geometry, exponents, and functions. Data generation, management and representation were guided by the notion of teaching experiments. Analysis was done through mapping learner's growth of mathematical understanding using Pirie-Kieren's (1994) model. Findings suggest that learner's growth in mathematical understanding can be observed, mapped and improved with the aid of probing questions.

CHAPTER 1 BACKGROUND

Introduction

In this chapter, I present an overview of the concept of understanding as seen by various researchers. I further outline the purpose of study, research question, and how the report is structured.

The concept of understanding

Different researchers have tried to find the meaning of mathematical understanding. Historically, various characterizations associated understanding with knowledge. It was later on, through the works of Brownell and Sims (1946), Polya (1962), Skemp (1976), Lehman (1977), and Sierpinska (1994), that clear attempts were made to distinguish between understanding and knowledge. Brownell (1946), for example, characterized understanding as an ability to act, feel or think intelligently with respect to a situation, varying with respect to degree of definiteness and completeness; varying with respect to the situation presented; requiring connections to real-world experiences and the inherent symbols; requiring verbalization although they may contain little meaning; developing from varied experiences rather than repetitive; influenced by the methods employed by the teacher; and inferred from observations of actions and verbalization. Two decades later, Polya (1962) identified four levels of understanding a mathematical rule as mechanical, inductive, rational knowledge involving concepts generalizations, and procedures or number facts. A decade and half later, Lehman (1977) equated understanding with three types of knowledge – application, meaning and logical relationships. Skemp (1976) categorized understanding into four levels: relational, instrumental, logical and symbolic. Sierpinska (1994), meanwhile, forwarded three different ways of looking at understanding. First, there is the act of understanding which is the mental experience associated with linking what is to be understood with the basis for that understanding. Examples of basis are mental representations, mental models, and memories of past experiences. Secondly, there is understanding which is required as a result of the acts of understanding. Thirdly, there are the processes of understanding which involve links being made between act of understanding through reasoning processes, including developing explanations, learning by example, linking to previous knowledge, linking to figures of speech and carrying out practical and intellectual activities.

Hiebert and Carpenter (1992) suggested that understanding occurs when a fact, ideas, or procedure is part of a network of interconnected facts, ideas, and procedures; and this network is connected to other networks in a meaningful way. They stated that a description of understanding encompasses structured internal representations, and connections both within internal and external representations and external representations, and between internal and external representations. In order to think about and manipulate mathematical ideas, they need to be represented internally, in a way that allows the mind to operate on them. They further proposed that understanding is generative in that new connections are made between new and existing knowledge.

Reading through the various views of understandings presented above, one has a sense that it is categories or types of understandings that are being presented. Interlinks of these categories is not always clear and therefore one cannot easily establish whether one form of understanding can evolve into the other. As such, it becomes a challenge for a classroom based teacher to assist learners' in deepening their understanding of concepts given that they are likely to be coming from different categories of understanding.

The literature also demonstrate that even though researchers tried to define understanding, there is no agreement within mathematics communities on its meaning since different authors approach it from diverse viewpoint (Schroeder, 1987). However, it is from the constructivist conceptualizations of understanding that I found an interpretation that mirrors the challenges of facilitating learning as in classroom situations. In particular, this relates to Pirie and Kieren's (1994) model of the growth in mathematical understanding. Pirie and Kieren's initial definition of mathematical understanding emanated from von Glasersfeld's constructivist definition of understanding (Kieren, 1990). Von Glasersfeld (1987) perceives understanding as the continual process of organizing one's knowledge structures. Using this definition as an advance organizer, Pirie and Kieren began to develop their theoretical position concerning mathematical understanding. They characterize mathematical understanding as a levelled but non-linear. It is a recursive phenomenon and recursion is seen to occur when thinking moves between levels of sophistication. Each level of understanding forms the base of and is therefore contained within succeeding levels. Any particular level is dependent on the forms and processes within and, further, is constrained by those without (Pirie and Kieren, 1989). According to Pirie and Kieren (1989), the learner actively participates in constructing understanding from the elements and situations provided in mathematical problems. This construction, according to them, is a process where learners continually engage in the act of organizing their own knowledge structures. As a result Pirie and Kieren describe understanding as a never-ending process (Byers et al, 1985, Pirie and Kieren, 1991, 1992, and 1994). It is Pirie and Kieren (1991) notion of understanding that this study has adopted. That is, mathematical understanding is an on-going and a recursive process whereby a learner when faced with a problem is able to use what is previously learned to solve the existing problem leading to a growth in that understanding.

The main aim of this study was to explore the growth through which learner's mathematical understanding is organized and re-organized for better understanding. An improved understanding of this phenomenon was vital for my research. As a teacher, this will help me understand how to facilitate the process thereof. As a developer of learning material, I will be in a position to expose learners to experiences that acknowledge their growth in understanding.

The study used Pirie-Kieren theory of understanding as a theoretical framework because the focus was on observations of the growth of learner's understanding as it unfolds in a classroom setting. The theory is grounded in a great deal of observations and teacher and learners' interactions in the context of learners actually doing mathematics (Pirie and Kieren, 1989, 1991). The phrase "as it is observed" was more important in the study as it was in Pirie and Kieren (1994).

The purpose of study

In this study my intention was not to measure mathematical understanding either quantitatively or statically. Rather, I set out to examine the breadth and depth of the learner's mathematical understanding, to investigate the complexity of their concepts, and to observe the facility, with which they integrated, distinguished, and assess their informal and formal mathematical realms of knowing. As a result, the purpose of this study is to explore the growth of grade 10 learner’s mathematical understanding.

Research questions

I spent several years teaching mathematics at high school. My observation is that most mathematics learners when advancing from one grade to another are unable to apply what was learnt previously to present situation. The cause may be how their mathematical understanding was developed. This persuaded me to understand how their mathematical understanding grows. Thus the research question was: In what ways are my grades 10 mathematics learner’s understandings growing?

The structure of the report

The report consists of five chapters. Chapter 1 deals with background to the study which includes definition of understanding, the purpose of the study, research questions, and how the study is structured. Chapter 2 deals with the literature review which relates to the growth of mathematical understanding. The chapter gives comprehensive information on Pirie and Kieren’s (1994) model of mathematical understanding. In addition, it presents how the model was used in classroom based studies. Chapter 3 gives details of the study’s research methodology which include: research design, participants, data collection, data analysis, quality criteria, limitation, and ethical considerations. Data were collected over three teaching experiments. Chapter 4 presents results and reflections of the study. Data collected from teaching experiments were analysed and reflected. Chapter 5 presents conclusion and recommendations. It evaluates whether the study had answered the research question and suggests areas for future research.

Conclusion

In this introductory chapter, mathematical understanding was outlined as a continuous non-linear and recursive process. This chapter provided the background to the study through which the notion of understanding that the study has adopted was clarified. I also attended to the purpose of the study, the research questions and a brief overview of how the report is structured.

CHAPTER 2 LITERATURE REVIEW

Introduction

In this chapter focus is on literature which relates to the growth in mathematical understanding. In particular, I pay special attention to Pirie-Kieren's model and studies that have used it in pursued of their learners' understanding.

Pirie-Kieren theory

The description of theory was first published in 1989 by the authors Pirie and Kieren (1989). The structure of the model was modified in response to suggestions and reactions to conference presentations. Word such as idea was replaced by image because researchers believed that evidence at these levels is frequently based on pictorial representation and also encompasses mental imagery. They felt that the concept of mental objects is firmly enough established to be comprehended within the theory. They also felt that image is less open to ambiguity than idea which carries a little of what they wished to describe. The Pirie-Kieren dynamical theory for the growth of mathematical understanding differs from other views of mathematical understanding in that it characterises growth as a whole, dynamic, levelled but non-linear, transcendently recursive process (Pirie-Kieren, 1991). This theory is compatible with the constructivist view outlined by von Glasersfeld (1987), according to which individuals must reflect on and reorganise their own personal constructs in order to build up new conceptual structures. However, the Pirie-Kieren (1994) theory views understanding as occurring in action and not as a product resulting from such actions.

According to Pirie and Martin (2000), the notion of recursion embedded in the definition is fundamental to the Pirie-Kieren view of the growth in mathematical understanding. This term is used to suggest that understanding can be observed as complex yet levelled and that each level is not the same as the previous level (Pirie and Martin, 2000). In developing this idea of mathematical understanding as a recursive process, Pirie and Kieren were influenced by the work of Maturana and Tomm (1986, 1989), and Maturana and Varela (1987), who see knowing as exhibited by effective actions as these are determined by an observer and the human knower of mathematics as self- referencing and self-maintaining in a particular niche of behavioural possibilities.

Pirie-Kieren's theory is a theoretical model. It has grown and evolved into a theory that can be used by a teacher or a researcher as a tool for listening and observing in the context of mathematical activity. It offers a theoretical way of looking at growing understanding as it is unfolding. It is a system by which an observer (a teacher or a researcher) can observe understanding not in terms of a personal acquisition or an acquired state but as an on-going process. Hence, it allows one to observe understanding in action and prompts the looking for relationships between less and more formal understanding actions. It is a theoretical thinking tool for a person who is observing mathematical understanding and who might be interacting with students who are engaging in understanding actions (Pirie and Martin, 2000).

The Pirie-Kieren theory provides a way of considering understanding which recognizes and emphasizes the interdependence of all the participants in an environment. It views learning and understanding as an interactive process. The location of understanding in the realm of interaction rather than subjective interpretation and recognition that understandings are enacted in our moment-to-moment, setting-to-setting movement (Davis, 1996) allows and requires the discussion of understanding not as a state to be achieved but as a dynamic and continuously unfolding phenomenon. Hence, it becomes appropriate not to talk about understanding as such, but about the process of coming to understand and about the ways that mathematical understanding shifts, develops, and grows as a learner moves within the world (Pirie and Martin, 2000).

According to Pirie and Martin (2000), Pirie-Kieren theory is a layered model. Layers are wrapped around and contain all previous layers. This set of unfolding layers suggests that any more formal or abstract layers of understanding actions enfolds, unfolds from, and is connected to inner, less formal, less sophisticated, less abstract, and more local ways of acting. The growth in understanding occurs through a continual movement back and forth through the levels of knowing, as the individual reflects on and reconstructs their current and previous knowledge. Although the rings of the model grow outward toward the more abstract and general, growth in understanding is not seen to happen that way. Instead, growth occurs through a continual movement back and forth through the levels of knowing, as an individual reflects on and reconstructs her current and previous knowledge. Pathways of growth drawn across this model illustrate the fact that growth in understanding need be neither linear nor unidirectional. Its purpose is for observing and describing the process through which knowledge is organised and re-organized, and how learners think about their understanding and build their understanding in an appropriate ways. Growth in understanding is seen as a dynamical and active process involving the building of and acting in a mathematical world. Acting involves a continual movement between different layers or ways of thinking and this continual movement does not move in one direction but back and forth through the layers. The movement of moving back to the layer, of revisiting and re-working the existing understanding and ideas for a mathematical concept is called folding back.

The Pirie-Kieren theory of understanding contains eight potential layers of action for describing the growth of understanding of a specific person for a specific concept. These layers are: *Primitive Knowing, Image Making, Image Having, Property Noticing, Formalizing, Observing, Structuring, and Investigating* (Thom and Pirie, 2006). Although they give a list of eight different levels, I described briefly below the first five, since I do not expect learners of high school to achieve the last three levels.

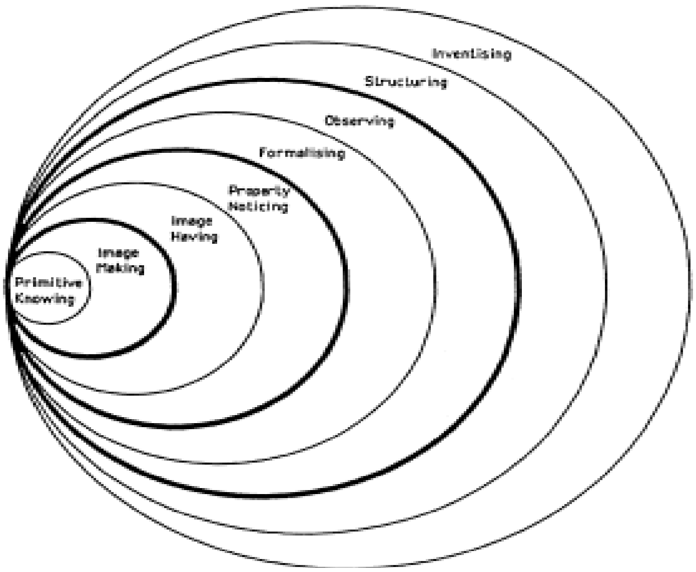


Figure 2: Growth in mathematical understanding

Primitive knowing

Primitive knowing is where the growth of understanding of any piece of mathematics starts. It does not imply low level mathematics, but is rather the starting place for the growth of any particular mathematical understanding. It is everything that the learner knows with the exception of knowledge of the new concept. It is only through assumption that the teacher or researcher know what the learner knows. This assumption can be drawn from learner's physical, verbal and written actions. When approaching the teaching of any topic, the teacher consciously or unconsciously, assume that learners possess certain prior understandings. For example, in teaching a linear graph, the teacher assumes that a learner holds an understanding of the co-ordinates and plotting points on a graph. Much of this knowledge may, of course, be irrelevant to the task in hand, but it can only assist in the development of new understanding. In Primitive Knowing, appropriate knowledge must be selected and used as a basis for growth in understanding (Pirie and Kieren, 1992; Pirie and Martin, 2000; Thom and Pirie, 2006).

Image Making

The learner is doing something and reviewing understanding. In doing, the learner might draw diagrams, or play with numbers, in order to get the idea of what the concept is all about. These images may be visual, pictorial in nature, expressed in languages, and in action. For example a learner who is engaged in plotting a linear graph, he or she may draw as many linear graphs as he or she could in order to get an understanding of graphs. In reviewing, the learner constructively alters previous understanding without seeing the pattern. The learner works with understanding. He or she is able to question his or her understanding. For example a learner who is engaged in plotting linear graphs may come to a point where he or she realizes that there is something wrong about the graph. He or she may utter words such as: the graph does not look right or it cannot be like that (Pirie and Kieren, 1992).

Image Having

At this level, the learner has the idea or picture inside his head. Image having comprises of image seeing and image saying. The learner can use a mental construct about an activity without having to do the particular activities which brought it about. In Image seeing, for example, still working with a linear graph, the learner will have a picture of a linear graph in his head. In image saying, given an equation of y= 2x + 3, a learner can articulates comment such as I thought the y-intercept = 3. At this level a learner is in a position to talk about his or her actions and carry them beyond the graphing situation. Image having does not necessarily having the right or complete picture. The interconnection of image seeing and image saying leads to the level of property noticing (Pirie and Kieren, 1992).

Property Noticing

Property noticing is an act of self- conscious reflecting of questioning one's understanding and of looking for what can be said more generally about the image. It is the level when one can manipulate or combine aspect of one's images to construct context specific, relevant properties. Working on linear graph, the learner can notice properties of a linear graph, for example: y intercept, x- intercepts, and the gradient and be able to work on them (Pirie and Kieren, 1992).

Formalizing

The learner is able to generalize what he has noticed in Property Noticing and no longer need to relate back to specific mathematical contexts that gave rise to his understanding.. The learner formulates rules. For example in equation of $y = ax + q$, if $a > 0$, the graph slopes to the right; if $a < 0$, the graph slopes to the left; if $q > 0$, the y- intercept is above the x- axis; if $q < 0$, the y- intercept is below the x- axis; if $q = 0$, the graph passes through the origin(Pirie and Kieren, 1992 et al).

Besides eight levels, this model has three different features. First feature involves the notion of *"don't need" boundaries* which are represented by thicker lines between images. This feature means that a student does not always need to be aware of inner levels of understanding in order to move to an outer level. The first "don't need" boundary occurs between image making and image having. When a person has an image of a mathematical idea, he or she does not need actions of image making. The next "don't need" boundary occurs between property noticing and formalizing. A person who has a formal mathematical idea does not need an image.

The second feature of this model is a built-in dynamic *“folding back”*. According to Martin, La Croix, and Fownes (2005), the growth of understanding does not occur through a simple linear pathway through the layers. Instead, mathematical understanding is seen to emerge through the continual movement back and forth through the layers of knowing, as individual reflect on and reconstruct their current understandings. These backward and forward movements across layers are called folding back. When a learner functions at an outer layer of understanding, and faced with a problem that is not immediately accessible, he will return to the inner layer to examine and modify their existing ideas about the concept. In that way the learner will be folding back. For example, a learner is able to draw a graph of $y= x^2$, when asked to draw the graph of $y= 2x^2$; he is unable to do so. He has to go back to the inner layer (Image Making) to re- work the problem based on existing knowledge of parabola. In so doing he will be building a thicker layer at that inner layer. Folding back to Image Making might involve physical actions such as drawing, working with manipulative or playing around with numbers. Folding back occurs with a purpose, namely to extend one’s existing understanding which have proved to be inadequate for handling a newly encountered problem. When the learner fold back to the inner layer, the actions in the inner layer are not the same as those previously performed. Folding back can be visualized as the folding of a sheet of a paper in which a thicker piece is created through the action of folding one part of the sheet onto the other. The folded sheet becomes thicker as it includes the existing sheet. The learner has a different set of structures, a changed and changing understanding of the concept, and this extended understanding acts to inform subsequent inner layer action. We can say that one now has a thicker understanding at the return – to level. This inner level action is part of a recursive reconstruction of knowledge, necessary to further build outer level knowing. Different students will move in different ways and at different speeds through the levels, folding back again and again to enable them to build broader but also deeper understanding. Folding back can be effective or ineffective. Effective folding back is when a learner is able to extend current inadequate mathematical understanding and ineffective folding back is when a learner is unable to apply his understanding to the outer layer (Martin, La Croix, and Fowns, 2005).

Third feature of this model is that of the structure within the levels themselves. Each of the levels in this model beyond primitive knowing has complementary aspects of "acting and expressing" which are essential before moving on from any level. Acting includes all previous understanding and provides continuity with inner levels and expressing entails looking at and articulate what was involved in the actions. Acting involves mental and physical activities and expression is to do with making overt to others or oneself the nature of those activities. Acting, the learner may reflect on how their previous understanding applies to a new learning situation. When expressing, however the learner makes it clear to oneself or others what knowledge was gained. Acting and expressing complementarities are represented by verbs such as doing and reviewing; seeing and saying; predicting and recording within the image making, image having and property noticing. In image doing, the student perform actions that might lead to formation of an image of an activity, while in reviewing, student sees some order within the activity she is engaged. Image seeing occurs when a student has collected together previous instances and has a pattern, while image saying occurs when student articulates the feature of that pattern (Pirie and Kieren, 1994).

Studies that have used Pirie-Kieren model

In this section I focussed on the application of Pirie-Kieren theory of understanding at different levels addressing different areas of mathematics.

Since its conception, the Pirie-Kieren (1994) model has been used in numerous studies (Pirie-Kieren, 1994; Martin and Pirie, 2003; Manu, 2005; Martin, Le Croik and Fownes, 2005; Mochon, 2006; Towers and Martin, 2006; Kyriakides, 2009; Nillas(2010). I discuss some few applications hereunder.

Pirie-Kieren (1994) in their study of the growth of mathematical understanding entitled: How can we characterize it and how can we represent it, they set out to examine mapping as a technique to record the growth of person's mathematical understanding. To illustrate this the work of Richard, who was one of six university mathematics student engaged for four hours in building a geometric shapes created by a computer was discussed and analyzed through coding and mapping his level of understanding on Pirie-Kieren model of understanding. They found that student's growth of mathematical understanding is not alike, and not age related, some students spent more time creating broad, rich images before moving to outward layers; and also that folding back can happen directly to any layer.

Manu (2005) in his study of the growth of mathematical understanding in bilingual context, his purpose of study was twofold: to illustrate an application of Pirie-Kieren theory within a bilingual context, as a language for and a way of observing and accounting for- the growth of understanding and secondly to examine the relationship between language switching and growth of understanding. In order to pursue his purpose a case study was undertaken. Two form three Tongan bilingual students: Maile and Alani participated in the study. The two students worked together under a topic pattern. Data was collected through the use of video recording, student's written work and student's follow up interviews about their recorded work. Analysis was done through mapping learner's growth of mathematical understanding using Pirie-Kieren model (1994). He noticed that students were able to progress in working with pattern despite their weak English skill. He concludes that mathematical understanding does rest with the ideas and images not with words.

Mochon (2006) in his study of the development of mathematical understanding in classrooms with a computer, the purpose of his study was to find out the changes that are propitiated in the learning and teaching of mathematics by the use of a computer and a projector in the classroom. To do this effectively, a parallel research project consisting of a dialectical experiment was conducted. Grades 2, 3, 4 and 5 students participated in the study. These students were from elementary middle economic class school. Their activities pertain to topics on additive problems, decimal systems, geometry, fractions, mental calculations and estimations. Data were collected from observations and interviews. Class sessions were audio and video taped. Throughout the study four aspects were analyzed whereby each aspect was analyzed based on different framework. The second aspect analysis was based on the categories of growth in mathematical understanding formulated by Pirie and Kieren (1994). His finding indicates that the mode of the teacher's instruction is a very critical factor in learning and teaching process.

Towers and Martin (2006) explored the interplay between the individual and collective in the mathematics classroom. They used a case study design to pursue the issue. Three elementary teachers, Mary, Shauna, and Hilary participated in the study. Teachers were invited over a series of one- hour lunchtime and engaged in a task that required them to label sixteen triangles as equilateral, isosceles or scalene. Data were collected through video recording. They employ in their analysis of data, element of Pirie and Kieren (1994) theory for the dynamical growth of mathematical understanding. Their finding was that student's growth of understanding can be developed or improved with the involvement of others.

Nillas (2010) in his study of characterizing pre-service teacher's mathematical understanding of algebraic relationships, his purpose of study was to examine pre-service teacher's mathematical understanding. In order to pursue his purpose, a qualitative study was conducted. Five pre-service teachers participated in the study. The pre-service teachers were either elementary or special education majors. Data were collected from pre-service teacher's written work, observations and interviews. Their written work pertains to topics on linear, exponential and quadratic relationships. Their class sessions were recorded by means of audio-tape. Analysis was done in two ways. Firstly with-in case (Miles and Huberman, 1994) was used to characterize teacher's mathematical understanding involving algebraic relationships. Secondly Pirie and Kieren (1994) model of growth of mathematical understanding was used to compare and identify feature of understanding evident in teacher's responses to each test item. His finding was that there is a need to engage pre-service teacher's mathematical understanding.

Wilson and Stein (2007) investigate the relationship between student's external representations of a mathematical concept and growth of understanding. Through a cased study design, they focused on eight undergraduate students enrolled in an introductory psychology course and two graduate students in mathematics education. Their generated data was coded using various levels of understanding from Pirie-Kieren (1994) framework. Next they were coded according to the types of representations that the participants were using during each instance from Cai and Lester's explanation of representations. Their finding was that there is an association between participant's level of understanding and the types of representations used.

In most of the studies outlined above, researchers adopted a qualitative case study approach. The studies were conducted in a classroom environment. Data were collected from learner’s written work, observations, and interviews. Data were recorded by means of journals, audio and video tapes. Transcripts, video-tape records, journals, and learners written work were used as source of data analysis. In their analysis of data, they employ element of Pirie and Kieren theory (1994). What I have learned from these studies will have an influence in moving forward in the design and analysis of my study.

Conclusion

In this chapter, I specifically focused on literature on Pierie-Kieren Model of understanding. First I focused on the origins and rationale for the model. This helped me to establish and acquaint myself with the terms used in the model. Secondly I focused on how the model was used by the owners and other researchers in trying to capture and interpret students’ understanding. This was very helpful in terms of how I was used the model in this report.

CHAPTER 3 RESEARCH METHODOLOGY

Introduction

Research methodology is the system of collecting data for research projects. The data may be collected for either quantitative or qualitative research. Some important factors in research methodology include validity of research data, ethics and reliability of measures. In addition, factors such as research design, data collection and data analysis are also included (De Vaus, 2001). In this study, data was collected for qualitative research.

Main issues to be addressed in this chapter are: design of the study, participant, data collection, data analysis, quality criteria, ethical considerations, and limitations of the study. In addition, I discuss how the above mentioned issues influenced my study.

Design of the study

Research design provides glue that holds the research project together. A design is used to structure the research, to show how all the major parts of research project work together to try to address the central research questions. It deals with a logical problem and not logistical problem (Yin, 1989). Before a builder can develop a work plan or order materials, he must first establish the type of building required, its uses and the needs of the occupants. Similarly, in research the issue of sampling, method of data collection (e.g. questionnaires, observations, and document analysis), and design of questions are all subsidiary to the matter of what evidence do I need to collect. In simple terms design deals with four problems: what questions to study, what data are relevant, what data to collect, and how to analyse the results. Designs are often equated with qualitative and quantitative research methods. Social surveys and experiments are frequently viewed as prime examples of quantitative research and are evaluated against the strengths and weaknesses of statistical, quantitative research methods and analysis. Case studies, on the other hand, are often seen as prime examples of qualitative research- which adopts an interpretive approach to data, study things within their context and considers the subjective meanings that people bring to their situation. The function of research design is to ensure that the evidence obtained enables us to answer initial questions as unambiguously as possible (De Vaus, 2001).

In this study I used qualitative case study approach. According to Merriam (1998), case study is an intensive, holistic description and analysis of single instance, phenomenon, or social unit. The most defining characteristic of a case study research lies in delimiting the object of study (bounded system) as outlined in Merriam (1998). It is seen as single entity, a unit around which there are boundaries. The case could be a person such as a student, a teacher, a classroom of children and so on. One technique for assessing the boundedness of the topic is whether there is a limit to the number of people involved who could be interviewed or a finite amount of time for observation. Another characteristic of qualitative case study is that of being particularistic and descriptive. Particularistic means that case study focuses on a particular situation, event, program, or phenomenon. It concentrates on a way particular groups of people confront specific problems, taking a holistic view of the situation. Descriptive means that the end product of a case study is rich, thick, and qualitative. Qualitative meaning that instead of reporting findings in numerical data, case studies presents documentation of events, quotes, sample and artefacts. The researcher is interested in insights, discoveries, and interpretations rather than hypothesis testing (Merriam, 1998).

The bounded system, or case, might be selected because it is an instance (interest) of some concern or issue. A case might be selected because it is intrinsically interesting; a researcher could study it to achieve as full an understanding of the phenomenon as possible (Merriam, 1998).

Unlike the experimental, survey, or historical research, case study does not claim any particular methods for data collection or data analysis. Any or all methods of gathering data can be used, although certain techniques are more suitable than others depending on type of research question one is pursuing. This is so because, by its nature, case study provides for thick and rich description of the case in hand. It would not be out of order for instance, to find both qualitative and quantitative data in some case study reports (Merriam, 1998).

Case study knowledge differs from other research knowledge because its knowledge resonates with our own experience, rooted in context, developed by reader interpretation, and based on reference population (Merriam, 1998)

There are different types of case studies such as exploratory, explanatory, and descriptive. An exploratory case study is aimed at defining the question and hypotheses of subsequent study. One can say that this version of a case study is a phase in a broader study. On its own, it has some sense of incompleteness. An explanatory case study presents data hearing on cause- effect relationships – explaining which causes produced which effects. A descriptive case study presents a complete description of a phenomenon. Exploratory and descriptive case study attempts to discover theory by directly observing a social phenomenon in its raw (Bassey, 2009).

Whilst I have noted the different versions of case studies in the preceding paragraph, I have opted to remain general in the way I approach the current study. I adopted the case study approach in order to provide a rich and detailed analysis, and deep and comprehensive description of the process of classroom interaction leading to the growth in mathematical understanding (Tower, 1998). In this study, the focus is on the growth in the participants' mathematical understanding. Considering the nature of understanding, it is important that I generate and collect different forms of qualitative data to provide for the rich and thick descriptions of the findings. In relation to the day to day activities related to the study, one can view the approach as following a teaching experiment approach.

A teaching experiment is a living methodology designed initially for the exploration and explanation of student's mathematical activity. It involves a series of teaching episodes. A teaching episode includes a teaching agent, one or more students, a witness, and the method of recording what transpire (Steffe and Thompson, 2000). Teaching episodes are recorded and analyzed. The analysis is then used to guide the next teaching episode. It is during this phase that the researcher's hypothesis is tested or perhaps abandoned based on responses given by the students. During the teaching episode, student's reasoning is the focus of attention (Ackermann, 1995).

The teaching experiment embraces both the learning and teaching cycle. Learning and teaching cycle consists of three stages for both the teacher and the student: an exploration phase, concept introduction phase, and concept application phase (Steffe and Thompson, 2000). In the exploration phase, students explore the concept under investigation through hand-on activities. . The teacher-researcher put aside his or her own concepts and operations and not insists that the students learn what he or she knows. The teacher researcher formulates and test research hypotheses as well as generates and tests them. Formulation and testing research hypotheses, the teacher researcher becomes familiar with student's mathematics and makes essential distinctions in student's ways and means of operating during teaching episodes. Generating and testing hypotheses, the teacher-researcher, through reviewing the records of one or more earlier teaching episodes, formulate one or more hypotheses to be tested in the next episode. In a teaching episode, the students' language and actions are a source of perturbation for the teacher-researcher. It is the job of the teacher-researcher to continually postulate possible meanings that lie behind students' language and actions. It is in this way that students guide the teacher researcher. The teacher-researcher may have a set of hypotheses to test before a teaching episode and a sequence of situations planned to test the hypotheses. But, because of students' unanticipated ways and means of operating as well as their unexpected mistakes, the teacher-researcher may be forced to abandon these hypotheses while interacting with the students and to create new hypotheses and situations on-the-spot. The teacher-researcher also might interpret the anticipated language and actions of the students in ways that were unexpected prior to teaching. Impromptu Interpretations occur to the teacher-researcher as an insight that would be unlikely to happen in the absence of experiencing the students in the context of teaching interactions. Here, again, the teacher researcher is obliged to formulate new hypotheses and to formulate situations of learning to test them. Boundaries of the students' ways and means of operating can be formulated through generating and testing hypotheses, where the students make what to us are essential mistakes. When a student makes what appears to be an essential mistake, our purpose is not "to judge or evaluate the child's performance in relation to performances of other children who might come up with the right answer" (Ackermann, 1995), but attempt to understand what the student can do; that is, the teacher-researcher must construct a frame of reference in which what the student can do seems rational (Steffe and Thompson, 2000).

Teaching actions occur in a teaching experiment in the context of interacting with students. However, interaction is not taken as a given-learning. How to interact with students is a central issue in any teaching experiment. The nuances of how to act and how to ask questions after being surprised are among, in our experience, the most central issues in conducting a teaching experiment. The researcher interacts with the student in a responsive and intuitive way as well as in analytical way. In responsive and intuitive interactions, the teacher researcher is usually not explicitly aware of how or why he or she acts as he or she does and the action appears without forethought. He or she acts without planning the action in advance of the action. In this role, teacher researcher becomes an agent of action (or interaction). As agents of action, he or she strive to harmonize themselves with the students with whom they are working to, that is, they become students and attempt to think like them(Thompson, 1982, 1991; van Manen, 1991). This stance is maintained throughout the experiment. In a teaching episode, as in a clinical interview, the students' reasoning is the focus of attention (Ackermann, 1995). When the students' reasoning proves to be rich and full of implications for further interaction by the teacher-researcher, he or she often turns to analytical rather than to responsive and intuitive action. Researchers' abilities to engage in analytical action frequently follow an insight into the mental operations that make students' language and actions possible. The teacher-researcher formulates an image of the students' mental operations and an itinerary of what they might learn and how they might learn it. Initially, this itinerary is articulated loosely or not at all. Nevertheless, the teacher-researcher has a sense of direction and a sense of possibilities for where he or she might try to take the students. The teacher-researcher now has initial goals along with a sense of possibilities for how the goals might be achieved in future teaching episodes. As the teacher-researcher engages the children in further teaching episodes, this goal structure becomes extended and articulated. The most important feature of the extension and articulation is that the teacher-researcher modifies the goal structure constantly while developing it to fit the students' mathematical activity. Extending and modifying the goal structure lasts until the students' schemes seem to be well established and the students seem to have reached a plateau (Steffe and Thompson, 2000).

The role of data in teaching experiment is twofold: on the one hand it has to provide summative knowledge to the researcher about the nature of the whole argument. In that case the data will be supporting quantitative approach. On the other hand it has to inform teachers about the next best step of instruction. The data needed is that which can evidence changes in the process of learning as well as interventions which might have caused them. In that case the data will be supporting qualitative approach. Consequently data are collected from written work such as tests, homework, and class work as well as the analysis of errors made by students. If we want a better picture of how students think, solve the problems, and understand mathematics, we have to create data that records these components. Both approaches, their analysis needs to be supported by mathematical knowledge as well as pedagogical craft knowledge of a teacher-researcher. Mathematical essays, paragraphs of explanations, or transcripts of mathematical conversations conducted with students either during class or during more elaborated interviews conducted outside the classroom are especially good for qualitative analysis (Czarnocha and Prabhu, 2006).

The structure of the interview resembles a Socratic dialogue. Students are repeatedly asked probing questions to try and elicit as much of their reasoning and thought processes as possible. The questions tend to be focused on around the activities or task that the students are asked to think about and explained (Ausubel, 1968).

A primary purpose for using teaching experiment methodology is for researchers to experience, first hand, student’s mathematical learning and reasoning. Without the experiences afforded by teaching, there would be no basis for coming to understand the powerful mathematical concepts and operations students construct or even for suspecting that these concepts and operations may be distinctly different from those of researchers. Teacher-researcher decides to conduct a teaching experiment when they want to respond to a learning problem in the classroom in a systematic and creative manner or when they feel the need of constant improvement of their workbench (Steffe, 1988).

In this study a minimum of three teaching experiments are presented which focussed on co-ordinate geometry, exponents, and functions. The first teaching experiment is made of two episodes. One episode comprised of three vignettes and the remaining one with one vignette. The second teaching experiment is made of one episode which comprised of three vignettes. The last teaching experiment comprises of two episodes and two vignettes. Classroom interactions are influenced by the underpinning theoretical framework, chosen by the researcher, for the teaching experiment (Steffe, 1988). In this teaching experiment the chosen theoretical framework supported the assumptions of constructivism, namely, conceptual knowledge will not be passively acquired by a student as it is passed by language from the teacher, but rather, it must be actively constructed within the social context of a classroom community (Doig, McCrae, and Rowe, 2003; von Glasersfeld, 1995). Social interaction in this teaching experiment was achieved through the class working in groups, discussing and sharing ideas on given activities. Mathematical concepts cannot be developed in the absence of language and students need to be afforded opportunities to talk about, share solutions and strategies, explain and clarify their own thinking (Australian Education Council, 1991; Doig et al., 2003; von Glasersfeld, 1995; Wood et al., 1995). In this teaching experiment, students were afforded an opportunity to explain and reflect their ideas through writing and interviews in order to make sense of their understanding of co-ordinate geometry, exponent, and function concepts. Learners were given activities to work on and submitted in writing. Hypotheses were formulated from learner’s written work. The authenticity of the hypotheses was tested through observations and interviews. Subsequently I engage the learner by asking prompting questions which could assist in her understanding. The primary aim of the teaching experiment in this study is to have an insight in learner’s mathematical understanding and to respond to their learning problems in the classroom which assists in understanding the growth in mathematical understanding. Resembling the study, teaching experiment supports qualitative approach. Data were collected from learner’s written work, observations, and interviews. Analysis was drawn from learner’s written work, observations, and transcripts of mathematical conversation conducted with the learner during interviews.

Participants

Two grade 10 mathematics learners participated in this study. I taught both learners since grade 8. Based on their performance I considered them to be of average ability. They were selected out of a total of 86 grade 10 mathematics learners. All learners were working in a classroom. For the reporting purposes, my focus was further reduced to one learner out of the two. I did that because my focus in the study is on what is happening in learner's mind. Having that in mind it is impossible to explore the growth in mathematical understanding of each and every learner. I selected the learner with reference that she gave information that is relevant to the purpose of study, hence purposive sampling (Hargrove and Jones, 2001).

Data collection

Data collection is an essential component in conducting research. In order to collect data, the researcher should be able to access the data that needs to be collected for the study. Data can be gathered from a number of sources including written documents, records, survey or interviews (Kajornboom, 2004). Taking in consideration of what is outlined above, learner's written work, observations, and interviews were utilised as techniques of data collection.

Written work

I collected data in classroom, whereby I gave learners written work in the form of class activities. They wrote their responses on piece of papers. I personally collected the response sheets and stored them in a safe box. The two learners were invited over a session of three hours after school over a period of seven days to work on mathematical problems, which I hoped, as well serving my research purpose, would also help them with their own mathematical and pedagogical knowledge. They worked together as a group on activities covering different areas of mathematics such as: Co-ordinate geometry, Algebra (Exponents), and Functions. These areas were chosen based on my observation that learners had certain understanding of mathematical concepts. Written work was used with the intention to understand the learner's mathematical understanding.

Observation

In everyday life we use observation to gain knowledge so that we can act in the world. Observation also informs, and enables us to test our common theories about the social world. In research observation has different purpose to everyday life. Its aim is the production of public knowledge about specific issues, which can be used by others in variety of ways. Furthermore observation in research is planned, conducted, interpreted, and analysed employing systematic and planned procedures, rather than happening spontaneously, haphazardly, and stored in personal memory as usually does in everyday life. The data produced by observational research are validated for accuracy unlike those produced routinely in everyday life (Sapsford & Jupp, 1996).

As part of research, observation can be used for different purposes. It may be employed in the preliminary stages of a research project to explore an area which can then be studied more fully utilizing other methods, or used towards the end of a project to provide a check on data collected in interviews or surveys. Where observation is the main research method employed, it may be used to enable qualitative description of the culture of a particular group (Sapsford and Jupp, 1996). At this stage, observation was used jointly with written work with the intension to supplement data gathered from learner's written work.

There are number of approaches to observational research. One important distinction is between more-structured and less-structured observation. I will present a background on less-structured observation for the mere fact that I claim to exercise an important observation technique that is of being a participant observer. Less-structured observation aimed to produce detailed, qualitative descriptions of human behaviour that illuminate social meanings and shared culture. These data are combined with information from conversations, discussions, interviews, and documentary sources to produce an in-depth and rounded picture of the culture of the group. One of the key aims of this type of observation is to see the social world as far as possible from the actor's point of view (detailed data), the main technique used is that of participant observation. The observer here participates with the group under study and learns culture, whilst at the same time observing the behaviour of group members. The usual method of recording data is in the form of field notes, and where possible, audio or video recordings. Less-structured observation produce far more detailed data on a behaviour of a particular individual or a group in particular setting. It involves the researcher spending long periods of time in the field, building relationships and participating in social interaction with participants. The aim is to build trust and accustom of her presence. Consequently, the data produced may be less influenced by the researcher and the research process (Sapsford and Jupp, 1996; Flick, 2002).

Throughout the study I played a role as participant observer. I was their teacher and this was part of my teaching. I observed learners working together on a given mathematical activity in a classroom. Simultaneously I ask them questions that assist them in their understandings. After each observation session, at home, I make notes of observations and record them in a journal. I did this because learners were curious about what I was writing and pretend to do the correct thing rather than being themselves. I recorded observations of one learner. These observations were conducted over a series of teaching experiments.

Interview

Interviews are systematic way of talking and listening to people and are another way to collect data from individuals through conversation. The interviewees are able to discuss their perception and interpretation in regard to a given situation. It is their expression from their point of view (Kajornboon, 2004). The focus in the study is on what is taking place in learner's mind. As a means of entering child's mind, I used clinical interview. What is happening in learner's mind can only be known through writing or talking. I went through learner's written work. I was drawn to particular activities where it appeared that there is underneath mathematical understanding that needs to be clarified or modified. I claim that the best method to clarify it is through clinical semi- structured interview. Clinical interview unlike other interviews is a powerful and appropriate tool for studying how people think (Ginsburg's, 1997). The main aim of clinical interview is to enter learner's mind to search for underlying structure. Clinical interviewing has a long tradition in educational research, dating back to Piagent, and has been used to study conceptual understanding in mathematics (Brown, 2004, Swanson, Schwartz, Ginsberg, and Kosan, 1981).

Interviews were conducted in the form of semi–structured questions. The interview was guided by the responses of the learner. It was conducted a week following the extended classes with the two learners. The primary interest is in clarity of the responses rather than performance. The sessions were audio taped.

Data analysis

Data analysis in qualitative research consist of preparing and organizing the data for analysis, then reducing the data into themes through a process of coding and condensing the codes, and finally representing the data in a figure, tables or a discussion (Creswell, 2007).

As stated by Boeije (2010), Qualitative data analysis is the segmenting of data into relevant categories and the naming of these categories with codes (reassembling) while simultaneously generating the categories from the data. In the reassembling phase, categories are related to one another to generate theoretical understanding of the social phenomenon under study in terms of the research questions. Two basic activities of data analysis, namely segmenting of data and reassembling categories are emphasised in the above definition. Segmenting refers to as unfolding, breaking up, separating or fragmenting. Reassembling is also referred to looking for patterns, searching for relationships between the distinguished parts, and finding explanations for what is observed.

Steps in analyzing qualitative data

Various steps are commonly used to analyze qualitative data. Steps are not always taken in sequence. Those steps are: preparing and organizing the data for analysis, engaging in an initial exploration of the data through the process of coding and condensing the codes, representing the findings through narrative and visuals such as in figures, tables, or discussion (Creswell, 2007, and 2012). This simply implies that firstly you collect data, transcribe data, read and read data several times in order to get a sense out of it, and mark or break into parts, and presents as figure or discussion. To analyze data, I will be informed by steps of analysis outlined above. The details of transcribing data and thereafter marking or coding it are addressed hereunder.

Transcribe data

Transcription of data is a process of converting audiotape recordings or field notes into data in the form of written text. In this study data were collected through interviewing learners and recorded on an audio- tape. I listened to the tape and transferred some interviews to a sheet of paper.

Coding of data

Data can be analyzed by means of a hand or by use of computer. Hand analysis of data refers to making use of a hand to mark and to divide it into parts. Computer data analysis refers to the use of computer program to facilitate the process of storing, analyzing, sorting and visualizing data. In this study, data was analyzed by hand writing on a sheet of papers. It is difficult to process large amount of diverse content at all once. Data are divided into relevant parts and chunks of meaning, within a holistic perspective and categorized according to an organizing system of topics (themes) derived from the data themselves. Coding categories of data mean that categories are marked with symbols, descriptive words or unique identifying names. The aim is to discover patterns (Creswell, 2012: McMillan and Schumacher, 1997). Boeije (2010), Saldana (2009), and Strauss and Corbin (1998) distinguished three types of coding: open coding, axial coding, and selective coding. Open coding is a process whereby all data that have been collected up to that point are read very carefully and divided into segments. Segments are compared among each other, grouped into categories dealing with the same subject, and labelled with a code. Open coding usually starts during the collection of the first round of data. Axial coding is a process of relating categories to subcategories, specifies the properties and dimensions of a category, and reassembles segmented data to give pattern to the emerging analysis. The primary purpose of axial coding is to determine the dominant and non-dominants elements in the research. Furthermore axial coding reduces and organizes data: synonyms are crossed out. Selective coding is a process whereby redundant codes are removed, and the best representative codes are selected. Although there are no guidelines for coding data, some general procedure may be followed such as making sense out of text, divide it into text or image segments, label the segments with codes, examine codes for overlap and redundancy, and collapse these codes into broad themes. Code can be stated in the participant’s actual words (in vivo codes) or expressed in your own words (lean codes) (Creswell, 2012).

Within five approaches of qualitative analysis, a qualitative case study analysis was the preferred method of analysis for the study. The main purpose for this selection is because its analysis consists of making a detailed description of the case and its setting, use categorical aggregation to establish themes or patterns, use direct interpretations, and develop an in-depth understanding that people can learn from the case either for their benefit or to apply to a population of cases (Creswell, 2007). In the study multiple sources of data such as learner’s written work, observations and interviews were used for a detailed description of the case. Data were categorised in to themes that helps in the interpretation. The results were of understanding the growth in learner’s mathematical understanding which might be of assistance to other mathematical teachers.

Having noticed different types of coding outlined above from Boeije (2010) and others, I associated myself with open coding. After completing interviews with the participant, I started the process of analyzing qualitative data. Firstly I listen and listen to all audiotapes that I recorded and select the one to be transcribed. I transcribed the selected audiotapes to a sheet of papers. At the end I had several pages transcribed. Simultaneously I read and read transcripts several times. There after mark and categorize the transcript employ Pirie-Kieren’s model (1994). I use brackets and inscribe Pirie-Kieren’s levels of mathematical understanding in bold along the margins. Pirie-Kieren’s levels of mathematical understanding are: Primitive knowing, Image making, Image having, Property noticing, and Formalizing (Pirie and Kieren, 1991). Furthermore I marked on the level of Pirie-Kieren levels of mathematical understanding growth (thicken) of mathematical understanding occurs. Lastly I represented the data in figures like mapping the participant’s growth in mathematical understanding on Pirie-Kieren model (1994).

Quality criteria

Qualitative researcher strives to obtain thick and rich meaning of a case. In order to obtain that, a qualitative researcher should validate a study. Many writers such as Lincoln and Guba(1985), Eisner(1991) and others are opposing the use of quantitative terms in qualitative work. Lincoln and Guba replace validity of the study, as one of qualitative terms, by trustworthiness of the study. They use unique terms such as: credibility, authenticity, transferability, dependability and confirmability to establish trustworthiness of the study. Among other techniques for trustworthiness, they suggested techniques such as prolonged engagement in the field, member checking, and the triangulation of data sources, methods, and investigators to establish credibility. Thick description is important for making sure that findings are transferable between the researcher and readers. Both dependability and confirmability are established through an auditing of the research process (Creswell, 2007).

In line with what is expressed above, I took note of the importance of trustworthiness of the study. In order to establish credibility, I made use of triangulation, prolonged engagement in the field, and member checking. For triangulation, I used three sources as data collection. I used written work as basic source of data; observation to confirm what has been written; and interviews to modify or clarify what has been written. With regard to prolonged engagement in the field, I was a participant observer in the study. I spent considerable time in the classroom together with the participant. In terms of member checking, I gave my head of department activity sheets to confirm the standard of questions. Furthermore I gave non-participants activity sheets in order to ensure their understanding of questions. After collection and analysis of data, I shared my interpretation of data and preliminary analysis with the participant in order not to misinterpret the participant.

Limitations

The study focussed on one participant, who might at any time not avail herself due to challenges such as getting ill or dead. Participant committed herself verbally than written. Verbal commitment does not bind the participant. It may influence the participant to take the study for granted. The observation in the classroom was recorded in the journal at home. My observation relied upon my memory, whereby I could sometimes omitted important observations. Only three teaching experiments were conducted in the study. This may lead to restriction of data collection.

Ethical considerations

A qualitative researcher faces many ethical issues that emerge during data collection, data analysis and giving out of qualitative reports. Taking that in consideration, a researcher must protect the anonymity of the participants, for example, by assigning numbers to individuals; develop case studies of individuals that represent a composite picture rather than individual picture; explain the purpose of study in general in order to build interest from participants; lastly share personal experiences with participants in an interview setting to reduce information shared by participants (Creswell, 2007). The study took place in the classroom, at school. It is important to ask permission from the principal and governing body of the school for utilising the classroom. I made an appointment with the principal and verbally outlined my study and further asked permission to utilize the classroom. He presented my request and discussed my study with the school governing body on my behalf, and a permission to utilize the classroom was granted. In my classroom, I explained the purpose of the study to the participants and verbally ensure them that what transpired during the process of the study lies between me and them. Additionally I assured them not to use their real names, rather use different names, for example names like Mpho and Thabo in its place. Although the study focus on growth in mathematical understanding of one learner, but what is learned from the study will affords me with the opportunity to have an in-depth view of how learner’s understanding grows.

Conclusion

In this chapter, research design, participants, data collection, data analysis, quality criteria, limitations, ethical considerations of a study, and their influence in my study were discussed. Qualitative case study approach was employed in order to provide a rich and detailed analysis of data. One grade 10 learner participated in the study. The learner was purposively sampled. Data were gathered from multiple and different techniques such as written work, observation, and interviews and analyzed by coding using Pirie and Kieren (1994) model.

CHAPTER 4 RESULTS AND REFLECTIONS

Introduction

Theoretically, once data has been collected via data collection processes research then proceed to data analysis (De Vaus, 2001), hence results and discussions of the study are discussed in the chapter.

Results and reflections refer to a detailed description, classification, interpretation and presentation of data. During the process of describing and classifying, a qualitative researcher develops categories and sort text (Creswell, 2007). Not all information is used in a qualitative study, some may be discarded (Wolcott, 1994). Interpretation involves making sense of the data. In the process of interpretation, the researcher step back and form larger meanings of what is going on in the situation (Lincoln and Cuba, 1985). Lastly, the researcher presents data either in the form of figure, table or discussion (Creswell, 2007).

Since the approach adopted in generating data for the study was that of teaching experiment, the results and discussions hereunder also follows the same pattern. In this chapter data is presented firstly as activities which are followed by learner's respond and interpretation. Secondly it is presented as short transcripts which are followed by interpretation. At last data is presented in a pictorial form where Mpho's growth in mathematical understanding is mapped on Pirie and Kieren (1994) model.

Teaching experiments

Three teaching experiments focusing on coordinate geometry, exponents, and functions were used in the study. The choice of the topics was not influenced by any other reason except that they fell within the period of data collection. Each of the three teaching experiments presented in this report had two episodes. The first teaching experiment had two episodes with the first comprising three vignettes and the second episode comprising one vignette. The second teaching experiment was made of one episode which comprised of three vignettes. The third teaching experiment was also made of two episodes each comprising of two vignettes.

The transcripts were coded guided by Pirie and Kieren (1994, 1991 & 1992) model of mathematical understanding. I categorized the growth into two themes: those caused by folding back to collect or by folding back to thicken. After each and every vignette an interpretation was done and the thickening effect outlined in the form of a summary. At the end of each episode a layered pictorial representation of the model of the growth of the learner was drawn. The aim was to produce in diagrammatic form, a map, of the growth of learner's understanding as it was observed by the teacher. Mapping entails plotting points on a diagram of the model, observable understanding acts and drawing continuous or discontinuous lines between these points, dependent on whether or not learner's understanding was perceived to grow in a continuous, connected fashion (Pirie and Kieren, 1991 and 1992). Lastly a figure was drawn to generalize learner's growth in mathematical understanding at the end of each teaching experiment.

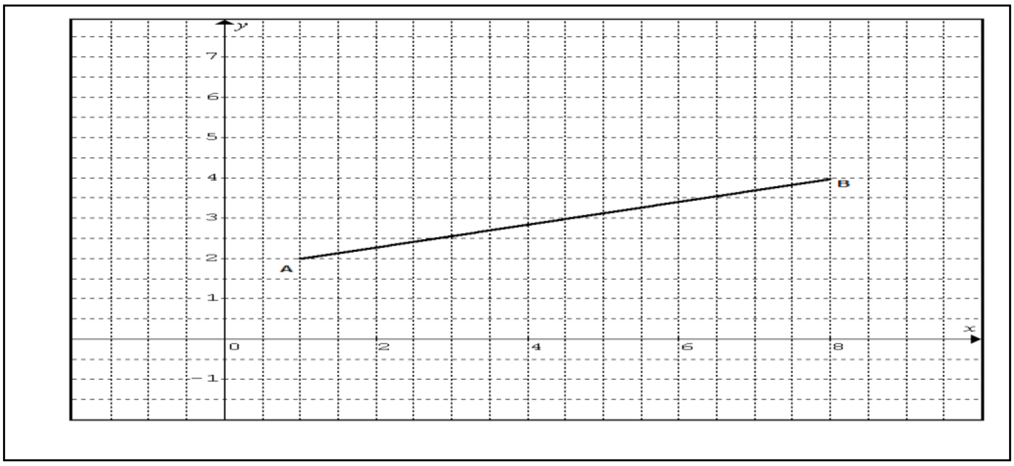
Teaching Experiment 1

The teaching experiment dealt with co-ordinate geometry. Learners had been introduced in the previous grade to the following understanding: plotting the Cartesian plane, distance between two points on a vertical or horizontal line, exponential laws, and properties of geometric figures and shapes.

At the end of the teaching experiment, learner's mathematical understanding will thicken (grow) in distance formula, properties of parallel and perpendicular lines.

Episode 1

In this episode an activity was designed for learners to create, manipulate, test and explore ideas about co-ordinate geometry. To do that, a line was drawn on a Cartesian plane which was accompanied by a set of questions to guide and validate their mathematical activity.



Learners were left to write the co-ordinate of point A and B, to determine the distance; midpoint; gradient of AB. The following task begins in a class as learners were left to work on the given activity. Mpho wrote the co-ordinates of A and B as: point A (1; 2) and B (8; 4). To determine distance, midpoint and gradient she wrote:

Distance AB

Let one block be = 0,5cm.
= 14blocks.
=7cm

Midpoint AB

$\frac{x+x}{2}$; $\frac{y+y}{2}$ = $\frac{8+1}{2}$; $\frac{4+2}{2}$ = $\frac{9}{2}$; $\frac{6}{2}$ *= 4,2 ; 3*

Gradient AB

$\frac{\Delta y}{\Delta x}$ = $\frac{2-4}{1-8}$ = $\frac{-2}{-7}$ *= 0,3*

Mpho started by calculating distance, midpoint and gradient of AB. She performed some actions which indicated evidence of image making. Mpho by writing the co-ordinates of point A and B respectively as (1; 2) and (8; 4), is an indication that she noticed the property of a point. Subsequently she noticed that a distance between blocks was 0,5. Her image is at property noticing. She moved to count the blocks and ended with fourteen blocks. She multiplied fourteen by 0,5 which gave her a distance of seven. We can see that Mpho was performing some physical activities like counting. When Mpho started to count, she moved out in her growth of understanding from property noticing to work on image making. Even though Mpho performed some physical activities on image making, I did not see her notice line AB as a slanting line which is formed by two lines, horizontal and vertical. She relied on the image she had of calculating the distance on a horizontal line and counts the distance of line AB to calculate distance AB. Mpho's growth of understanding was at image having. Without Mpho noticing that line AB is not a horizontal or vertical line, she would not be able to determine distance AB. The teacher wanted to ascertain whether Mpho does not sees line AB as formed by two lines or that her understanding is not adequately developed to allow her to see that. The previous activity was given to Mpho and Thabo to work in the present of the teacher. The dialogue unfolds as follows:

Vignette1

Thabo: We are going to count blocks
Mpho: From point A, how many blocks to this point with co- ordinate 8 and how Many block up to point B? (***image making)***
Thabo: We need a ruler to measure and count how many blocks do the ruler occupy. Let me show you (take a ruler and count). Is 1, 2, 3,...14 plus this(referring to the remaining space).
Mpho: Is seven blocks to the right and two blocks up. This distance is from right up. ***(property noticing)***
Thabo: I have another idea, lets drop this line.
Mpho: Ok we say 1 block = 1cm, then we count 1, 2, 3, ...7 and 8, 9, up, so distance is 9cm. This line will fall on point 8 and leave a space. ***(property noticing)***
Thabo: Can you see it falls on point 8 and leave one block.
Mpho: You know what, lets add the co- ordinates of A and B, that is 8- 1 = 7, 4 – 2 = 2 and here is 9cm, 7cm to the right and 2cm up. **(*property noticing)***

Mpho started by counting blocks. She was working at image making. She noticed that line AB composed of seven blocks to the right and two blocks up. When she indicated directions as right and up, she moved from image making to property noticing. She was uncertain about what she got. She went back to count and used a ruler to pull down the line. She moved back from property noticing to image making. While on image making, she noticed that the line falls between eight and nine. She cannot account for the remaining space, she moved back again to image making and subtracted as well as added the co-ordinates of point A and B. She noticed that the distance to the right is 7cm and 2cm up together made 9cm. Mpho's articulation of right and up indicates that she was out to perform at property noticing because she could notice that line AB is made up of horizontal and vertical line. What she has accumulated from image making did not enable her to perform well at the property noticing level. Mpho's growth did not allow her to notice that line AB could be one side of a right angle triangle which might lead to application of theorem of Pythagoras. Addition of 7 and 2, indicated that her growth at property level was not well enough developed. Unfortunately Mpho needs to have an understanding that would allow her to see those two lines forming a right angle. Having such an understanding would enable her to determine distance AB.

Vignette 2

Having realized that Mpho's growth in orientations around a right angle is not adequately developed; I interviewed her with the intention of taking her back to image making level to enhance her understanding. The interview unfolded as follows:

> Teacher: look at this picture. What kind of picture can you draw? (**image making)** ***(fold back to collect)***
> Mpho: Right angle triangle ***(property noticing)***
> Teacher: Can you show me? (**image making)** ***(fold back to collect)***
> Mpho: Like this (she draws horizontal and vertical lines connecting to line AB). ***(property noticing)***
> Teacher: How can you calculate one side of right angled triangle if given 2 sides? **(image making)** ***(fold back to collect)***
> Mpho: Ok this is hypotenuse, adjacent and opposite. Is theorem of Pythagoras (**thickening)**, we are going to substitute:
> $AB^2 = 7^2 + 2^2$
> $= 49 + 4$
> $= 53$, the square root of 53 = 7,3 ***(property noticing)***

The teacher engaged Mpho in a directed image making activity. The teacher asked her to draw a picture using line AB, Mpho carried out this physical action and draws a right angle triangle. In doing that, I suggested that Mpho has now folded back to image making and do something physical that would allow him to modify her inappropriate understanding of theorem of Pythagoras. Mpho noticed that the triangle was a right angled triangle. That indicates that she had moved out in her growth of understanding to work on property noticing. She further noticed that the line was made up of horizontal and vertical line. I saw her drawing horizontal and vertical line. She again illustrated evidence of property noticing. The teacher gave light by asking her, how one side of a right angled triangle could be calculated? The teacher's question prompted Mpho to fold back eventually noticed the sides as hypotenuse, adjacent, and opposite and lastly theorem of Pythagoras. She applied her understanding at property noticing and finds distance AB.

Throughout the activity, Mpho never applied a distance formula to calculate distance AB. The teacher assumed that Mpho's growth of understanding at property noticing level is not well enough developed to allow the use a distance formula, which is at formalizing. Pirie and Kieren (1992) refer to such understanding as disjoint-understanding which occurs when a student work with information that does not become connected to her own constructed knowledge. It is represented with unattached cross, to indicate that the understanding at this point was not connected to or based on the student's current understanding. To introduce new understanding so that Mpho's growth at property noticing level connects to formalizing; she presents a distance formula. The dialogue unfolds as follows:

> Teacher: there is another way of calculating distance AB. We can use a distance formula like this. (the teacher writes distance formula & substitutes the values of x and y)
>
> Mpho: where did you find the values of x and y?
>
> Teacher: from the co-ordinates of A and B. (the teacher continue calculating)
>
> Mpho: Ah! The answer is the same as using theorem of Pythagoras!
>
> Teacher: yes. A distance formula is used to find any distance on a straight line.

The teacher's intervention folded Mpho back to image making and used a distance formula to calculate distance AB. During working at image making, Mpho realized that the previous answer she got using theorem of Pythagoras is the same as that of distance formula. Mpho say "ah the answer is the same". Mpho has moved out to formalizing.

In this episode Mpho's growth of understanding the distance between two lines started at image having. She had an image of a horizontal line. She noticed that line AB is formed by horizontal and vertical lines. She moved out to property noticing. For her growth to grow, the teacher; through questioning, folded her back to image making to re-work her growth which was not well developed at property noticing level. Working at image making, Mpho noticed a formation of right angled triangle as well as application of theorem of Pythagoras. The teacher presents a distance formula as a way of connecting the existing growth with the new ones. She used a distance formula to calculate distance AB. Mpho's growth moved out to formalizing. The map of Mpho's growth in understanding of the distance between two points would look like figure 2

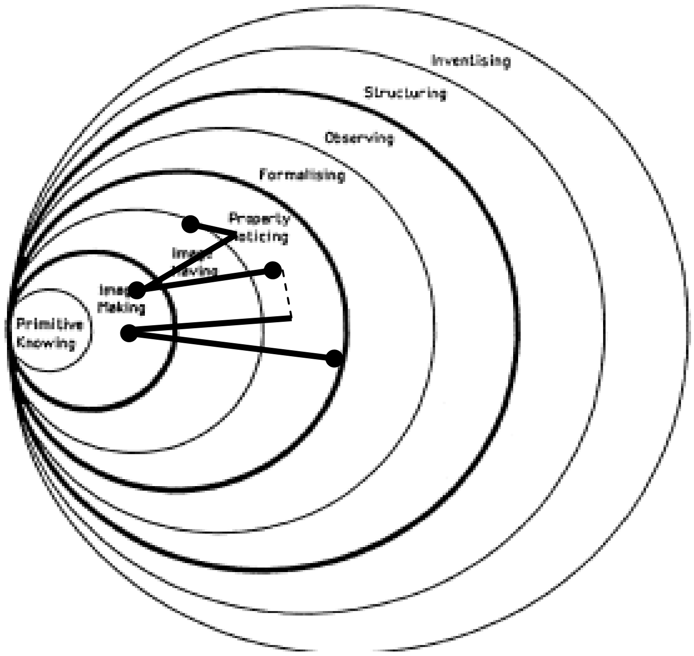


Figure 3: Mpho's growth in understanding of distance between two points

Episode 2

In this episode an activity was once more designed for learners to create, manipulate, test and explore ideas about co-ordinate geometry. To do that, a four sided figure with A(3,7); B(1,2); C(5, 1); D(7,5) was drawn on a Cartesian plane which was accompanied by a set of questions to guide and validate their mathematical activity. Learners were left to determine the distance; midpoint; gradient, the shape and lastly justify it.

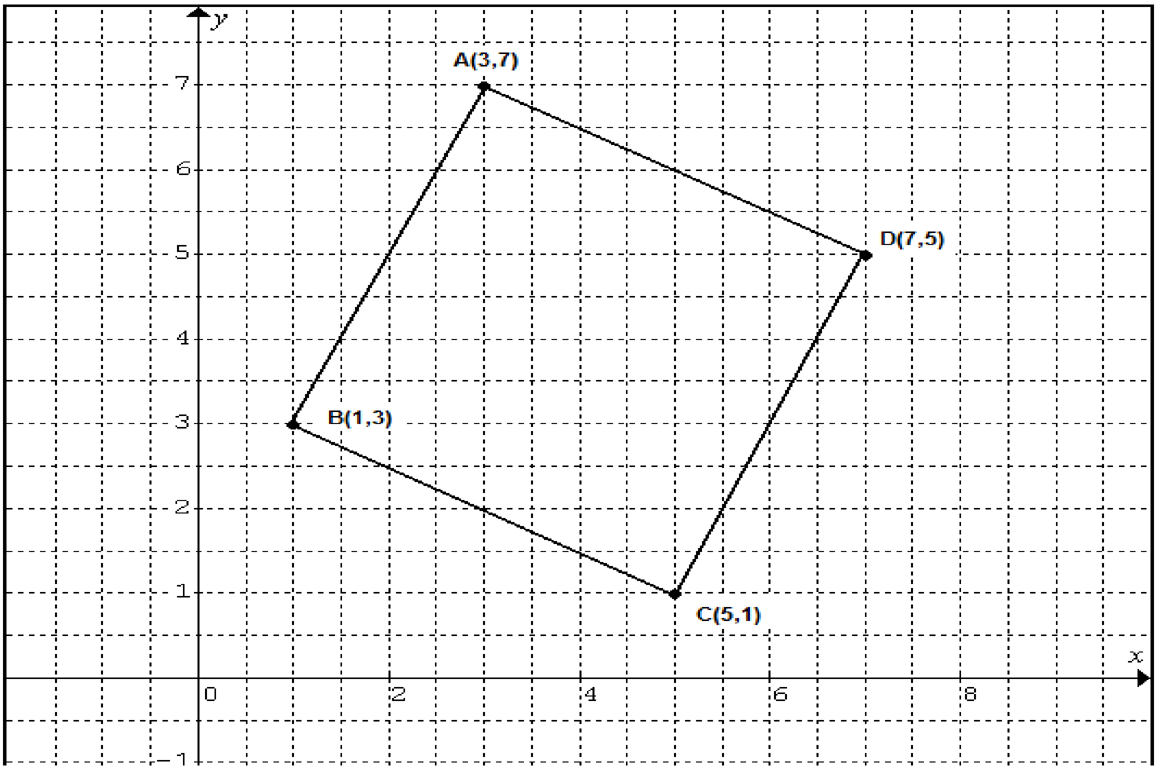


Figure 3: quadrilateral ABCD drawn on a Cartesian plane

This activity started when Mpho determined the distance, midpoint, gradient, and justifies the shape in the classroom. She started by calculating distance, midpoint and gradient. Her calculations were as follows:

Distance AB

$=\sqrt{(x-x)-(y-y)}=\sqrt{(1-3)+(3-7)}=\sqrt{20}$ = 4,5 $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Distance BC

$\sqrt{(x-x)-(y-y)}=\sqrt{(5-1)+(1-3)}=\sqrt{20}$ = 4,5

Distance CD

$\sqrt{(x-x)-(y-y)}=\sqrt{(7-5)+(5-1)}=\sqrt{20}$ = 4,5 $\frac{s}{p}$ =

Distance AD

$\sqrt{(x-x)-(y-y)}=\sqrt{(7-3)+(5-7)}=\sqrt{20}$ = 4,5

Gradient AD

$\frac{y-y}{x-x}=\frac{5-7}{7-3}=\frac{-2}{4}$ = - 0,5

Gradient CD

$\frac{y-y}{x-x}=\frac{5-1}{7-5}=\frac{4}{2}$ =2

Gradient AB

$\frac{y-y}{x-x}=\frac{3-7}{1-3}=\frac{-4}{-2}$ = 2

Gradient BC

$\frac{y-y}{x-x}=\frac{1-3}{5-1}=\frac{-2}{4}$ = - 0,5

The quadrilateral shape is a rhombus. According to the calculations of the gradients AB//DC; AD//BC and distance AB is equal to distance DC; distance AD is equal to BC.

The action of calculation indicated evidence of image making. During her working at image making, she had an image of a rhombus. She said that the quadrilateral is a rhombus. Mpho moved out from image making to image having. She noticed that opposite side are equal and parallel. She wrote that AB= DC, AD= BC, and AB//DC, AD//BC. Mpho illustrated working on property noticing. She based her reason for parallel lines on calculations of gradients. She wrote that according to calculations of the gradients AB//DC and AD//BC. She did not come clear what she meant by 'according to calculation of gradient'. She is unable to reason why lines are parallel. I was not certain whether she meant that gradients are equal. But by looking at her calculations of gradients, gradient AB and DC; AD and BC are not equal. I suggest that Mpho either not had the necessary understanding of relationship between gradients and parallel lines or that his understanding was not well developed enough to allow her to use it. Instead, prompted by the teacher, she folded back to image making to perform more images to build a new image and to enhance an existing one. The teacher wanted to move Mpho's growth to another level. The dialogue unfolds as follow:

Vignette 1

Teacher: You said the shape is Rhombus, agree!
Mpho: It is a rhombus because all sides are equal, but no right angles. ***(property noticing)***
Teacher: why are you saying all sides are equal?
Mpho: According to our calculations, AB = DC, AD = BC, and AB // DC, AD // BC ***(property noticing)***

Teacher: why is AB // DC?
Mpho: From the picture. You can see that this line does not meet. (Indicating by hand). **(property noticing)**
Teacher: By looking at the picture we can conclude that AB // DC? Really? **(image Making)**
Mpho: AH! Yes
Teacher: When lines are parallel, their gradients are equal. Like if AB // DC, the gradient of AB must be equal to the gradient of DC. ***(image making) [fold back to thicken]***
Mpho: Yes they are equal (referring to her calculations). **(formalizing)**
Teacher: But only parallel lines can determine that this shape is a Rhombus?

Mpho: No, it has no right angle. There are two Obtuse and acute angles. ***(property noticing)***
Teacher: why?
Mpho: because angle C is more than 90^0. ***(property noticing)***
Teacher: How did you find to be more than 90^0?
Mpho: They told us that obtuse, acute, right angles stand like this. (make drawings of angles). DC is not straight. ***(property noticing)***
Teacher: So it means an angle of 90^0cannot be formed by slanting lines? Look at

This: (teacher draws right angled in different shapes) Do you think there is no angle equal to 90^0?

Mpho: There are, just that they have changed their positions. **(image having)**

Teacher: If there is an angle of 90^0, product of gradients must be equal to -1.

Like $M_{AB} \times M_{DC} = -1$, $M_{AD} \times M_{BC} = -1$ **(image making) [*fold back to Thicken*]**

Mpho calculated the gradients, and finds their products equals to -1. ***(image making)***

Mpho: Oh! The shape is a square! Jo! Jo! Jo! **(formalizing)**

The intervention of the teacher caused Mpho to fold back to image making and noticed the shape as rhombus with opposite sides equal and parallel. In addition to that, she said that there was no right angle. Her explanation indicated the evidence of property noticing. Mpho derive a reason for parallel lines from the image she holds of parallel lines. She said that lines do not meet. Her understanding of parallel lines as lines that do not meet illustrates working at property noticing. Looking at Mpho's explanation of parallel lines, one may conclude that her growth in understanding at that level is at lowest which denied her an opportunity to notice properties of parallel lines. Such understanding Pirie and Kieren (1992) referred to as disjoint-understanding. To make connections of the previous understanding with the new understanding, the teacher presented a statement that lines are parallel if their gradients are equal, which is one property of parallel lines. The statement caused Mpho to fold back to image making to rework her understanding on parallel lines. Mpho moved out in her growth in understanding at image having to collect information at image making. During her collection she noticed that gradients of opposite lines are equal. She indicated by saying that they are equal. Mpho has moved out to property noticing.

Mpho still maintained that the shape is a rhombus because there is no right angle. This also illustrated evidence of property noticing. She had an image of right angle formed by vertical and horizontal lines. She illustrated by saying DC is not straight and also drawing a right angle with vertical and horizontal lines. The action indicated evidence of property noticing. This illustrated that Mpho's growth in understanding right angle would not afford her opportunity to notice properties of right angle. She had a disjoint understanding (Pirie and Kieren, 1992). To make connection of previous understanding with new ones, the teacher presents the statement that an angle of 90 degrees is formed when the product of gradients is equal to minus one. The aim was to fold Mpho back to image making to rework her understanding on orientation of a right angle. Mpho calculated the gradients. She performs actions at image making. During her working at image making, she noticed that the product of gradients is equal to minus one. Mpho has moved out to property noticing. She concluded that the shape is a square no longer a rhombus. She said “oh! The shape is a square! Jo! jo! jo!" I assume that Mpho has moved to formalizing by justifying the shape as a square.

In this episode Mpho’s growth in understanding of parallel and perpendicular lines started at image having. She referred the shape of a quadrilateral to rhombus. The growth moved to property noticing. She noticed that opposite sides are equal, parallel, and with no right angle. Her growth of understanding parallel lines and right angle was not well developed. The teacher presented two statements, lines are parallel if their gradients are equal and a right angle is formed if product of gradients is equal to -1, in order to connect Mpho’s existing growth with the new one. The statements folds Mpho back to image making. During her working at image making, she noticed that lines are parallel if their gradients are equal and also that a right angle is formed if product of gradients is equal to -1. Mpho moved out to property noticing. She concluded by saying the shape is a square. This illustrated evidence of formalizing. Mpho’s growth in understanding of parallel and perpendicular lines would be mapped like figure 4

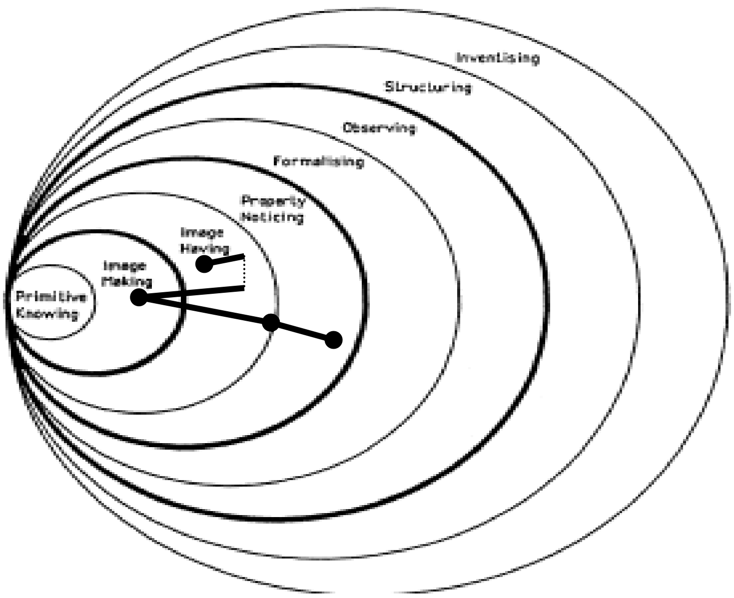


Figure 4: Mpho's growth in understanding of parallel and perpendicular lines

The growth of understanding of Mpho throughout the teaching experiment would look like:

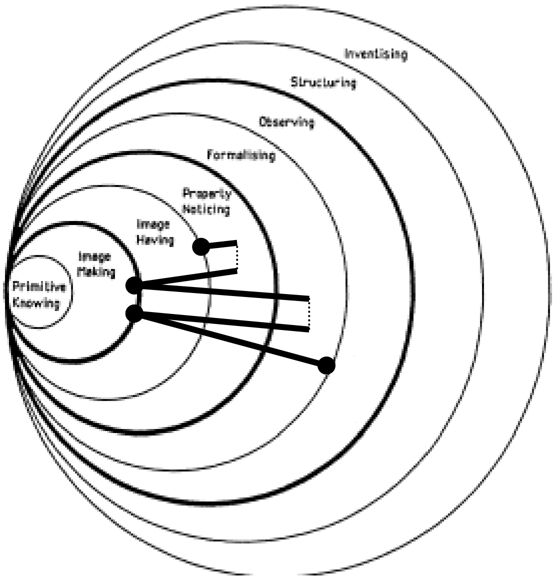


Figure 5: Mpho's growth in understanding of distance between two points, parallel and perpendicular lines

Teaching Experiment 2

This teaching experiment dealt with exponents. Learners have been introduced in the previous grade to the following understanding of exponents: laws of exponents; addition, subtraction, multiplication, and division of integers; language of algebra e.g. $2 = 2^1$; $2y = 2(y)$; $2y^2 = 2(y^2)$; factorise (common factor)

At the end of the teaching experiment a learner's mathematical understanding will thicken (grown) in language of algebra.

Episode 1

For learners to engage mathematically, the following activity was given to them to simplify: $2a^2$ X $2a^2$. Learners were left to work on activity and submit on completion. The activity was carried out in a classroom.

The episode started after Mpho has submitted her paper sheet. She wrote $2a^2$ X $2a^{-2}$ = $2a^{2 + (-2)}$ = $2a^0$ = 2. During her working at image making, Mpho noticed 2a as a single base with 2 and -2 as its exponent. She applied exponential law of multiplication and adds 2 and -2 to get $2a^0$. The actions illustrated working at property noticing. Taking 2a as a single base might originated from lack of language of algebra (understands what is meant by 2a). In short we can say she lack meaning making. This illustrated that Mpho's growth in understanding exponents at that level is not well developed. Mpho needs to have an understanding of meaning making which would help her to realize single and composite base in order to work successfully on this activity.

Vignette 1

In this vignette, the same activity was given to Mpho with an assistance of Thabo. The aim was to certify whether Mpho's growth of understanding exponents is not well developed in meaning making. The teacher listens to their conversation and engages them in their discussions. The following dialogue occurred:

Mpho: $2a^2 = (2a \times 2a)$ ***(property noticing)***
Thabo: Hm! Hm! Is this square for a or 2? This squared is for a.
Mpho: If we have $4^2 = 16$ is 4 X 4. ***(property noticing)***
Thabo: If is a^2, it means two a's (a X a). What is 2a X a =?
Mpho: $2a^2$
Thabo: Thank you, why did you put 2(square) on 2?
Mpho: If it is 41^2, it means 1^2, that is 1 X 1 ***(property noticing)***
Thabo: This a is a variable
Teacher: Mpho $2a^2$ is the same as 2 X a^2, is it correct? ***(image making)*** ***(fold back to thicken)***
Mpho: Yes
Teacher: This square is for 2 or a?***(image making, fold back to thicken)***
Mpho: Is for a

Mpho's answer on $2a^2$ as 2a X 2a confirmed that her growth at property noticing was not well developed in meaning making. In the case of single base like 4^2 she noticed the base as 4 and exponent as 2 while in composite base like $2a^2$ she noticed base as 2a (single base) and exponent as 2. Taking 2a as single base is an indication that Mpho lacks algebraic meaning. The actions indicated evidence of property noticing. Thabo's question invoked the folding back to image making which was not effective. Mpho still was unable to differentiate a single base and a composite one. She said 41^2 is 1 X 1. The teacher's intervention causes Mpho to fold back to image making. The teacher asked her whether 2 X a^2 is $2a^2$? Mpho noticed that $2a^2$ is a composite number made up of 2 and a^2. The exponent of two is for base a not base two. Mpho again illustrated working at property noticing.

Vignette 2

In this vignette the focus is still on Mpho's written work: $2a^2$ X $2a^2$ = $2a^{2+(-2)}$, particularly on base 2. The teacher wanted to modify Mpho's growth of algebraic meaning. The following dialogue occurred:

> Teacher: What is happening here? Are you happy with $2a^{2 + (- 2)}$?
> Mpho: If you have the same base, add the exponents.**(property noticing)**
> Teacher: If you have the same base you add the exponent. Have you done that?
> Mpho: Yes we write the base, this 2a **(property noticing)**
> Teacher: Then you add the exponent of a? ***(image making)***
> ***(fold back to thicken)***
> Mpho: No, oh here is 2 and the exponent is 1. $2^{1 + 1}$ $a^{2 + (- 2)} = 2^2 a^{2- 2} = 4a^0 = 4$.

Mpho noticed property of exponent, which was: multiplying numbers on the same base we add the exponents. She added the exponents of a. Mpho's actions illustrated working at property noticing. The teacher realized that Mpho still lack meaning making. She did not notice that 2 is the same as 2^1.The question from the teacher "then you add the exponent of a" folds Mpho to image making. During her working at image making she noticed that two is a base, has an exponent of one. We see Mpho adding the exponent of two and arriving at the answer. The actions indicated evidence of moving out to property noticing.

Mpho's growth in understanding of algebraic meaning in this episode started at property noticing level. She noticed 2a as a single base as well as law of exponent of multiplication. She added 2 and -2. The teacher through questions folded her back to image making to re-work her understanding of algebraic meaning. During her working at image making she noticed that 2a is made up of 2 and a, which is a composite base. Even though her growth of understanding has been developed in composite base, her growth is not completely developed in algebraic meaning. She still insisted on adding the exponent of a. She folded back to image making. She noticed that 2 had an exponent of 1. Mpho's growth in understanding of meaning making is mapped like in figure 6

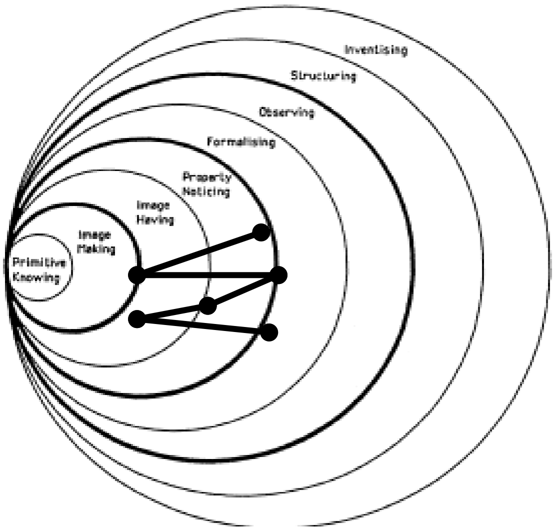


Figure 6: Mpho's growth in understanding of algebraic meaning

Teaching experiment 3

The teaching experiment dealt with function of y = ax^2 + q. It consists of two episode and four vignettes. Learners have been introduced in the previous grade to the following understanding: plotting point on a Cartesian plane, ordered pairs, y and x-intercept. At the end of teaching experiment learner will be able to draw a graph without a table, notice the effect of the value of a and q on the graph.

Episode 1

In this episode an activity was designed for learners to create, manipulate, test and explore ideas about the function y = ax^2 + q. To do that, learners were given a table to complete and thereafter draw graphs, lastly answer several questions based on the graph. The table includes the x- values from -4 to 4. Learners were left to complete the table, draw the graph and answer the questions.

Table 1: Activity 1

Copy and complete the following table and draw the graph of given functions on the same set of axes.

x	-4	-3	-2	-1	0	1	2	3	4
$y = x^2$									
$y = 2x^2$									
$y = 3x^2$									
$y= (1/2)x^2$									

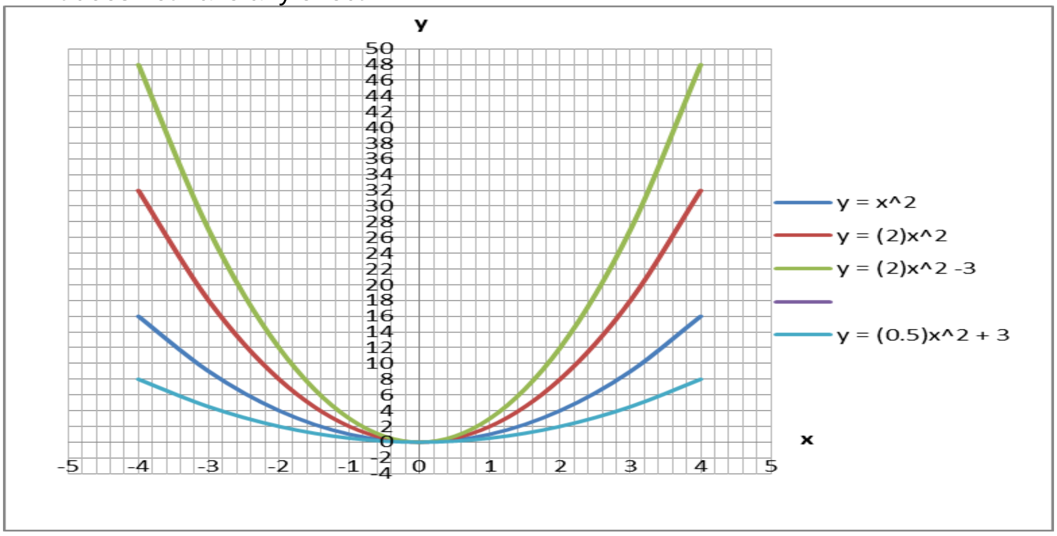
1.1. Now look at the table again. Once all the entries of the row $y=x^2$ are entered, is it possible to complete the rest of the table using those values. Explain by means of examples.
1.2. Do you notice anything about the shape of the graphs? Explain.
1.3. What happens to the value of y as x increases or decreases?
1.4. Without completing the table. Explain how the graph of $y=4x^2$ would be like as compared to that of $y=x^2$.
1.5. What effect does the value of **a** have on the graph of $y=x^2$?

Table 2: Mpho's responds on activity 1

Mpho starts by completing the table, draw the graphs and answer questions on an answering sheet and graph paper. Her work was as follows:

X	-4	-3	-2	-1	0	1	2	3	4
$y= x^2$	16	9	4	1	0	1	4	9	16
$y= 2x^2$	32	18	8	2	0	2	8	18	32
$y= 3x^2$	48	27	12	3	0	3	12	27	48
$y= (1/2)x^2$	8	4,5	2	0,5	0	0,5	2	4,5	8

1.1 Yes, the shape of the graph is a curved.
1.2 The value of y also increases or decreases.
1.3 $y = 4x^2$ is going to be curved graph.
1.4 It does not have any effect.



Mpho performed some physical actions such as completing the table and drawing the graph. Mpho had moved out of her growth in understanding to work on the image making. The image she had of a graph is a curved graph. She moved out to image having. She noticed that as the value of x increases or decreases, the value of y increased and decreases respectively. The actions indicated evidence of property noticing. During her working at property noticing she concluded that the value of a did not have any effect on the graph. She had not moved to formalizing.

Mpho's answer on 1.4 did not give a clear explanation. Mpho's graph did not support what she wrote on 1.4. That the value of a do not affect the graph. I noted some changes on the graph but she said there was no change. I assume that Mpho might have a difficulty in the meaning of the word effect. The teacher wanted to be certain whether Mpho had a difficulty in meaning making. The teacher return the same activity to Mpho only focused mainly on the effect of the value of a on the graph. She wrote that:

1.1 Yes, they all have a minimum turning point for the curve which is (0; 0).
1.2 The value of y also increases or decreases
1.3 The graph of $y = 4x^2$ will also be curved graph as compared to that of $y = x^2$, due to the effect of a where a >0 and the two graphs has minimum turning point
1.4 The graph of $y = x^2$ has the value of a as 1, which means the graph will have a turning point and also face downwards as the value of a > 0.

Mpho noticed that the graph had minimum turning point of (0; 0), graph is curved, and when a > 0 the graph faces downwards. Mpho's action indicated evidence of property noticing. Mpho sees many graphs represented on the same Cartesian plane, not one graph represented differently. When asked about the effect of a on the graph $y=x^2$, she thinks that the question refers to the graph of $y=x^2$ only. She gave the value of a as 1. She should have presented different values of a. This confirms that she has a difficulty in meaning making. The teacher interviewed her to enhance her understanding on the effect of a on the graph.

Vignette 1

In this vignette the teacher worked together with Mpho. The teacher collected Mpho's written work and they both looked at the graphs. They both started working at image making level. They started with the graph on table 4 which dealt with the value of a whereby a is greater than one. The teacher wanted to move Mpho's growth to other levels. The dialogue unfolds as follows:

Teacher: look at the graph of y= x^2
Mpho : here it is (she shows by a hand)
Teacher: look again at the graph of y= $2x^2$. What do you notice?
Mpho : both graphs faced upward, but the graph of y= $2x^2$ is inside the graph of y=x^2 **(property noticing)**
Teacher: ok look again at the graph of y= $3x^2$. What do you notice?
Mpho : they both faced upward. The graph of y= $3x^2$ is inside both graphs. **(property noticing)**
Teacher: these graphs are heading somewhere. Where are they heading to?
Mpho : yes they are heading towards the y- axis. **(property noticing)**
Teacher: what do you think about the graph of y= $4x^2$? Where will the graph be?

Mpho : the graph will be closer to y= axis than the graph of y= $3x^2$. The graph will be in front of the graph of y= $3x^2$. **(formalizing)**
Teacher: what do you think about the graph of y= $6x^2$?
Mpho : the graph will be much closer to the y- axis. **(formalizing)**
Teacher: ok what can you say about the value of a? Is the graph narrower or wider?
Mpho : as the value of a increases the graph becomes narrower. **(formalizing)**

In this vignette, the teacher engaged Mpho in a direct image making activity. The teacher told Mpho to look at the graph and Mpho carries out this physical action. In doing this, I suggest that Mpho has folded back to do something physical that would allow him to modify his inappropriate image of meaning making. During her working at image making, she noticed that when a is greater than 1 the graph faces upward no longer downwards as previously noted, the graph come closer to y- axis. She illustrated evidence of property noticing. She generalized how other graphs would look like for example the graph of $4x^2$ and $6x^2$. She said that the graph will be closer to y-axis. She moved out to formalizing. The teacher realized that term such as narrower did not emerge in Mpho's growth in understanding. She brought it as a question. This allows Mpho to generalize that as the value of a increases, the graph becomes narrower. She again illustrated evidence of formalizing.

Vignette 2

In this vignette, the teacher worked again with Mpho. They are again using the graph in table 4 which deals with the value of a. The focus is on the value of a less than one. The teacher here has the same motive as in vignette 1 that is moving Mpho's growth to other levels. The dialogue unfolds as follows:

Teacher: let's go back to the graph of y= x^2. Look at the graph of y= $1/2x^2$. What do You notice?
Mpho : the graph is below the graph of y= x^2. **(property noticing)**
Teacher: is the graph closer to the y- axis or the x- axis?
Mpho : the graph is closer to the x- axis. **(property noticing)**
Teacher: let's draw the graph of y= $1/4x^2$ (they both draw the graph on a separate page). What do you notice? **(image making)**
Mpho : the graph is below the graph of y=$1/2x^2$and y= $1/4x^2$ and more closer to

the x- axis. **(property noticing)**

Teacher : what do you think about the graph of y= $1/8x^2$?

Mpho : the graph will be much closer to x- axis. **(formalizing)**

Teacher : what can you say about the value of a? Is the graph narrower or wider?

Mpho : the value of a decreases. The graph becomes wider. **(formalizing)**

In this vignette, the teacher engaged Mpho in a direct image making activity. The teacher told Mpho to look at the graph and Mpho carried out this physical action. In doing that, I suggest that Mpho has folded back to do something physical that would allowed him to modify his inappropriate image of interpret the graph. During her working at image making, she noticed that when a decreases the graph faces upwards and comes closer to the x- axis. She illustrated evidence of property noticing. She generalized how other graph would look like for example the graph of $1/8x^2$ would be closer to the x- axis. She moved out to formalizing. The teacher used a question in order to replace the word closer with the word wider. She wanted to know whether the graph will be narrower or wider. The question made it easier for Mpho to generalize that as the value of a decreases, the graph becomes wider. She again illustrated evidence of formalizing.

Mpho's growth in understanding the effect of a on the graph of $y = ax^2 + q$ in this episode started at image making. She completed a table and drew graphs. Her growth moved to image having. She said the graph is curved. She noticed that when $a > 0$, the graph faces upwards. When $a > 1$, the graph comes closer to the y- axis and when $a < 1$ the graph comes closer to the x- axis. She moved out to property noticing. She had a disjoint understanding of terms which the teacher brings in through questions. Is the graph narrower or wider? She generalizes that if $a > 1$ the graph becomes narrower and if $a < 1$ the graph becomes wider. Mpho's growth may be mapped like in figure 7

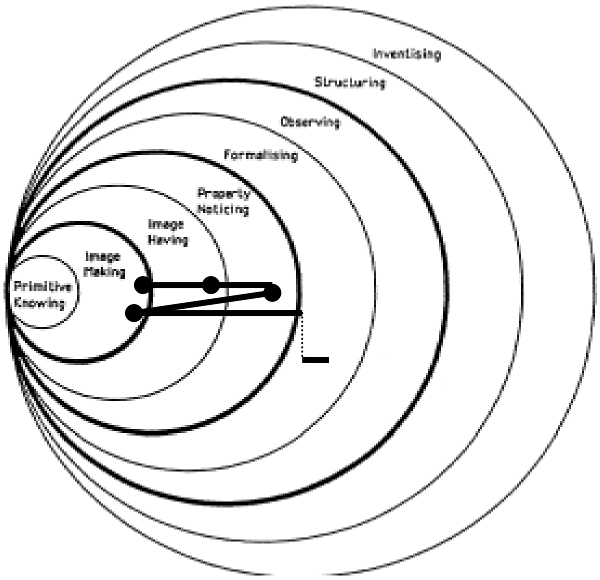


Figure 7: Mpho's growth in understanding of the effect of a on $y = ax^2 + q$

Episode 2

In this episode an activity was again designed for learners to create, manipulate, test and explore ideas about the function $y = ax^2 + q$. To do that, learners were given a table to complete and thereafter draw graphs, lastly answer several questions based on the graph. The table includes the x- values from -4 to 4. Learners were left to complete the table, draw the graph and answer the questions.

Table 3: Activity 2

Copy and complete the following table and draw the graphs of the given function on the same set of axes.

X	-4	-3	-2	-1	0	1	2	3	4
$y = x^2$									
$y = x^2+1$									
$y = x^2+2$									
$y = x^2+ 3$									
$y = x^2- 2$									

2 Now look at the table again. Once all the entries of the row $y = x^2$ are entered, is it possible to complete the rest of the table using those values. Explain by means of examples.

2.1 Do you notice anything about the shape of the graph? Explain.

2.2 What happens to the value of y as x increases or decreases?

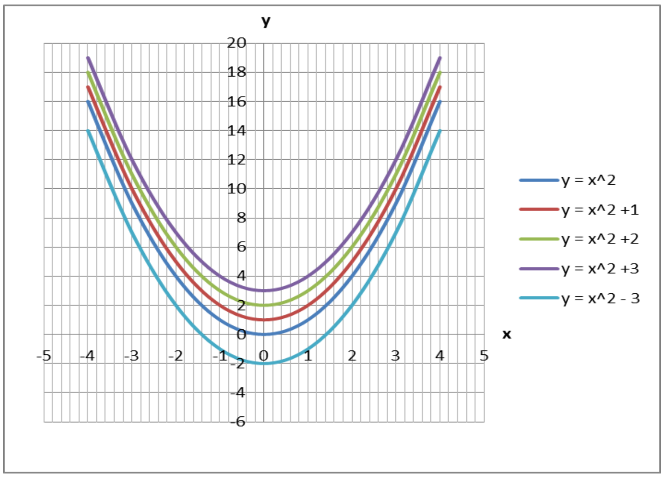
2.3 Without completing the table, explain how the graph of $y = x^2 + 6$ would be like compared to the graph of $y = x^2$.

2.4 What effect does the value of q have on each graph?

Table 4: Mpho's responds on Activity 2

Mpho completed the table and her answer to the activity was as follows:

X	-4	-3	-2	-1	0	1	2	3	4
$y = x^2$	16	9	4	1	0	1	4	9	16
$y = x^2+1$	17	10	5	2	1	2	5	10	17
$y = x^2+2$	18	11	6	3	2	3	6	11	18
$y = x^2+ 3$	19	12	7	4	3	4	7	12	19
$y = x^2- 2$	14	7	2	-1	-2	-2	2	7	14



2.1 Yes, the shape of the graph is going to be a curved or more V shape, as the Value of x decreases the value of y decreases

2.2 It also decreases x values e. g. (-4; 16) and (-3; 0)

2.3 The graph will be V shaped graph

2. 4 it does not have any effect as there is no value of q

Mpho started by completing the table and drawing different graphs on the same Cartesian plane. She was performing some action at the image making. She noticed that the graph is curved, that x and y are directly proportional, and no values of q. Mpho has moved out to property noticing. Looking at Mpho's graphs, I saw some changes on the graphs but she said that q did not affect the graph because there is no value of q. Mpho had a difficulty in meaning making. She did not know what q meant on the graph. Understanding of the value of q would enable her to notice changes on the graph. The teacher interviewed her in order to enhance in meaning making.

Vignette 1

This vignette resemble vignette one and two in episode 1. The teacher worked together with Mpho. They collected Mpho's written work and both looked at the graphs. In this instance, they started working with the graph on table six that deals with the value of q whereby the value of q is greater than zero. The teacher wanted to shift Mpho's growth to another levels. They both started at image making level. The dialogue unfolds as follows:

Teacher : look at the graph of y= x^2 and y= x^2+ 1. What do you notice?
Mpho : both graphs looked upward. The graph of y= x^2+ 1 has moved upward. **(property noticing)**
Teacher : by how many units from the graph of y= x^2?
Mpho : by 1 unit. **(property noticing)**
Teacher : it is the only thing that you have noticed?
Mpho : hmm! Oh the turning point. **(property noticing)**
Teacher : what about the turning point?
Mpho : the graph of y= x^2 its turning point is (0;0) and of y= x^2+ 1 is (0;1). **(property noticing)**
Teacher : where does this 1 come from?
Mpho : from the equation y= x^2+ 1 (she circle 1 in the equation) **(property noticing)**
Teacher : and this 1 is the value of a or q? **[thickening]**
Mpho : is the value of q. **(property noticing)**
Teacher : look again at those graphs and tell me another thing you notice.
Mpho :oh my God! no! no! no! Nothing.
Teacher : what about their distance from y- axis? (the teacher use hand to explain)
Mpho : The distances are equal. There is a line that cut the graph. What do we call this line? (she indicated the line by hand) **(property noticing)**
Teacher : the axes of symmetry. The axes of symmetry of the graph y= x^2 is (0; 0) and the x value on the y-axis = 0. Ok look again at the graph of y= x^2 and y= x^2+ 2. What do you notice?
Mpho : the graph of y= x^2+ 2 has moved upward by two units. Turning point is (0; 2). **(property noticing)**
Teacher : what about the graph of y= x^2+ 6?
Mpho : oh the graph will move upward by 6 units. Turning point is (0; 6). **(formalizing)**

In this vignette, the teacher engaged Mpho in a direct image making activity again. The teacher told Mpho to look at the graph and Mpho carried out this physical action. In doing that, I suggest that Mpho has folded back to image making to do something physical that would allowed her to modify her inappropriate image of meaning making. During her working at image making, she noticed that both graphs faced upwards, the value of q determines the upward movement and the turning point of the graph. She says the graph of y = x^2 + 1, has moved up by 1 unit and turn at (0; 1). She indicated that one comes from the value of q. Mpho has moved out to property noticing. I saw Mpho stating that the graph of y = x^2 + 6 would face upward and turn at (0; 6) without working on it. This illustrated that Mpho has moved out to formalizing.

Vignette 2

The teacher worked together with Mpho. They collected Mpho's written work and both looked at the graphs. In this instance, they start working with the graph on table six that deals with the value of q whereby the value of q is less than zero. The teacher once more wanted to shift Mpho's growth to another level. The dialogue unfolds as follows:

Teacher : look at the graph of y= x^2 and of y= x^2 – 2. What do you notice?

Mpho : the graph of y= x^2 – 2 has moved down by two units. The turning point is (0; -2). **(property noticing)**

Teacher : what about the graph of y= x^2 – 4?

Mpho : the graph will move down by four units. The turning point is (0; -4) **(formalizing)**

In this vignette, the teacher engaged Mpho in a direct image making activity again. The teacher told Mpho to look at the graph and Mpho carried out this physical action. In doing that, I suggested that Mpho had folded back to do something physical that would allowed him to modify his inappropriate image of interpreting the graph. During her working at image making, she noticed that both graphs faced upwards, the value of q determines downwards movement and the turning point of the graph. Mpho said the graph of y = x^2 – 2 has moved down by two units, and turning point was (0; -2). Mpho has moved out to property noticing. We see Mpho stating that the graph of y = x^2 + 6 will face upward and turn at (0; 6) without working on it. This illustrated that Mpho has moved out to formalizing.

Mpho's growth of understanding in this episode started at image making. She completed a table and drew graphs. She had an image of a curved shape. She moved out to image having. She noticed that the graph has a turning point of 0 and the value of q. She further noticed through the intervention of the teacher that the graph has axes of symmetry. The term axes of symmetry did not emerge from Mpho's growth in understanding. She had a disjoint understanding. Lastly she generalizes that when q > 0, the graph moves up by q and when q < 0, the graph moves down by q. Mpho moves out to formalizing. Mpho's growth may be mapped like in figure 7.

Mpho's growth in understanding the effect of a and q on the graph of $y = ax^2 + q$ may be mapped like in figure 8

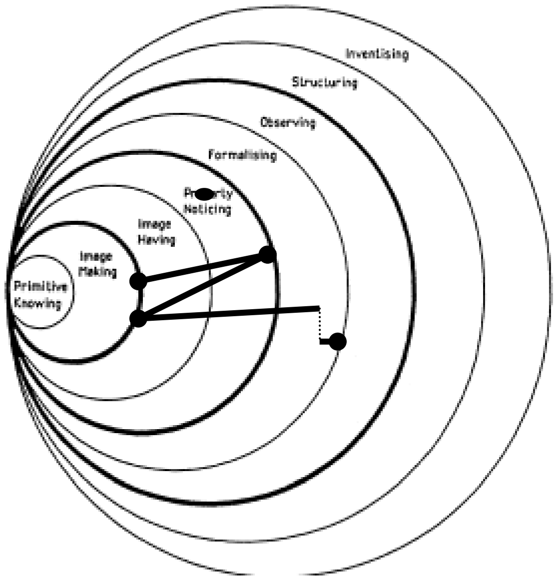


Figure 8: Mpho's growth in understanding the effect of a and q on $y = ax^2 + q$

Reflections on the three teaching experiments

The purpose of the study was to explore the growth of a learner's understanding through three teaching experiments which dealt with co-ordinate geometry, exponents, and function of the graph $y= ax^2 + q$. Pirie and Kieren (1991), characterised growth as a whole, dynamic, levelled but non- linear, transcendently recursive process.

The findings in this study coincided with Pirie and Kieren's characterization of growth as a dynamic, levelled but non- linear path. The level at which Mpho's growth in mathematical understanding started at each teaching experiment demonstrates that the growth is dynamic. This is demonstrated as she works with co-ordinate geometry, Mpho's growth started at image having; exponents, her growth in understanding started at property noticing; and with functions, her growth started at image making. Mapping of Mpho's growth in understanding in all teaching experiments clearly illustrated that the growth in understanding is levelled and not in linear path. In teaching experiment 1, Mpho's growth in understanding started at image having level moved back to image making, moved forward to property noticing back again to image making level and preceded forward to formalizing. In teaching experiment 2, Mpho's growth of understanding started at property noticing level moved back to image making level moved forward to property noticing level back again to image making level preceded forward to property noticing. In teaching experiment 3, Mpho's growth of understanding started at image making level moved to image having level and to property noticing level back to image making level preceded forward to formalizing. Mpho's growth of understanding formed a zig sag pattern.

Mpho's levels of mathematical understanding spanned the five levels, which are primitive knowing, image making; image having; property noticing and formalizing, of Pirie-Kieren (1994) model. The order of these levels was not based on the complexity of mathematics involved in the task. In most of teaching experiments, Mpho demonstrated the attainment of formalizing level. This was demonstrated in teaching experiment 1 and 3. However in teaching experiment 2 there was no evidence of attainment of formalizing level. The nature of the task required in each teaching experiment somewhat predefined the five levels spanned by Mpho's mathematical understanding. The activities given in all teaching experiments involved use of mathematical skills that required understanding involving formation of a right angled triangle and parallel lines; algebraic and function meanings.

Mpho's mathematical understanding did not always demonstrate direct movement between levels. Instead, some partial or full indirect movement between levels of understanding which is represented by dotted line were demonstrated by Mpho in teaching experiment 1 and 3. Pirie-Kieren (1994) refers to such an understanding as disjoint understanding.

The dynamic feature of folding back was evident in the ways Mpho developed her mathematical understanding. This feature was demonstrated in all teaching experiments. Pirie (1992, 1994, and 2002) argued that folding back was crucial to student's growth of mathematical understanding. In the case of Mpho folding back was crucial in conjecturing, validating and generalizing her solution to the problem. Mc Gaffrey at al. (2001) argued that instructional activities which emphasized problem solving, communication, reasoning and mathematical connections enabled students to develop complex cognitive skills and process. These forms of mathematical activities were included in each teaching experiment.

The feature of acting and expressing were also evident in the levels of Mpho's mathematical understanding. These were characterized by continuity with inner levels of understanding beyond primitive knowing (Pirie-Kieren, 1994). Mpho demonstrated understanding that encompassed engagement in mental or physical activities (acting) and ability to show (expressing) to herself or others the nature of those activities. Specific mathematical actions such as doing, reviewing, seeing, saying, predicting, and recording were demonstrated in various levels of understanding. Although Pirie-Kieren's model has the "don't need" boundaries features between image making and image having; property noticing and formalizing, the levels of mathematical understanding always gave rise to outer level of understanding. Examples of this can be found in teaching experiment 3 as Mpho worked through the function.

Results suggests that one way teachers may be able to help learners grow in their understanding of mathematical tasks is by encouraging them to fold back by asking prompting questions that provoke understanding. Such questions are demonstrated in teaching experiment 2. Pirie and Kieren (1991, 1994) asserts that folding back to previous level thickened growth in understanding. In Teaching experiment 1, 2, and 3 the teacher prompted Mpho, she folded back to image making and her understanding was thickened in: formation of a right angled triangle and parallel lines; algebraic meanings; and meaning making in functions.

Research on mathematical understanding (Pirie and Kieren, 1994) suggested that repeated access to image making activities is a necessary foundation for the growth of mathematical understanding. This thought was confirmed in all teaching experiments. Pirie and Kieren (1991) considers mathematical understanding as an on- going process in which a learner responds to the problem of reorganizing his or her knowledge structure by continually revisiting existing understanding. In all classrooms episodes described previously, what is consistent amongst many of episodes is that the learner revisits image making several times in order to generate thicker understanding. Mpho's understanding was characterized by a change in her ability to think about the mathematical relationships involved in the problem. Simon

(2002, 2006) called this a “key developmental understanding”. This form of understanding is important in her ability to develop conceptual understanding of mathematics.

This leads us to consider the effect and importance of teacher intervention in helping learner to fold back and rework their understanding. The reasons for intervention in both classroom episode is twofold: to direct the learner explicitly to what they need to do when they are stuck and to introduce new understanding which does not emerge in the growth of learner’s understanding. Examples of this can be found in teaching experiment 1 and 3, as Mpho worked through formation of right angle and function. Tower (1998) has developed a number of intervention themes that describe teacher’s action-in-the moment. Intervention themes such as showing, telling, shepherding reinforcing, inviting and glue- giving were demonstrated by the teacher in all teaching experiment.

Conclusion

In this chapter three teaching experiments were presented. Data was presented in the form of learner’s activities, transcripts and pictorial diagrams of Pirie and Kieren (1994) model. A reflection was done where quite a few findings were noted: growth in understanding is changing, moves back and forth within the levels, and recurring. An element of folding back played an important role in the growth of understanding. It thickens understanding. Image making is the only level where the growth moves back. These back and forth movement is caused by prompting questions.

CHAPTER 5 CONCLUSION AND RECOMMENDATION

Introduction

In this chapter a summary of the study is enfolded and recommendation given.

Summary

The purpose of the study was to explore the growth in mathematical understanding.
Research question:
In what ways are my grade 10 mathematics learner's understandings growing?

- The results of the study confirmed that the growth in understanding is changing, moves back and forth within the level of understanding, and it is an on- going process.
- When the growth folds back it thickened understanding.
- The results further indicated that the movement of growth is facilitated by prompting questions.

The review of literature led to the identification of Pirie and Kieren theory model as an appropriate tool for the study. Further review of application of the model by other researchers provided me with some insights as to how to use the model in my own classroom setting. The study was a qualitative case study allowing me to have an in-depth approach to issues under investigation. Mpho, a grade 10 mathematics learner became the focal point. Using a teaching experiment technique, Mpho was observed through a series of episodes on three different content areas such as co-ordinate geometry, exponents, and functions. Data were collected in the form of Mpho's written work, observations, and interviews. Data were recorded through the use of audio recorder. Analysis was done through mapping Mpho's growth in mathematical understanding employing Pirie and Kieren (1994) model. The results were that learner's growth in understanding is ever changing. The growth moves back and forth within the levels of understanding and is recurring. The results further shows that it is through prompting question that understanding grows. In figure 4, Mpho's growth in mathematical understanding started at image having level, moved back to image making level, proceeded forward to property noticing level, back to image making level, and lastly proceeded forward to formalizing. In figure 5, Mpho's growth in mathematical understanding started at property noticing level, moved back to image making level, forward to property noticing level, back again to image making level, lastly moved forward to property noticing level. In figure 7, Mpho's growth in mathematical understanding started at image making level, moved to image having level to property noticing level, back to image making level, and lastly moved forward to formalizing.

Recommendations

In the study, the growth in mathematical understanding was observed through the use of activities, questions, and teacher interventions in teaching experiments. The nature of the activities required the use of pre-knowledge and continuity. It allowed a learner to move from simple to abstract. In teaching experiment 1, the first activity used was to determine the distance, midpoint, and gradient on one straight line. The second task used four jointed straight lines to determine the distance, midpoint, gradient, and to define the shape with reasons. I recommend that teachers compose and make use of activities that encourages use of pre-knowledge in their daily lessons.

The questions used were prompting questions. Questions that compel a learner to give more of what she understood. Teachers need to be able to use such questions in teaching and learning situation. In order to do that, I recommend that teachers should be taken for in-service training.

The growth of mathematical understanding was explored through the use of three teaching experiment which dealt with three mathematical topics such as co- ordinate geometry, exponents, and functions. I recommend that further research be conducted on other areas of mathematics for a further exploration.

Even though teacher intervention was not my focus of study, it played a major role as far as Mpho's growth in mathematical understanding. It redirected her understanding. In teaching experiment 1 Mpho noticed the shape as a rhombus, through teacher's intervention her growth transformed and noticed the shape as a square. In teaching experiment 3, Mpho noticed that the value of a widen or narrowed the graph and the value of q moved the graph vertically up or down. I recommend that further research be conducted on the impact of teacher's intervention on growth in mathematical understanding.

Conclusion

In this chapter a brief summary of the study was outlined. Recommendations were further provided.

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