PROPERTIES AND CALCULUS ON PRICE PATHS IN THE MODEL-FREE APPROACH TO THE MATHEMATICAL FINANCE

by

Lesiba Charles Galane

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Supervisor: Dr. Farai J. Mhlanga
Co-supervisor: Prof. Rafal M. Łochowski
(Warsaw School of Economics)

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Declaration

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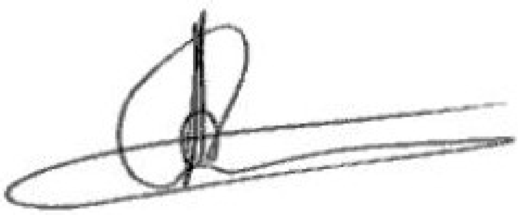

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Abstract

Vovk and Shafer, [41], introduced game-theoretic framework for probability in mathematical finance. This is a new trend in financial mathematics in which no probabilistic assumptions on the space of price paths are made. The only assumption considered is the no-arbitrage opportunity widely accepted by the financial mathematics community. This approach rests on game theory rather than measure theory. We deal with various properties and constructions of quadratic variation for model-free càdlàg price paths and integrals driven by such paths. Quadratic variation plays an important role in the analysis of price paths of financial securities which are modelled by Brownian motion and it is sometimes used as the measure of volatility (i.e. risk). This work considers mainly càdlàg price paths rather than just continuous paths. It turns out that this is a natural settings for processes with jumps. We prove the existence of partition independent quadratic variation. In addition, following assumptions as in Revuz and Yor's book, the existence and uniqueness of the solutions of SDEs with Lipschitz coefficients, driven by model-free price paths is proven.

Acknowledgements

First and foremost, I would like to give the glory and honour to God Almighty for the gift of life. His protection and blessings have seen me through challenging times during the writing of this thesis.

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Lastly, a special thanks to my family: my parents and my brother and sisters for supporting me spiritually throughout writing this thesis and my life in general. It has always perceived as impossible dream until it is done. When Sir Issac Newton (1676) wrote to his friend Robert Hooke, “If I have seen further, it is by standing on the shoulders of giants“.

Dedication

This thesis is dedicated to my parents, Frans Resimate Baloyi and Maria Raesibe Baloyi, and my beloved children ‘Kgadi Happy’ and ’Moloko Dylan’ Moremi.

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Chapter 1

Introduction

1.1 Background Information

Global financial crisis is always characterized by unique factors. For instance, the recent 2020 financial crisis is believed to be rooted from the outbreak of coronavirus pandemic. This pandemic has a negative impact on the world economy. A huge impact was felt on trade and tourism industries that were closed in order to contain the virus. This led to financial markets experiencing shock. The S&P 500 drops by 4.4% on the 28 February 2020. Also, the Shanghai stock market experienced 8% drop on the same date. It is believed that the spread of coronavirus pandemic increases the financial volatility and its persistence may lead to a new international financial stress [1].

The 2008-2009 financial crisis resulted from the wrong pricing of risk related to mortgages [5]. In particular, risks related to mortgages were much higher than the risk models used by the banks envisaged. Several strategies based on the diversification of wealth showed pessimistic results during this event, [15, 16]. In particular, equity markets performed poorly, with returns recorded at -50%. The Johannesburg Stock Exchange also recorded a 40% drop on the All Share Index [5].

Once the imminent danger has subsided to each financial crisis, the natural reaction is to retrospect, determine the course, and develop strategies to avoid or mitigate the impact of future crisis with similar characteristics. The poor performance of the global economy lead to the asset management industry trying to restore the situation and turned on the risk parity scheme [39].

Prior to the global financial crisis of the 2008-2009, more than $400 trillion worth of securities were priced by methods inspired by the Black-Scholes formula [8]. This approach assumes that security prices follow some probabilistic models such as Brownian motion. Under this assumption, options are priced by a formula that contains a parameter representing volatility (often considered a risk) of the stock price. This parameter is usually estimated from the past fluctuation of security prices [40]. However, the financial crisis raised an alarm based on the reliability and validity of this methodology.

In the recent years, the revival of the game-theoretic probability free approach has gained interest both in practise and academic [41, 50]. In these books, an outer-measure (often referred to as *Vovk's outer-measure*) of a set of price paths in continuous time has been defined which corresponds to a minimal superhedging price. The *outer-measure* is one of the trendiest characteristics of the game-theoretic probability free approach. It is perceived as a measure which precludes the wrong probabilistic assumptions in the development of financial models. It is emphasised by the appearance of the Brownian motion in the financial context [18].

This research follows the same trend in financial mathematics, where no assumptions on the probability measure on the space of price paths, representing possible prices of financial assets on some time interval, are made [11]. The only assumptions follow from the notions of no arbitrage opportunities, like “no free lunch with vanishing risk" or “no very large profit with a small risk", widely accepted by the mathematical finance community as essential properties which prices of financial assets should fulfil. It appears that from these essential assumptions follow quite strong consequences about the path properties of the price trajectories in continuous-time, which resemble the properties of local martingales. Quadratic variation of price paths of an asset corresponds in real world to (realized) volatility of the asset and plays crucial role in pricing derivatives underlined by this asset [12, 21].

1.2 Problem Statement

Over the 20$^{\text{th}}$ century, probabilistic methods have had a significant impact on the field of finance. However, questions arising from events such as financial crisis have stimulated research trend in probability. Several academic researchers and practitioners have turned on the model-free approach, often coined probability-free or model independent [41, 50]. The intuition behind is to avoid the wrong models, based on the wrong probabilistic assumptions. In the model-free approach, one uses the only widely accepted, basic assumptions of no existence of arbitrage opportunities. From these essential assumptions, one derives what properties the continuous-time price processes “typical price paths" should have. The word “typical" is understood in the following game theoretic sense: Investor may get infinitely rich by considering investment in those paths where the considered property fails [46].

Quadratic variation of price paths of an asset plays a crucial role in pricing derivatives underlined by this asset [12, 21]. The existence of quadratic variation of typical continuous price paths was proven in [46]. Existence of this quantity was also proven for typical càdlàg price paths with mildly restricted jumps in [47].

Furthermore, in [31] it was proven for non-negative price paths with jumps.

The main drawback of the existing results on the existence of quadratic variation with jumps in particular, is that it is defined along very favourable but also very special sequence of partitions of time intervals (Lebesgue partitions). Natural question arises, what happens if we allow other partitions to be used in the definition of the quadratic variation?

Pathwise local times and pathwise general Itô formula (Ito-Tanaka formula) are used for example to derive arbitrage free prices for weighted variance swaps in model-free settings [21]. In [34] the authors proved the existence of local time of typical continuous price paths. More precisely, they proved that it should be possible to make an arbitrary large profit by investing in the one-dimensional continuous price paths which do not possess local times. Again, the main drawback of the definition is that the local times are defined along the Lebesgue partitions.

Some insight of what happens if we allow other partitions to be used in the definition of local times was discussed in [13]. However, the results apply to continuous semimartingales and it is not known to what extend they may be applied to typical price paths. Yet another direction would be the introduction of local times for typical price paths with jumps [30].

In our research we were focussed on the properties and calculus of the price paths using model-free approach. In particular, we investigated the consequences of no-arbitrage opportunities for càdlàg price paths. Our main aim was to provide rigorous direct proof for the existence of quadratic variation for the typical càdlàg price paths with mildly restricted downwards jumps and its uniqueness (independence on the partition). In addition, we showed how the quadratic variation can be used to measure price volatility. Lastly, we proved the existence and uniqueness solution of the stochastic differential equations driven by continuous model-free price paths.

1.3 Literature Review

In the 20$^{\text{th}}$ century, most of the models for evolution of the stock prices in the financial industry were based on the probabilistic models and their measure theoretic framework. The measure theoretic framework is rooted in works of Fréchet, Bernoulli, Laplace, Cournot among others by Wald [51], Ville [42] and von Mises [43]. Its formalization was led by Kolmogorov in his trendiest article, [26].

Two principles were considered in the attempt to formalize and interpret probability, see Figure 1.1. Firstly, we briefly describe Principle B. The principle is related to the measure theoretic approach which assumes that prices of securities in a financial market follow some probabilistic model such as geometric Brownian motion. The version was published by Cournot [10], and dates back the work of Bernoulli in 1713. In all these contributions, the theory of martingales was exploited. This includes the law of iterated logarithm, the central limit theorem and the law of large numbers. The most exciting innovations in the modern history of financial mathematics are rooted in the discovery of the Brownian motion by a Scottish botanist Robert Brown in 1827.

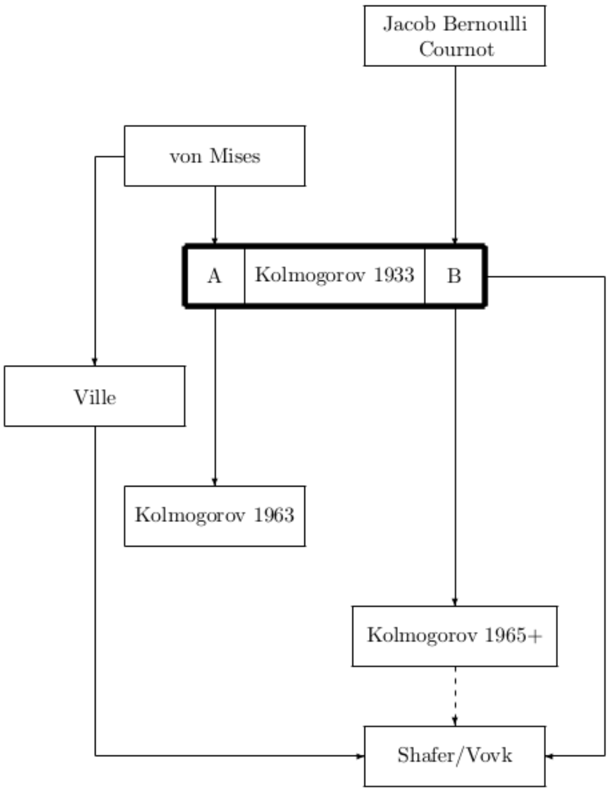
The first mathematical model of the Brownian motion was through the Danish astronomer and actuary Thorvald Nicolai Thiele in 1880. Bachelier improved this work by introducing Brownian motion as tool for modelling price fluctuations in his Doctoral thesis [3]. Kolmogorov's 1933 formulation of the measure theoretic foundation begins with the notion of probability measure P on a σ-algebra $\mathscr{F}$ of subsets of Ω, and then proceeded to prove results on the triple $(\Omega, \mathscr{F}, P)$. It rely on axioms and definitions outlined by Kolmogorov in 1933. The extension of Kolmogorov's axioms to continuous time was codified in Doob's 1953 book [48] and improved in subsequent decades. Restricting prices of financial security to be positive, Samuelson considers geometric Brownian motion which then became a standard reference. An interesting history on the emergence of measure theory and martingales may be found in [7] and [48].

In 1973 Fisher Black and Myron Scholes made a major breakthrough in the financial industry by introducing a model for pricing of a call option coined "Black-Scholes equity model" [8]. This model considers a replicating portfolio containing a risky asset and a riskless asset. Since then, the model serves as a foundation for various derivative pricing models with some calibration of parameters such as volatility estimation.

Principle A is related to the game theory. It dates back to Pascal, Von Mises and Ville's ideas which say probability reflects frequency of occurrence of some event. In particular, it reflect the odds of winning in a fair game. The game theoretic version of the principle acknowledges that probabilities and other betting offers can be tested by a series of bets that avoid risking bankruptcy whether or not the tests boils down to a bet on a single event and even whether or not the betting follows a pre-specified strategy [48]. Kolmogorov's necessary and sufficient conditions for convergence are elaborated by the game theoretic framework by a strategy for Skeptic that succeeds if the conditions are satisfied and a strategy for Reality the succeeds if the conditions are not satisfied.

In recent years, Shafer and Vovk improved model-free approach [41]. This approach is centred on sequential game with two players, i.e. Investor (Skeptic) and Markets (Reality), over a specific time interval. Players in this game alternate moves and only one player wins. These moves are fulfilled using the non-standard analysis in order to accommodate continuity. The game may have infinitely many rounds and at each round an investor decides what investment to hold and the market decides how the price should evolve. Each game is a perfect information game and the market in this case is assumed to be efficient in order to preclude arbitrage

Figure 1.1: Theoretical Attempts at the Formalization and Intepretation of Probability both in the Game and Measure Theory [49]



opportunity. Investor in this case requires a winning strategy.

Vovk [45] proposed a new definition of continuous-time events of model-free approach represented by semi-infinite interval. The greatest incentive of using model-free approach is natural emergence of probability, which is a postulate in measure-theoretic framework. In [41],[44], [45], and [46] the class of allowed trading strategies which investors may consider is defined in two steps, i.e elementary trading strategy and positive capital process. The elementary trading strategy consists of the initial capital and the bet on the security. This betting is based on the analysis of the price paths of an idealised financial security. Using the notion of diversification, investors ensure that their account never goes into debt by splitting their initial capital into a countable number of accounts and on each account running a simple trading strategy.

Several results based on the model-free approach have already been established. For example, Vovk [45] showed that the continuous price paths of a security have certain properties of the Brownian motion which include the absence of zeros and the points of strict increase and decrease while in [44] showed that the strong variation index of the typical price paths is at most 2.

Vovk in [47] shows that the price paths of the underlying financial security possess quadratic variation when price paths are càdlàg functions with mild restriction on the size of jumps. For instance, if $\omega \in [0,T] \to \mathbb{R}$ is a càdlàg function which represents the price path of a financial security, then the jumps of ω are constrained as follows:

$$|\omega(t) - \omega(t-)| = |\Delta\omega(t)| \leq \psi(\sup_{s\in[0,t)} |\omega(s)|), \tag{1.3.1}$$

where $\omega(t-) := \lim_{s\to t, s<t} \omega(s)$ and ψ is a non-decreasing function over $[0,\infty)$. The indirect proof of the existence of quadratic variation also exists for continuous price paths [46].

Föllmer [17] proved the pathwise Itô's formula associated with the quadratic variation for a function $f \in C^2$ along the sequence of Lebesgue partitions. In particular,

$$f(x_t) - f(x_0) = \int_{[0,t)} f'(x_s)dx_s + \frac{1}{2}\int_{[0,t)} f''(x_s)d[x]_s, \tag{1.3.2}$$

where the integral $\int_{[0,t)} f'(x_s)dx_s$ is defined as the limit of non-anticipated Riemann sums and exists whenever the quadratic variation $[x]_t$ exists.

Following Vovk's approach in [46], Perkowski and Prömel in [35] presented two general solutions to pathwise integration problem. The first solution was proposed in [14] where they restrict themselves to trading strategies (integrands) of bounded variation. This assumption is not very natural in the model-free approach. The second solution rests on the idea which goes back to Lyons in [32]. They restrict the set of possible price paths to those admitting quadratic variation to allow the application of Föllmer's pathwise Itô calculus, [17].

Davis, Obłój and Siorpaes [13], studied the notion of local times for "typical continuous price paths" and derived pathwise change of variable formulas in the spirit of Tanaka's formula. We denote by π a partition of $[0,\infty)$, i.e., $\pi = (t_j)_{j\in\mathbb{N}}$ where $t_j \in [0,\infty]$, $t_0 = 0$, $t_j < t_{j+1}$ if $t_{j+1} < \infty$, and $\lim_{j\to\infty} t_j = \infty$. For a continuous function $x = (x_s)_{s\geq 0}$, $s \in [0,t]$ and a partition $\pi = (t_j)_j$, the discrete local times (along π) of x is given by:

$$L_t^\pi(x) := 2\sum_{t_j\in\pi} \mathbb{1}_{[\![x_{t_j\wedge t}, x_{t_{j+1}\wedge t})\!)}(u)|x_{t_{j+1}\wedge t} - u|, \tag{1.3.3}$$

where

$$[\![u,v)\!) := \begin{cases} [u,v), & \text{if} \quad u < v \\ [v,u), & \text{if} \quad u > v. \end{cases} \tag{1.3.4}$$

Thus, the Tanaka Meyer for $f \in C^2$ is given by

$$f(x_t) - f(x_0) = \sum_{t_j \in \pi} f'_-(x_{t_j})(x_{t_{j+1} \wedge t} - x_{t_j \wedge t}) + \frac{1}{2}\int_{\mathbb{R}} L_t^{\pi}(u) f''(u) du, \tag{1.3.5}$$

where for $a, b \in \mathbb{R}$ we denoted by $a \wedge b := \min(a, b)$ and f'_- denote the left-derivative of the convex function f.

As noted in [41], the literature on game-theoretic approach is not a comprehensive treatise on mature and finished mathematical theory, ready to be shelved for posterity. It is only an invitation to participate. Although variation of security prices have been studied in the literature, for example in [44], [45] and [46], they need to be studied further. In particular, we consider price process of an idealised financial security to be a càdlàg function, rather than continuous.

Our main contribution in this thesis is the independence of quadratic variation of model-free, càdlàg price paths obtained from the sequence of stopping times as long as the "vertical meshes" (see for example Theorem 3.41) of the sequences of these stopping times tend to 0. Similarly as for semimartigales, we show that this quantity may be also expressed using the truncated variation. Turning our attention to the continuous price paths, we prove the existence and uniqueness of the stochastic differential equations using a model-free version of the Burkholder-Davis-Gundy inequality, which is also our contribution.

1.4 Structure of the Thesis

The rest of the thesis is organised as follows. In Chapter 2, we review the classical construction of quadratic variation of a semimartingale and its application in the stochastic calculus. The quadratic variation in real world application can be used to measure price volatility.

Chapter 3 is dedicated to the basic fundamentals of game theory (model-free) to mathematical finance. In particular, various properties and definitions involved in the sequel are discussed. In addition, we present various constructions of quadratic variations for typical càdlàg price paths in the model-free setting.

We proved that the model-free typical (in the sense of Vovk) càdlàg price paths with mildly restricted downward jumps possess quadratic variation, which does not depend on any specific sequence of partitions as long as these partitions are obtained from stopping times such that the oscillations of the path on the consecutive (half-open on the right) intervals of these partitions tend (in a specific sense) to zero. Moreover, we define sequences of quasi-explicit, partition-independent quantities (normalized truncated variations) that tend to this quadratic variation.

In Chapter 4, which is also the main part of this work, we address the model-free continuous price paths. Using the assumptions as in Revuz and Yor's [37], we proved the existence and uniqueness of the solution of the SDE's with Lipschitz coefficients. The main tool in our reasoning is the Burkholder-Davis-Gundy inequality for integrals driven by model-free continuous price paths. Lastly, our conclusion is in Chapter 5.

Chapter 2

Classical Approach to Stochastic Integration

In this Chapter, we review some basic facts on the construction of stochastic integration mainly from the classical probability theories. The role of quadratic variation and its application in the Itô formula is also discussed. It is more convenient to study model-free mathematical finance while having a background of how it is done in the classical probabilistic sense.

We borrow some definitions and notations from [36] and [38] in our settings. In the classical approach, one considers a continuous-time stochastic process X which is a family of $\mathbb{R}$ or $\mathbb{R}^d$-valued random variables $\{X(t) = X_t, t \in [0,\infty)\}$ defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

For a fixed t, X_t is a random variable. If we fix ω, X is a function of time $[0,\infty) \ni t \to X_t(\omega) = X(\omega,t)$. This function is called a sample path, trajectory or realization of a stochastic process X. In addition, mathematical finance community often associate this path with the evolution of prices of a financial security. Considering filtration $(\mathscr{F}_t)_{t\in[0,\infty)}$, a family of σ-algebras $\mathscr{F}_t, t \in [0,\infty)$ that is increasing: $\mathscr{F}_s \subset \mathscr{F}_t$ if $s \leq t$, one has the following definition.

Definition 2.1. *A filtered complete probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ is said to satisfy the usual hypothesis if the following conditions hold:*

i. *$\mathscr{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathscr{F}$,*

ii. *$\mathscr{F}_t = \cap_{u>t}\mathscr{F}_u$, for all $t \in [0,\infty)$, i.e. filtration $(\mathscr{F}_t)$ is right continuous.*

The introduction of filtered probability space became successful in mathematical finance due to the effectiveness and usefulness of concepts such as stopping times, martingales and adaptation. These concepts will be defined below.

Definition 2.2. *A process $X = (X)_{t\geq 0}$ is said to be càdlàg if a.s. (almost surely), that is with probability* 1*, its sample paths are right continuous with left limits (RCLL).*

Figure 2.1 illustrates an example of a realisation of a continuous process over the domain $[0,10]$.

Figure 2.1: Example of a Realisation of a Continuous Process

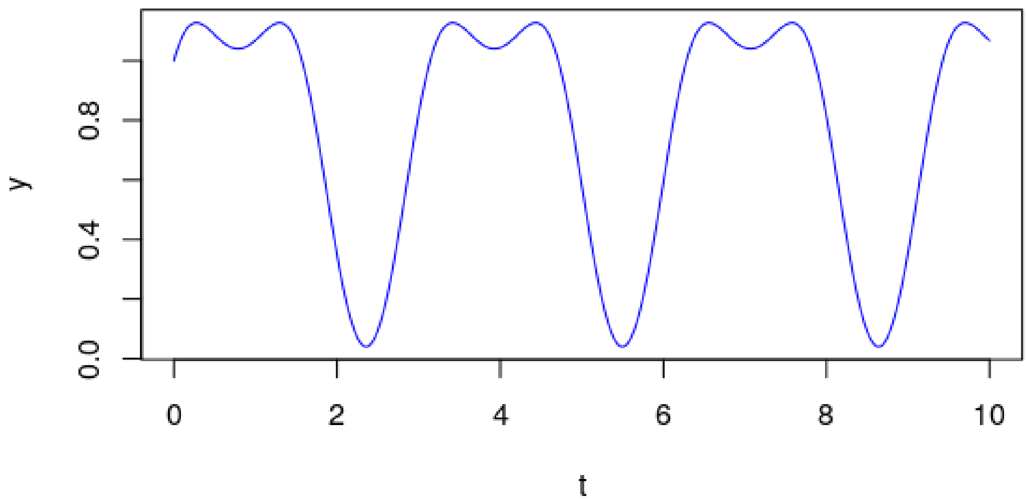


Figure 2.2 illustrates an example of a realisation of a càdlàg process over the domain $[0,10)$ generated in R studio, see Appendix A for the code. The blue (respectively, red) shaded dots denote closed (respectively, open) end points. The term càdlàg is a French acronym which stands for a right continuous process with left limits. The values of the realisation are in blue, while red dots are the values of left limits at the points of discontinuity (i.e. jumps) of the realisation. The approximation of a càdlàg function with step functions is illustrated in Figure 2.3. Alternative to càdlàg process, we have làdlàg (i.e. a left continuous process with right limits), see the example in Figure 2.4 generated over the domain $(0,10]$.

Figure 2.2: Example of a Realisation of a Càdlàg Process

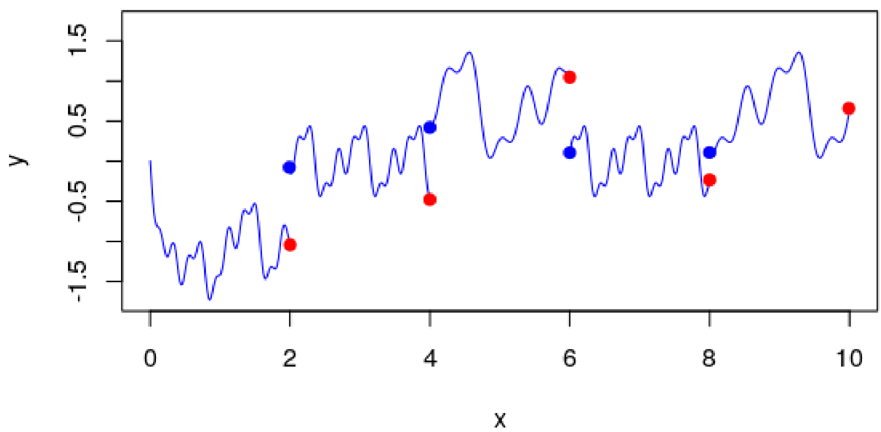


Figure 2.3: Example of Approximation of a Càdlàg Function by a Piecewise Constant Function

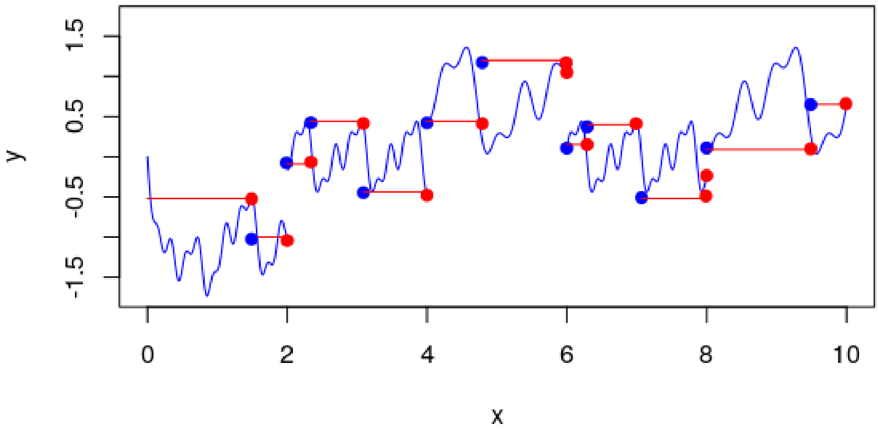
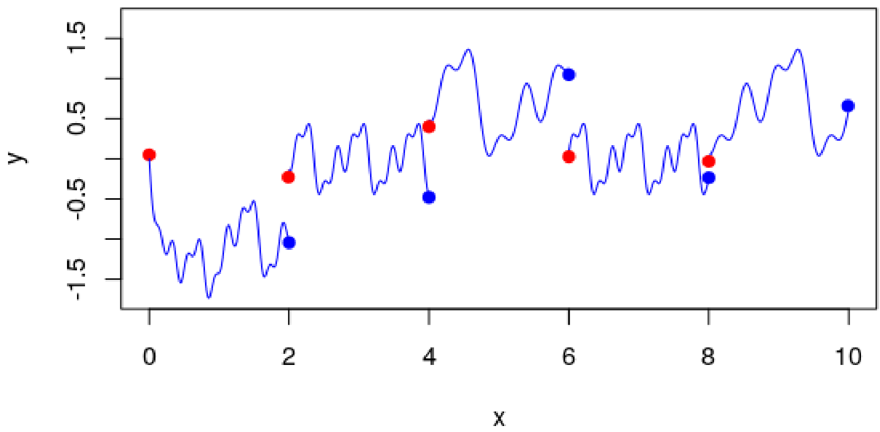


Figure 2.4: Example of a Làdlàg Function



Definition 2.3. *A process $X = (X)_{t\geq 0}$ is said to be adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$ if $X_t \in \mathscr{F}_t$, i.e X_t is $\mathscr{F}_t$- measurable for each t.*

Definition 2.3 details that a process (X_{t-}) is predictable. In particular, for each stopping time $T > 0$, X_{T-} is known immediately before time T.

2.1 Riemann and Lebesque-Stieltjes Integrals

In this subsection we present predecessors of stochastic integral, i.e. the Riemann, Riemann-Stieltjes and the Lebesgue-Stieltjes integral. The area under a smooth function, see Figure 2.5a, was, among others, investigated by Georg Friedrich Bernhard Riemann. In his approach he considers partitions, i.e grids (or subdivisions) of the domain $[a,b]$, denoted by

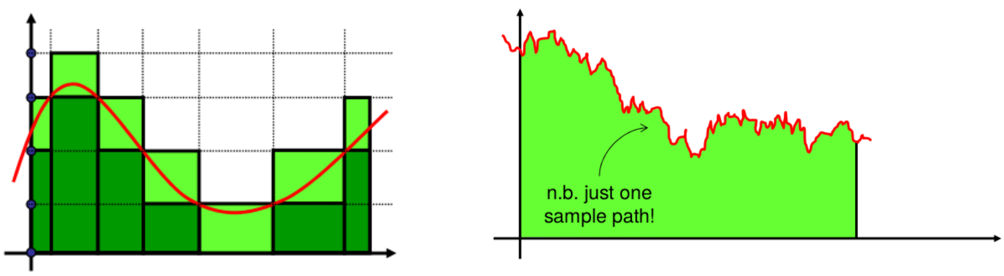
$$\pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

and evaluate the lower and upper sums given by:

$$O_\pi(f) = \sum_{j=1}^{n} \left(\sup\{f(t) : t_{j-1} \le t \le t_j\}\right)(t_j - t_{j-1})$$
$$U_\pi(f) = \sum_{j=1}^{n} \left(\inf\{f(t) : t_{j-1} \le t \le t_j\}\right)(t_j - t_{j-1}),$$

respectively. In this case, the Riemann integral exist if the infimum of the lower sums coincide with the supremum of the upper sums. The generalization of the Riemann integral is the Riemann-Stieltjes integral defined below.

Figure 2.5: Stochastic versus Smooth Function



(a) Smooth function **(b)** Stochastic Process

Definition 2.4. *Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be the deterministic functions. Then the Riemann-Stieltjes integral of f (integrand) with respect to another function g (integrator) is the limit of sums (if it exists) given by:*

$$\sum_{i=1}^{n} f(s_i)\{g(t_i) - g(t_{i-1})\}, \tag{2.1.1}$$

where $n \in \mathbb{N}$, $s_i, \in [t_{i-1}, t_i]$ for $i = 1, 2, \cdots, n$ and $\pi = \{a = t_0 < t_1 < t_2 < \cdots < t_n = b\}$ as the mesh of the partition π,

$$mesh(\pi) = \max_{i=1,2,3,\cdots,n} |t_i - t_{i-1}|,$$

goes to zero.

The Riemann-Stieltjes integral works well only for integrator with bounded total variation. Thus, what follows is the definition of bounded total variation integrator.

Definition 2.5. *Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be the deterministic functions and π a finite partition over the interval $[a,b]$ with the mesh tending to zero.*

1. *The first variation of the integrator $g : [a,b] \to \mathbb{R}$ with respect to the partition π is*

$$V_{\pi}^{1}(g) = \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})|. \tag{2.1.2}$$

2. *The total variation of the integrator g on the interval $[a,b]$ is given by*

$$TV(g) = \sup_{n} \sup_{a \leq t_0 < t_1 < \cdots < t_n \leq b} \sum_{i=0}^{n} |g(t_i) - g(t_{i-1})|. \tag{2.1.3}$$

3. *A function g is said to be of bounded total variation if its total variation is finite, i.e $TV(g) < \infty$.*

The total variation of g on the interval $[a,b]$ is the supremum of the first variation of g over all possible partitions of $[a,b]$.

Theorem 1. *Let f be continuous function and g a path with finite total variation on the domain $[a,b]$. Then the Riemann-Stieltjes integral*

$$(RS) \int_{[a,b]} f dg \tag{2.1.4}$$

exists and $f \mapsto (RS) \int_{[a,b]} f dg$ is a continuous operation on the space of continuous functions on $[a,b]$ equipped with the supremum norm.

Figure 2.6 below illustrates the infinite variation of a “typical" path of a Brownian motion process and a function with finite total variation over the domain $[0,1]$.

The Riemann-Stieltjes integral may not exist for Borel-measurable integrand f and finite total variation integrator g. This occurs when the jumps of the function f coincide with the jumps of g.

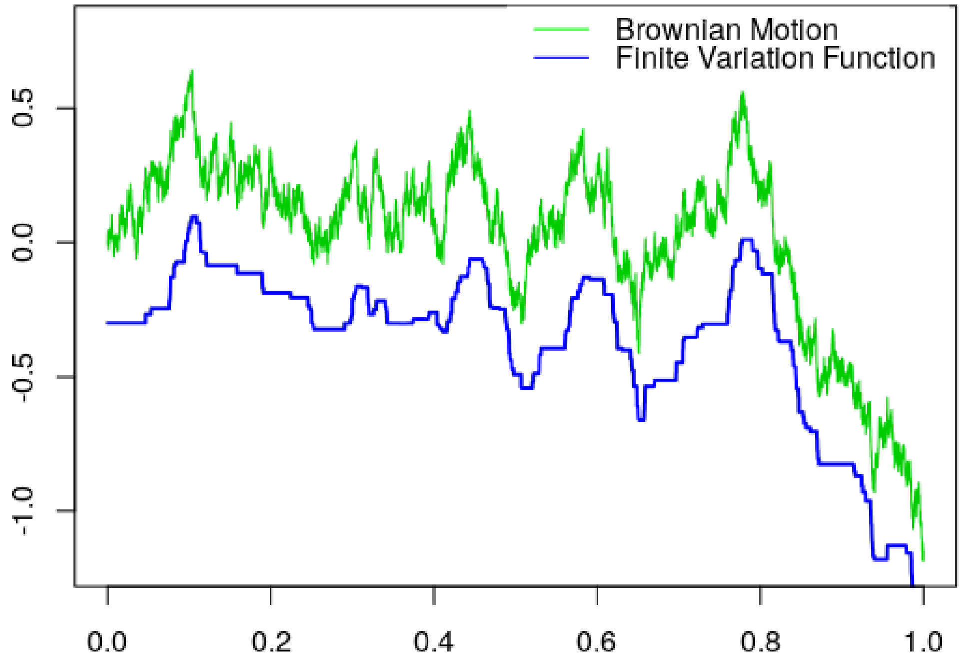
Example 2.6. *Consider the following functions $f, g : [0,1] \to \mathbb{R}$.*

$$f(t) = g(t) = \begin{cases} 0 & \text{if} \quad t \in [0,1) \\ 1 & \text{if} \quad t = 1, \end{cases}$$

For any partition of the domain $[0,1]$, say $0 = t_0 < t_1 < \cdots < t_n = 1$, the Riemann-Stieltjes sum

$$\sum_{i=1}^{n} f(s_i)\{g(t_i) - g(t_{i-1})\} = 0 \tag{2.1.5}$$

Figure 2.6: Typical Finite versus Infinite Variation Processes



if $s_n < 1$ *since* $f(s_1) = f(s_2) = \cdots = f(s_{n-1}) = f(s_n) = 0$ *for all* $s_i \in [t_{i-1}, t_i]$, $i = 1, 2, \cdots, n-1$, *while if* $s_n = t_n = 1$, *the Riemann-Stieltjes sum*

$$\sum_{i=1}^{n} f(\tilde{s}_i)\{g(t_i) - g(t_{i-1})\} = f(\tilde{s}_n)(g(t_n) - g(t_{n-1})) = 1 \times 1 = 1, \tag{2.1.6}$$

where $\tilde{s}_i \in [t_{i-1}, t_i]$ *and* $\tilde{s}_i := \frac{s_{t_{i-1}} + s_{t_i}}{2}$. *Thus, the Riemann-Stieltjes integral does not exist in this case.*

The refinement of the Riemann-Stieltjes integral is the Lebesgue-Stieltjes integral introduced on the measure space $([0,T], \mathscr{B}([0,T]), \mu_g)$. The notation $\mathscr{B}([0,T])$ denotes the σ-field of Borel subsets of $[0,T]$ and μ_g is a signed, σ-finite measure on $[0,T]$.

Definition 2.7. *Let g be a càdlàg function and* $[0 \leq c \leq d \leq T]$. *Then the measure* μ_g *is given by*

$$\mu_g(c, d] = g(d) - g(c). \tag{2.1.7}$$

Extending the measure μ_g to the Borel subsets of $[0,T]$, the Lebesgue-Stieltjes integral may be defined as the usual Lebesgue integral of f with respect to the measure μ_g. That is, the integral

$$(LS)\int_{[0,T]} f dg = \int_{[0,T]} f d\mu_g. \tag{2.1.8}$$

The existence of the Integral (2.1.8) is guaranteed by the finiteness of the total variation of g on $[0,T]$ together with the boundedness and Borel-measurability of the integrand f. Note that in Example 2.6, the Lebesque-Stieltjes integral is given by

$$(LS)\int_{[0,1]} f dg = \int_{[0,1]} f d\mu_g = f(1)\mu_g\left(\{1\}\right) = 1 \tag{2.1.9}$$

since the measure μ_g is given by

$$\mu_g(I) = \begin{cases} 0 & \text{if} \quad 1 \notin I, \\ 1 & \text{if} \quad 1 \in I \end{cases}$$

for $I \subseteq [0,1]$.

2.2 Integration by Parts Formula for Lebesque-Stieltjes Integral

Proposition 2.8. *If f and g are both càdlàg functions and have finite total variation, then we have the following integration by parts formula,*

$$\begin{aligned} f(T)g(T) - f(0)g(0) = (LS)\int_{[0,T]} f(t-)dg(t) + (LS)\int_{[0,T]} g(t-)df(t) \\ + \sum_{0<s\leq T} \Delta f(s)\Delta g(s), \end{aligned} \tag{2.2.1}$$

where $\Delta f(s) = f(s) - f(s-)$, $\Delta g(s) = g(s) - g(s-)$ and $f(0-) = f(0), g(0-) = g(0)$.

Unfortunately, for an integrator which is a “typical" Brownian motion path, the integral (2.1.8) does not exist since the Brownian paths have a.s. (almost surely) infinite variation, see Figures 2.5b and 2.6.

2.3 Quadratic Variation

In this Section we detail the properties of quadratic variation and review some constructions for various paths. We first define the following convergence.

Definition 2.9. *Let X_n be a sequence of random variables. Then X_n converges in probability to X, denoted $X_n \to^{\mathbb{P}} X$ (or $X = (\mathbb{P}) \lim_{n\to\infty} X_n$), if*

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0 \tag{2.3.1}$$

for all $\varepsilon > 0$.

Definition 2.10. *Let X_n be a sequence of random variables. Then X_n converges in L^2 to X, denoted by $X_n \to^{L^2} X$ (or $X = \left(L^2\right) \lim_{n\to\infty} X_n$) if*

$$\lim_{n\to\infty} \mathbb{E}\left(|X_n - X|^2\right) = 0. \tag{2.3.2}$$

Let X be a real-valued càdlàg adapted process on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. Suppose that a process X is jointly measurable with respect to $\mathscr{B}(\mathbb{R}^+) \times \mathscr{F}$. Thus X is adapted if X_t is $\mathscr{F}_t$-measurable.

Definition 2.11. *For any finite partition (possible random), say*

$$\pi = \{0 = t_0 \leq t_1 \leq \cdots \leq t_n = t\}$$

of $[0,t] \subset [0,\infty)$, the discrete quadratic variation till time $t \geq 0$ of the process X along the partition π is given by the following random variable

$$[X]_t^\pi = \sum_{i=1}^n \left(X_{t_i} - X_{t_{i-1}}\right)^2.$$

Definition 2.12. *Let $t \geq 0$. Assume that for any sequence of deterministic partitions π^n of $[0,t]$ such that mesh $|\pi^n| \to 0$ (as $n \to +\infty$) there exists the limit (in probability) of discrete quadratic variations $[X]_t^{\pi^n}$ along the partitions π^n. Then the quadratic variation till time t of a stochastic process X, denoted by $[X]_t$ is equal to this limit,*

$$[X]_t = (\mathbb{P}) \lim_{n\to+\infty} [X]_t^{\pi^n}. \tag{2.3.3}$$

Definition 2.13. *Stochastic process $X = (X_t)_{t\geq 0}$ is of finite quadratic variation if it has finite quadratic variation till time t for any $t \geq 0$.*

Note that functions such as continuously differentiable function, have zero quadratic variation. Any continuous function with non-zero quadratic variation must have infinite total variation.

Remark 2.14. *The definition of quadratic variation, 2.12, requires that the limit on the right hand side of (2.3.3) is independent of the choice of the sequence of partitions π^n. Naturally, such a quantity does not need to exist for any given process. If it exists for all $t \geq 0$, then it has the following properties:*

i. $[X]_0 = 0$*, a.s.*

ii. $[X]_t$ *is non-decreasing function and therefore has finite variation.*

iii. $[X]_t$ *is* $\mathscr{F}_t$*-measurable for any* $t \geq 0$*.*

In particular, for any $0 \leq s \leq t$*, we have*

$$[X]_s \leq [X]_t.$$

Since $[X]_t$ is non-decreasing, it has countable number of positive jumps and it may be decomposed into continuous and jump part. Thus, the quadratic variation in this case becomes

$$[X]_t = [X]_t^c + \sum_{0<s<t} ([X]_{s+} - [X]_{s-}) + [X]_t + [X]_{t-}, \tag{2.3.4}$$

where $[X]_t^c$ is the continuous part of the quadratic variation. As was proven in [29] it is possible to define quadratic variation of a semimartingale without considering a sequence of partitions. In [29], a sequence of semi-explicit quantities was considered, which do not depend on any partition. It has been shown that for any process $X_t, t \geq 0$ which is a real càdlàg semimartingale on $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$, these quantities tend almost surely to the continuous part of the quadratic variation of X.

Definition 2.15. *A process X is of finite variation if its total variation is a.s (almost surely, or with probability one) bounded.*

Proposition 2.16. *Let* $X = (X_t)_{t \geq 0}$ *be a real-valued continuous process. If X is a finite variation process, then its quadratic variation is identically zero.*

Proof. Fix $t \geq 0$. For any partition $0 \leq t_0 < t_1 < \ldots < t_n \leq t$ we have

$$\sum_{i=1}^{n} \left(X_{t_i} - X_{t_{i-1}}\right)^2 \leq \sup |X_{t_i} - X_{t_{i-1}}| \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|.$$

Since X_t has finite variation, for almost all $\omega \in \Omega$ there exist $M(\omega)$ such that

$$\sum_{i=1}^{n} |X_{t_i}(\omega) - X_{t_{i-1}}(\omega)| \leq M(\omega) < +\infty$$

and by continuity of X, $\sup_i |X_{t_i}(\omega) - X_{t_{i-1}}(\omega)| \to 0$ as the mesh of the partition, $\max_i |t_i - t_{i-1}| \to 0$. Combining these two observations, we get the thesis.

□

2.4 Quadratic Variation of Standard Brownian Motion

Owing its name to the Scottish Botanist, the notion of Brownian motion in mathematical finance was first introduced by Louis Batchelier. To define the standard Brownian motion, let us recall the definition of normal distribution.

Definition 2.17. *A real random variable X has normal (Gausian) distribution if there is μ and $\sigma > 0$ such that its density function f_X is given by*

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \mathbb{R}. \tag{2.4.1}$$

Since μ and $\sigma^2 > 0$ are real parameters, we write $X \sim N(\mu, \sigma^2)$ to mean that X is normally distributed with these parameters. It is possible to prove that μ is then equal the mean of X and σ^2 is equal to the variance of X. If the mean, μ, is zero and the variance, σ^2, is one, we can write Equation (2.4.1) as follows

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right). \tag{2.4.2}$$

Definition 2.18. *A stochastic process B on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ is called a one dimensional Brownian motion if the following conditions are satisfied:*

i. *$B_0 = 0$ a.s.*

ii. *$B_t - B_s$ is independent of the filtration $\mathscr{F}_s$ for all $t \geq s$.*

iii. *$B_t - B_s$ has a normal distribution with zero mean and variance $t - s$ for all $t > s$.*

iv. *B_t is adapted to $\mathscr{F}_t$.*

v. *The sample paths $B(\omega)$ of B are continuous functions (i.e. they have no jumps) almost everywhere (a.e). This means that for almost all $\omega \in \Omega$, the trajectory $B(\omega)$ is continuous.*

Proposition 2.19. *Let T be the horizon and let $\pi^n = \{0 = t_0^n < t_1^n < t_2^n < \cdots < t_{k(n)}^n = T\}$ with $n \geq 1$ be a sequence of deterministic partitions. Then, if $mesh(\pi^n) \to 0$ we have*

$$(L^2) \lim_{n\to\infty} \sum_{i=1}^{k(n)} \left(B_{t_i^n}(\omega) - B_{t_{i-1}^n}(\omega)\right)^2 = T \tag{2.4.3}$$

Proof. Let the partition π of subdivision $[0,T]$ be given by:

$$0 = t_0^n < t_1^n < \cdots, t_{k(n)}^n = T$$

with $0 \leq s \leq t$. Using the independence of increments

$$\mathbb{E}\left(\sum_{i=1}^{k(n)} (B_{t_i^n} - B_{t_{i-1}^n})^2 - T\right)^2 = \sum_{i=1}^{k(n)} \mathbb{E}\left((B_{t_i^n} - B_{t_{i-1}^n})^2 - (t_i^n - t_{i-1}^n)\right)^2$$

$$= \sum_{i=1}^{k(n)} \mathbb{E}\left((B_{t_i^n} - B_{t_{i-1}^n})^4 + (t_i^n - t_{i-1}^n)^2 - 2(B_{t_i^n} - B_{t_{i-1}^n})^2(t_i^n - t_{i-1}^n)\right). \quad (2.4.4)$$

Since the increment $B_t - B_s$ has normal distribution $N(0, t-s)$, it implies that for any natural number, say k, we have

$$\mathbb{E}[(B_{t_i^n} - B_{t_{i-1}^n})^{2k}] = \frac{(2k)!}{2^k k!}(t_i^n - t_{i-1}^n)^k. \quad (2.4.5)$$

Thus, Equation (2.4.4) becomes

$$\sum_{i=1}^{k(n)} \left[\frac{4!}{2^2 2!}(t_i^n - t_{i-1}^n)^2 - 2\frac{2!}{2}(t_i^n - t_{i-1}^n)^2 + (t_i^n - t_{i-1}^n)^2\right]$$

$$= 2\sum_{i=1}^{k(n)} (t_i^n - t_{i-1}^n)^2 \leq 2 \max_{i=1,2,\cdots,n} (t_i^n - t_{i-1}^n) \sum_{i=1}^{k(n)} (t_i^n - t_{i-1}^n)$$

$$= 2 \max_{i=1,2,\cdots,n} (t_i^n - t_{i-1}^n) T \to_{n\to\infty} 0. \quad (2.4.6)$$

□

Since convergence in L^2 implies convergence in probability, we get

Theorem 2. *Let B be a standard Brownian motion on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, then the standard Brownian motion is of finite quadratic variation and*

$$[B]_t = t \quad \textit{for all } t \geq 0 \quad a.s. \quad (2.4.7)$$

Considering the continuity of Brownian motion on the time interval $[0,t], t > 0$, its total variation is infinite. That is

$$TV\left(B, [0,t]\right) = \sup_{\pi} \sum_{i=1}^{n} \left|B_{t_i} - B_{t_{i-1}}\right| = +\infty. \quad (2.4.8)$$

This follows from Proposition 2.16 and Theorem 2. Otherwise, if it was finite, we would have $[B]_t = 0$, but $[B]_t = t > 0$. Theorem 2 justifies that the quadratic variation of a Brownian motion is equivalent to change in time, i.e. increment of the

$$(dB_t)^2 = dt. \quad (2.4.9)$$

2.5 Existence of Quadratic Variation of a Semimartingale

In this Section we introduce important family of stochastic processes which possess quadratic variation and are "good" integrators. These are essential requirement in the development of stochastic integral relative to a local martingale which is of infinite variation, see [38, Page 2]. We will assume the hypothesis as stated in Definition 2.1.

Definition 2.20. *A real valued stochastic process $(X_t)_{t\geq 0}$ is called a martingale (respectively supermartingale, submartingale) with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ if the following conditions hold:*

i. $X_t \in L^1$, *that is* $\mathbb{E}\{|X_t|\} < \infty$.

ii. *For each $s < t$, the expectation $\mathbb{E}\{X_t|\mathscr{F}_s\} = X_s$, a.s. (respectively $\mathbb{E}\{X_t|\mathscr{F}_s\} \leq X_s$, $\mathbb{E}\{X_t|\mathscr{F}_s\} \geq X_s$).*

Definition 2.21. *A random variable $T : \Omega \to [0,\infty]$ is a stopping time with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ if the event $\{T \leq t\} \in \mathscr{F}_t$, for every $t \geq 0$.*

Definition 2.22. *Let $\tau : \Omega \to \mathbb{R}_+$ be a stopping such that $\{\tau \leq t\} \in \mathscr{F}_t$ for all $t \geq 0$. Then*

$$\mathscr{F}_\tau = \{A \in \mathscr{F} : \left(\mathbb{1}_A \cap \{\tau \leq t\}\right) \in \mathscr{F}_t\}. \tag{2.5.1}$$

Definition 2.23. *A stochastic process M is a local martingale with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ if the following conditions hold:*

i. *M_0 is $\mathscr{F}_0$ measurable.*

ii. *There exists an increasing sequence of stopping times T_n with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ with $T_n \to \infty$ a.s. such that stopped process*

$$\{M_{T_n} - M_0\} \quad n = 1, 2, \cdots$$

is a martingale.

Definition 2.24. *A real valued stochastic process $H = \{H_t\}_{t\geq 0}$ is called simple predictable process if it can be written as:*

$$H_t(\omega) = \phi_0(\omega)\mathbb{1}_{\{0\}}(t) + \sum_{i=1} \phi_i(\omega)\mathbb{1}_{(\tau_i,\tau_{i+1}]}(t), \tag{2.5.2}$$

where $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n$ and ϕ_i is bounded $\mathscr{F}_{\tau_i-}$ measurable random variable for $i = 1, 2, \cdots, n$ and $\phi_0 \in \mathscr{F}_0$.

Definition 2.25. *Let H be a simple bounded predictable process and M a local martingale. Then the integral $(H \cdot M)$ is defined as*

$$(H \cdot M)_t = \sum_{i=0}^{\infty} \phi_i (M_{\tau_i \wedge t} - M_{\tau_{i-1} \wedge t}). \tag{2.5.3}$$

From [24, Theorem 7.12] it follows that the integral $(H \cdot M)$ preserves local martingale property. Note that $\forall a, b \in \mathbb{R}$, we denote by $a \wedge b := min(a,b)$.

Example 2.26. *Let M be a martingale and S,T be two stopping times such that $S \leq T$ a.s. Then the integral*

$$(H \cdot M)_t = M_{T \wedge t} - M_{S \wedge t}$$

is a martingale, where H is given by

$$H(t,\omega) = \begin{cases} 1 & \text{if} \quad S(\omega) < t \leq T(\omega) \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.4}$$

We state without proof the following theorem.

Theorem 3. *Let X be a local martingale with càdlàg paths. Then, X possesses quadratic variation.*

Remark 2.27. *An elementary proof of Theorem 3 may be found in [25] for càdlàg martingales. In addition, if M is a càdlàg local martingale and $[M]$ denotes its quadratic variation, then $M^2 - [M]$ is again a local martingale. Note that $[\cdot]$ denotes the quadratic variation.*

In what follows is the extension of the Integral (2.5.3) to the square predictable processes in L^2. Let M be a square integrable martingale. For any $T \geq 0$ we denote by $L^2(M)$ a family of predictable processes which satisfy the following condition

$$\mathbb{E}\left(\int_{[0.T]} H_s^2 d[M]_s \right) < +\infty. \tag{2.5.5}$$

Remark 2.28. *If $H \in L^2(M)$, then H is a limit of a sequence of simple predictable processes H^n in the sense that for any $T > 0$,*

$$\lim_{n \to \infty} \mathbb{E} \int_{[0,T]} (H_s^n - H_s)^2 d[M]_s = 0. \tag{2.5.6}$$

Theorem 4. *Let M be a square integrable martingale. If $H \in L^2(M)$ and H^n is a sequence of simple predictable processes in $L^2(M)$ such that $H^n \to H$ in $L^2(M)$, then for any $T > 0$, the sequence of integrals $(H^n \cdot M)_T$ converges in L^2 and we identify this limit as $H \cdot M$. That is*

$$\mathbb{E}\left((H^n \cdot M)_T - (H \cdot M)_T \right)^2 \to_{n \to \infty} 0. \tag{2.5.7}$$

Proof. Note that for any $T > 0$, H is a limit of a simple predictable process

$$H_t^n = \phi_0^n \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{\infty} \phi_i^n \mathbb{1}_{(\tau_{i-1}^n, \tau_i^n]}(t), \tag{2.5.8}$$

and the following holds

$$\mathbb{E} \int_{[0,T]} \left(H_s - H_s^n\right)^2 d[M]_s \to_{n\to\infty} 0. \tag{2.5.9}$$

Thus, it suffices to show that the integral $(H^n \cdot M)_T$ is a Cauchy sequence in L^2. In particular, that for every $\varepsilon > 0$, we have

$$\mathbb{E}\left((H^n \cdot M)_T - (H^m \cdot M)_T\right)^2 < \varepsilon \tag{2.5.10}$$

for sufficiently large n and m. The integrals

$$(H^n \cdot M), (H^m \cdot M)$$

are square integrable martingales and their difference can be expressed as

$$\left((H^n \cdot M) - (H^m \cdot M)\right) = (H^m - H^n) \cdot M. \tag{2.5.11}$$

The quadratic variation of the Integral (2.5.11) is then given by

$$\left[(H^n - H^m) \cdot M\right]_T = \int_{[0,T]} (H^n - H^m)_s^2 \, d[M]_s. \tag{2.5.12}$$

It follows from Remark 2.27 that

$$\left((H^n - H^m) \cdot M\right)^2 - \left[(H^n - H^m) \cdot M\right] \tag{2.5.13}$$

is again a martingale. Thus,

$$\begin{aligned}
\mathbb{E}\left((H^n - H^m) \cdot M\right)_T^2 &= \mathbb{E}\left[(H^n - H^m) \cdot M\right]_T \\
&= \mathbb{E} \int_{[0,T]} (H^n - H^m)_s^2 \, d[M]_s \\
&\leq 2\mathbb{E} \int_{[0,T]} (H^n - H)_s^2 \, d[M]_s + 2\mathbb{E} \int_{[0,T]} (H^m - H)_s^2 \, d[M]_s \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned} \tag{2.5.14}$$

provided that

$$\mathbb{E} \int_{[0,T]} (H^n - H)_s^2 \, d[M]_s < \frac{\varepsilon}{4} \quad \text{and} \quad \mathbb{E} \int_{[0,T]} (H^m - H)_s^2 \, d[M]_s < \frac{\varepsilon}{4}.$$

Taking ε arbitrary small, we get the assertion. □

In the sequence of estimates (2.5.14) we used elementary inequality

$$\begin{aligned}
(a-b)^2 &= 2(a-c)^2 + 2(b-c)^2 - (a+b-2c)^2 \\
&\leq 2(a-c)^2 + 2(b-c)^2
\end{aligned}$$

which is valid for any real a, b and c.

2.6 Stochastic Integral for Semimartingales

In this Section we turn our attention to the integral with respect to a semimartingale.

Definition 2.29. *M is a locally square integrable local martingale if there exists a sequence of stopping times $\tau^n \to \infty$ a.s. such that for any n*

$$M^n = M_{\tau^n \wedge \cdot} \tag{2.6.1}$$

is square integrable.

Equality (2.6.1) means that for any $t \geq 0$, $M^n_t = M_{\tau^n \wedge t}$.

Definition 2.30. *Let M be a locally square integrable local martingale and $\tau^n \to \infty$ a.s. is an increasing sequence of stopping times such that $M_{\tau^n \wedge \cdot}$ is a square integrable martingale. Then for $H \in L^2(M)$ we define integral*

$$(H \cdot M)_t = \lim_{n \to \infty} (H \cdot M)_{\tau^n \wedge t} = \lim_{n \to \infty} (H \cdot M_{\tau^n \wedge \cdot})_t \,. \tag{2.6.2}$$

Definition 2.31. *A càdlàg semimartingale is a càdlàg adapted process X admitting a decomposition*

$$X = X_0 + M + A, \tag{2.6.3}$$

where A is a càdlàg adapted process with locally finite total variation starting from 0, M is a càdlàg local martingale process and X_0 is $\mathscr{F}_{0^-}$ measurable.

Remark 2.32.
1. *If M is a local martingale with bounded jumps, say bounded by one, then M is locally square integrable.*
2. *Each càdlàg process can have only finite number of jumps with size bigger than* 1.

As a consequence of Remark 2.32, we can assume that the local martingale M appearing in Decomposition (2.6.3) is locally square integrable local martingale.

Definition 2.33. *Let M be a locally square integrable local martingale such that the Decomposition (2.6.3) holds. For $H \in L^2(M)$ we define the stochastic integral with respect to semimartingale X as*

$$(H \cdot X)_t = (H \cdot M)_t + \int_{[0,t]} H_s dA_s, \tag{2.6.4}$$

where $\int_{[0,t]} H_s dA_s$ is the usual Lebesgue-Stieltjes integral.

Remark 2.34. *Although the decomposition (2.6.3) may not be unique, the integrals defined by (2.6.4) for two different such decomposition a.s. coincide.*

Proposition 2.35. *Let A be a càdlàg finite variation process. Then the quadratic variation of A is simply the sum of squares of jumps. That is*

$$[A]_t = \sum_{0<s\leq t} (\Delta A_s)^2, \tag{2.6.5}$$

where $\Delta A_s = A_s - A_{s^-}$.

Proof. Consider the càdlàg process

$$A_t = A_t^c + \sum_{0<s\leq t} \Delta A_s$$

where A_t^c denotes the continuous part of A_t over $[0,t]$ and $\sum_{0<s\leq t} \Delta A_s$ is the sum of jumps between 0 and t. We denote the latter by A_t^d. Thus, the quadratic variation along the partition $\pi = \{0 = t_0 < t_1 < t_2 < \cdots < t_{k(n)} = t\}$ of a process A_t is given by

$$\begin{aligned}
[A]_t &= \sum_{i=1}^{k(n)} (A_{t_i^n} - A_{t_{i-1}^n})^2 \\
&= \sum_{i=1}^{k(n)} \left(\left(A^c_{t_i^n} - A^c_{t_{i-1}^n} \right)^2 + \left(A^d_{t_i^n} - A^d_{t_{i-1}^n} \right)^2 \right) \\
&= \sum_{i=1}^{k(n)} \left(A^c_{t_i^n} - A^c_{t_{i-1}^n} \right)^2 + 2\sum_{i=1}^{k(n)} \left(A^c_{t_i^n} - A^c_{t_{i-1}^n} \right) \left(A^d_{t_i^n} - A^d_{t_{i-1}^n} \right) + \sum_{i=1}^{k(n)} \left(A^d_{t_i^n} - A^d_{t_i^n} \right)^2 .
\end{aligned} \tag{2.6.6}$$

We will consider the terms in Equation (2.6.6) separately. The first term is bounded by

$$\begin{aligned}
\sum_{i=1}^{k(n)} \left(A^c_{t_i^n} - A^c_{t_{i-1}^n} \right)^2 &\leq \sum_{i=1}^{k(n)} \left(\max_{1\leq i\leq k(n)} \left| A^c_{t_i^n} - A^c_{t_{i-1}^n} \right| \right) \left| A^c_{t_i^n} - A^c_{t_{i-1}^n} \right| \\
&\leq \left(\max_{1\leq i\leq k(n)} \left| A^c_{t_i^n} - A^c_{t_{i-1}^n} \right| \right) \sum_{i=1}^{k(n)} \left| A^c_{t_i^n} - A^c_{t_{i-1}^n} \right| .
\end{aligned} \tag{2.6.7}$$

The sum in Equation (2.6.7) is bounded by the total variation of A_t^c over the domain $[0,t]$. That is

$$\sum_{i=1}^{k(n)} \left| A^c_{t_i^n} - A^c_{t_{i-1}^n} \right| \leq TV(A^c, [0,t]) < +\infty. \tag{2.6.8}$$

However, the first product in Equation (2.6.7) tends a.s. (almost surely) to zero as the *mesh* of the partition π tends to zero.

The second term in Equation (2.6.6) tends to zero since it can be bounded with the help of Cauchy-Schwartz inequality represented as follows

$$\begin{aligned}\left|\sum_{i=1}^{k(n)}\left(A^c_{t^n_i}-A^c_{t^n_{i-1}}\right)\left(A^d_{t^n_i}-A^d_{t^n_{i-1}}\right)\right| &\leq \sum_{i=1}^{k(n)}\left|A^c_{t^n_i}-A^c_{t^n_{i-1}}\right|\left|A^d_{t^n_i}-A^d_{t^n_{i-1}}\right| \\ &\leq \sqrt{\sum_{i=1}^{k(n)}\left(A^c_{t^n_i}-A^c_{t^n_{i-1}}\right)^2}\sqrt{\sum_{i=1}^{k(n)}\left(A^d_{t^n_i}-A^d_{t^n_{i-1}}\right)^2} \\ &\to 0, \end{aligned} \tag{2.6.9}$$

and follows from Equation (2.6.7) and because the sum $\sum_{i=1}^{k(n)}\left(A^d_{t^n_i}-A^d_{t^n_{i-1}}\right)^2$ is bounded by $TV(A^d,[0,t])^2$. Finally, the last term $\sum_{i=1}^{k(n)}\left(A^d_{t^n_i}-A^d_{t^n_i}\right)^2$ tends to $\sum_{0<s\leq t}(\Delta A_s)^2$ as the mesh of the partition π tends to zero. □

Theorem 2.36. *Let A be a càdlàg finite variation process, M be a locally square integrable local martingale. Then the quadratic variation of the process $X = X_0 + M + A$ denoted by $[X]$ is finite and given by*

$$[X]_t = [M]_t + 2\sum_{0<s\leq t}\Delta M_s \Delta A_s + \sum_{0<s\leq t}(\Delta A_s)^2. \tag{2.6.10}$$

Proof. First notice that the càdlàg finite variation process A is expressed as follows

$$A_t = A^c_t + \sum_{0<s\leq t}\Delta A_s, \tag{2.6.11}$$

where A^c is continuous part of A. Using the Decomposition 2.6.3, we have

$$\begin{aligned}[X]_t &= (\mathbb{P})\lim_{n\to\infty}\sum_{i=1}^{k(n)}\left(X_{t^n_i}-X_{t^n_{i-1}}\right)^2 \\ &= (\mathbb{P})\lim_{n\to\infty}\sum_{i=1}^{k(n)}\left(\left(M_{t^n_i}-M_{t^n_{i-1}}\right)+\left(A_{t^n_i}-A_{t^n_{i-1}}\right)\right)^2 \\ &= (\mathbb{P})\lim_{n\to\infty}\left(\sum_{i=1}^{k(n)}\left(M_{t^n_i}-M_{t^n_{i-1}}\right)^2+2\sum_{i=1}^{k(n)}\left(M_{t^n_i}-M_{t^n_{i-1}}\right)\left(A_{t^n_i}-A_{t^n_{i-1}}\right)+\sum_{i=1}^{k(n)}\left(A_{t^n_i}-A_{t^n_{i-1}}\right)^2\right)\end{aligned}$$

The first term $\sum_{i=1}^{k(n)}\left(M_{t^n_i}-M_{t^n_{i-1}}\right)^2$ tends (in probability) to $[M]_t$ as $n\to\infty$. The last term tends to $[A]_t = \sum_{0<s\leq t}(\Delta X_s)^2$, see Proposition 2.35. The middle term

$$\sum_{i=1}^{k(n)}\left(M_{t^n_i}-M_{t^n_{i-1}}\right)\left(A_{t^n_i}-A_{t^n_{i-1}}\right) \tag{2.6.12}$$

can be proved that it converges to $\sum_{0<s\leq t} \Delta M_s \Delta A_s$ using the decomposition $A = A^c + A^d$ and the Cauchy-Schwartz inequality as in the proof of Proposition 2.35. □

2.6.1 Integration By Parts Formula

The summation by parts formula for partition $\pi^n = \{0 = t_0^n < t_1^n < t_2^n < \cdots < t_{k(n)}^n = t\}$ and semimartingales X and Y gives

$$X_tY_t - X_0Y_0 := \sum_{i=1}^{k(n)} X_{t_{i-1}^n}\left(Y_{t_i^n} - Y_{t_{i-1}^n}\right) + \sum_{i=1}^{k(n)} Y_{t_{i-1}^n}\left(X_{t_i^n} - X_{t_{i-1}^n}\right) + \sum_{i=1}^{k(n)} \left(X_{t_i^n} - X_{t_{i-1}^n}\right)\left(Y_{t_i^n} - Y_{t_{i-1}^n}\right). \quad (2.6.13)$$

The first two terms on the right of Equation (2.6.13) tend (in probability) to the stochastic integrals $(X_- \cdot Y)_t$ and $(Y_- \cdot X)_t$ respectively as the vertical mesh, that is

$$\max_i \sup_{s,t\in[t_{i-1}^n, t_i^n)} (|X_t - X_s| + |Y_t - Y_s|)$$

of the partition π tends to zero. From this it follows that the last term converges as well and its limit is called quadratic covariation and is denoted by $[X,Y]_t$.

$$[X,Y]_t = (\mathbb{P}) \lim_{n\to\infty} \sum_{i=1}^{k(n)} \left(X_{t_i} - X_{t_{i-1}}\right)\left(Y_{t_i} - Y_{t_{i-1}}\right). \quad (2.6.14)$$

Remark 2.37. *Notice that if X and Y are semimartingales (with respect to the same filtration) then $X+Y$ and $X-Y$ are also semimartingales. Let $[X+Y]$ and $[X-Y]$ denote their respective quadratic variations. From elementary identity $ab = \frac{1}{4}\left((a+b)^2 - (a-b)^2\right)$ and Equation (2.6.14) we have the following polarisation formula*

$$[X,Y] = \frac{1}{4}\left([X+Y] - [X-Y]\right).$$

Proceeding to the limit in Equation (2.6.13), we get the following integration by parts formula

$$X_tY_t - X_0Y_0 = (X_- \cdot Y)_t + (Y_- \cdot X)_t + [X,Y]_t. \quad (2.6.15)$$

If $Y = X$, then from (2.6.15) we get the following formula

$$X_t^2 - X_0^2 = 2(X_- \cdot X)_t + [X]_t. \quad (2.6.16)$$

A generalised Itô formula for semimartingale X is given below, see [24, Theorem, 26.7].

Theorem 5. *Let $X = (X^1, \cdots, X^d)$ be a semimartingale in $\mathbb{R}^d$, and a function $f \in C^2(\mathbb{R}^d)$. Then*

$$f(X_t) = \sum_{i=1}^{d} \int_{(0,t]} f_i'(X_{s-})dX_s^i + \sum_{i,j=1}^{d} \int_{(0,t]} f_{i,j}''(X_{s-})d[X^i, X^j]_s^c + \sum_{0<s\leq t} \{\Delta f(X_s) - \sum_{i=1}^{d} f_i'(X_{s-})\Delta X_s^i\}, \tag{2.6.17}$$

where f_i' is the first partial derivative with respect to X_i and $f_{i,j}''$ is the second partial derivative with respect to X_i and X_j.

Notice that Equation (2.6.16) is a special form of generalised Itô formula when $f(x) = x^2$.

2.7 Integration by Parts Formula and Brownian Motion

In this section we turn our attention to the special case when X is a one-dimensional standard Brownian motion. Let us recall that the standard Brownian motion has the following quadratic variation

$$[B]_t = t.$$

Theorem 6. *Let B denotes the standard Brownian motion and $f \in C^2$. Then it holds that*

$$f(B_t) - f(B_0) = \int_{(0,t]} f'(B_s)dB_s + \int_{(0,t]} \frac{1}{2} f''(B_s)ds \quad a.s., \tag{2.7.1}$$

where the first integral in (2.7.1) is the stochastic integral. It may be also obtained as the suitable limit (in probability) as follows

$$\lim_{n\to\infty} \sum f'(B_{t_{i-1}^n})(B_{t_i^n} - B_{t_{i-1}^n}) \tag{2.7.2}$$

and the second integral is a Riemann integral for each ω.

The differential form of equation (2.7.1) under the rules

$$(dB_t)^2 = dt \quad \text{and } (dB_t)^3 = 0$$

is given by

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2} f''(B_t)dt. \tag{2.7.3}$$

If f is a function of two variables, then the differential form of (2.7.3) is given by

$$df(t, B_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial B}dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial B^2}dt. \tag{2.7.4}$$

Remark 2.38. *Recall from classical calculus that for functions, say $f, g \in C^1$, we have*

$$\frac{d}{dt} f(g(t)) = f'(g(t))g'(t).$$

The integral form is given by

$$f(g(t)) = f(g(0)) + \int_{(0,t]} f'(g(s))g'(s)ds \tag{2.7.5}$$

which implies that

$$f(g(t)) = f(g(0)) + \int_{(0,t]} f'(g(s))dg(s). \tag{2.7.6}$$

Thus, in the stochastic setting, from the Itô formula (2.7.1), we get additional the term $\int_{(0,t]} \frac{1}{2} f''(B_s)ds$ which is called the correction term.

2.8 Application of Quadratic Variation

In this Section we highlight the application of quadratic variation in mathematical finance. In practice, this important mathematical quantity corresponds to realised volatility, often coined undesirable thing in asset allocation, [39].

We assume that the log price of a security is a semimartingale and that its continuous diffusible process is given by

$$dX_t = \mu_t dt + \sigma_t dB_t, \tag{2.8.1}$$

where $X_t = \ln S_t$ denotes the observed log price at time t, μ_t is the càdlàg drift process, σ_t is the instantaneous adapted càdlàg volatility process and B_t is a one-dimensional Brownian motion. The integral form of Equation (2.8.1) is

$$X_t = X_0 + \int_{(0,t]} \mu_s ds + \int_{(0,t]} \sigma_s dB_s, \tag{2.8.2}$$

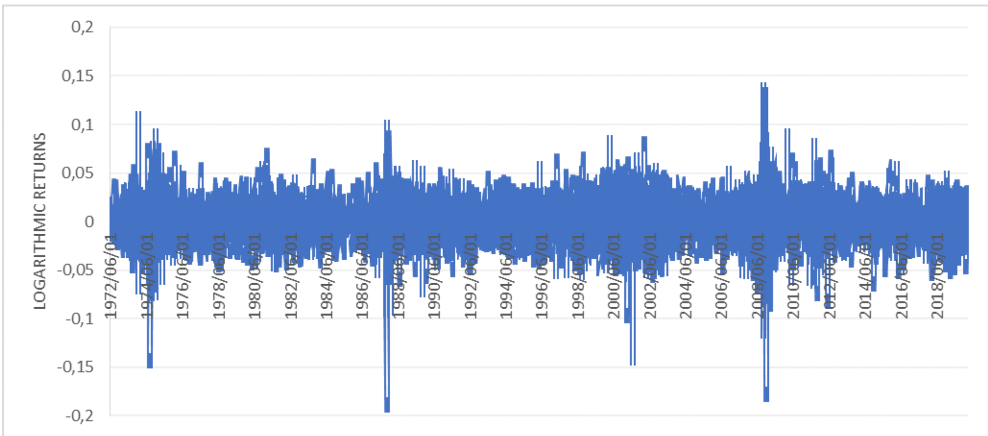
where the first integral in Equation (2.8.2) denotes the classical Riemann integral and the second integral is the Itô stochastic integral. From Equation (2.8.2) it follows that

$$[X]_t = \int_0^t \sigma_s^2 ds. \tag{2.8.3}$$

Figure 2.7 shows the observed daily prices of EMR (i.e. Emerson Electric Company listed in New York Stock Exchange) stock ticker from 01 June 1972 to 22 February 2020 and their corresponding logarithmic returns are given in Figure 2.8.

Figure 2.7: Typical Prices of EMR

Figure 2.8: Logarithmic Price Returns of EMR



We turn our attention to the Black-Scholes formula for option pricing. The Black-Scholes formula is derived under the assumption that the logarithm of stock price follows the dynamics (2.8.1) with constant μ and σ. Volatility is one of the important parameter incorporated in this formula.

We let C_{price} be the premium for the call option, S_0 denotes the initial stock price of the underlying security, T is the term to maturity, K the option strike price, r the risk-free rate, σ the constant instantaneous volatility of the underlying asset and $N(x)$ the standard normal cumulative distribution function cumulated to the point x. Then the pricing formula is given by

$$C_{\text{price}} = S_0 N(d_1) - N(d_2) K e^{-rT}, \tag{2.8.4}$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \tag{2.8.5}$$

$$d_2 = d_1 - \sigma\sqrt{T}. \tag{2.8.6}$$

Quadratic variation may also be used in swap's contracts [21]. Variance swap is a derivative with a path-dependent payoff which allows investors to take positions on the future variability of an asset. Payoff may be replicated using a portfolio of puts and calls and a dynamic position in the asset.

Chapter 3

Model-Free Itô Calculus

Following the financial crisis in 2008, see [5], there has been an increase in the model-free mathematical finance research aiming to understand the essential properties of the evolution of the prices quoted on financial markets. More recent research focusses on the establishment of price paths properties in the continuous time, see [41, 44, 45, 46, 50]. Moreover, several properties of local martingales have been proven for typical continuous time model-free price paths [45].

In this chapter, we introduce model-free mathematical finance both in continuous and càdlàg cases. We show the model-free version of formulas such as (2.6.15). In addition, various constructions of quadratic variations are also discussed and we also prove the generalised Itô's formula.

3.1 Model-Free Approach with Continuous Price Paths

Our approach is to establish more properties that build on the same ground, see [46], and later generalise the results for càdlàg price paths. These properties build on securities analysis and the behaviour of market participants (i.e. the trader and financial market).

3.1.1 Quadratic Variation and other Variations

In this Subsection, we define Lebesgue partitions and quadratic variation along sequence of Lebesgue partitions for continuous price paths. In addition, we give the proof of the existence of this quantity for 1-dimensional and extend to d-dimensional case. The quadratic variation of a price path is a crucial ingredient in the construction of model-free Itô integral.

First we consider a basic game between two players, i.e. reality (financial market) and Sceptic (Trader) in continuous time. Trader in this game will want to beat the market by betting against reality's move over time $[0,\infty) = \mathbb{R}_+$. This is called a perfect information game in the sense that each move by either reality or trader is immediately revealed to the other player. First Sceptic chooses his trading strategy and then reality chooses continuous function $\omega : [0,T] \to \mathbb{R}^d$, where $d \in 1,2,\cdots$ and $T \in (0,+\infty)$.

Let $S_0 \in \mathbb{R}^d_+$ and $C_{S_0}([0,T],\mathbb{R}^d)$ denote the space of all continuous functions $\omega : [0,T] \to \mathbb{R}^d$ starting from S_0. $\Omega = C_{S_0}([0,T],\mathbb{R}^d)$ will be our sample space with coordinate process

$$S_t(\omega) = \left(S^1_t(\omega), S^2_t(\omega), \ldots, S^d_t(\omega)\right) := \omega(t) = \left(\omega^1(t), \omega^2(t), \ldots, \omega^d(t)\right), \quad (3.1.1)$$

$t \in [0,T]$, and natural filtration $\mathscr{F}_t$, $0 \le t \le T$ ($\mathscr{F}_t$ is the smallest σ-algebra which makes all variables S_u, $0 \le u \le t$, measurable). This is generalised in [22] where Ω is called sets of beliefs or prediction sets.

Let $\tau : \Omega \to [0,T] \cup \{+\infty\}$ denote stopping times with respect to $\mathscr{F}$ and the corresponding σ-algebras $\mathscr{F}_\tau$ are defined as usual.

Definition 3.1. *A real valued process $G : [0,T] \times \Omega \to \mathbb{R}^d$ is called a simple process (or simple trading strategy) if it can be written as*

$$G_t(\omega) = g_0(\omega)\,\mathbb{1}_{\{0\}}(t) + \sum_{l=0}^{+\infty} g_l(\omega)\,\mathbb{1}_{(\tau_l(\omega),\tau_{l+1}(\omega)]}(t). \quad (3.1.2)$$

where $0 = \tau_0 \le \tau_1 \le \ldots$ are the stopping times and $g_l : \Omega \to \mathbb{R}^d$, are bounded $\mathscr{F}_{\tau_l}$-measurable functions such that for every $\omega \in \Omega$, $\tau_l(\omega) = \tau_{l+1}(\omega) = \ldots \in [0,T] \cup \{+\infty\}$ from some $l \in \{1,2,\ldots\}$.

Remark 3.2. *The second term on the right of Equation (3.1.2) may be interpreted as diversified trader's portfolio where g_l denotes position taken at time τ_l.*

Definition 3.3. *Let G be a simple trading strategy. Then the corresponding capital (integral) process $(G \cdot S) : [0,T] \times \Omega \to \mathbb{R}$ is*

$$\begin{aligned}(G\cdot S)_t(\omega) &:= \sum_{l=0}^{\infty} g_l(\omega)\cdot(S_{\tau_{l+1}(\omega)\wedge t}(\omega) - S_{\tau_l(\omega)\wedge t}(\omega)) \\ &= \sum_{l=0}^{\infty} g_l(\omega)\cdot S_{\tau_l(\omega)\wedge t,\tau_{l+1}(\omega)\wedge t}(\omega),\end{aligned} \quad (3.1.3)$$

where $S_{u,v} := S_v - S_u$ for $u,v \in [0,T]$.

The Integral process (3.1.3) is well defined for all $\omega \in \Omega$ and $t \in [0,T]$. The family of simple strategies is denoted by $\mathscr{G}$.

Definition 3.4. *Let G be a simple trading strategy and $\lambda > 0$. Then G is called strongly λ-admissible if*

$$(G \cdot S)_t(\omega) \geq -\lambda \tag{3.1.4}$$

for all $\omega \in \Omega$ and $t \in [0,T]$.

The family of λ-admissible strategies is denoted by $\mathscr{G}_\lambda$.

Definition 3.5. *Let $A \subseteq \Omega$ be a set of possible scenarios. Then the outer measure $\bar{\mathbb{P}}$ of A is the minimal superhedging price for $\mathbb{1}_A$ such that*

$$\bar{\mathbb{P}}(A) := \inf\left\{\lambda > 0 : \exists (G^n)_{n\in\mathbb{N}} \subseteq \mathscr{G}_\lambda \text{ s.t. } \forall \omega \in \Omega \ \liminf_{n\to\infty}(\lambda + (G^n \cdot S)_T(\omega)) \geq \mathbb{1}_A(\omega)\right\}. \tag{3.1.5}$$

A set $A \subseteq \Omega$ is called a *null set* if it has outer measure zero. A property (P) holds for *typical price paths* if the set A where (P) is violated is a null set.

Definition 3.6. *For each $\omega \in \Omega$ and $p \in (0,\infty)$, the strong p-variation is*

$$\mathbf{v}_p^{[a,b]}(\omega) = \sup_n \sup_{a\leq t_0<\cdots<t_n\leq b} \sum_{i=1}^n |\omega(t_i) - \omega(t_{i-1})|^p. \tag{3.1.6}$$

Definition 3.6 is a generalisation of total variation defined in Chapter 2.

Remark 3.7. *Vovk showed that the strong p-variation, $\mathbf{v}_p^{[a,b]}(\omega)$, is finite if there exists a unique number called variation index of ω, denoted by $vex(\omega) \in [0,\infty]$, such that $p > vex(\omega)$, see [44]. Also, $\mathbf{v}_p^{[a,b]}(\omega)$ is infinite when $p < vex(\omega)$ and $vex(\omega) \notin (0,1)$.*

In his paper, [46], he also defined quadratic variation over a sequence of stopping times. This sequence is defined inductively over semi-infinite interval, $[0,\infty)$, where ω is a continuous function. The following definition state the quadratic variation over this sequence of stopping times.

Definition 3.8. *Let $t \in [0,\infty)$ and $\omega \in \Omega$. Then the n^{th}-approximation of quadratic variation is*

$$[\omega]_t^n = \sum_{k=0}^{\infty} \left(\omega(T_k^n \wedge t) - \omega(T_{k-1}^n \wedge t)\right)^2 \quad n = 0,1,\cdots, \tag{3.1.7}$$

where T_k^n is the sequence of stopping times (known as the Lebesgue partition) defined inductively over the set of grids $\mathbb{D}_n$.

In particular, for $k = -1, 0, 1, 2, \cdots$, the sequence of Lebesgue partition is defined by:

$$T^n_{-1}(\omega) := 0,$$
$$T^n_0(\omega) := \inf\{t \geq 0 | \omega(t) \in \mathbb{D}_n\},$$
$$T^n_k(\omega) := \inf\{t \geq T^n_{k-1} | \omega(t) \in \mathbb{D}_n \text{ and } \omega(t) \neq \omega(T^n_{k-1})\}, \quad k = 1, 2, \cdots$$

and the set of grids is defined as follows,

$$\mathbb{D}_n := \{k2^{-n} | k \in \mathbb{Z}\}.$$

Proposition 3.9. *For typical ω, the function*

$$[\omega] : t \in [0, \infty) \to [\omega]_t := \overline{[\omega]}_t = \underline{[\omega]}_t \tag{3.1.8}$$

exists, and is an increasing element of Ω, satisfying $[\omega]_0 = 0$.

Proposition 3.9 follows from Theorem 3.25 which is proved in Section 3.2. It states that quadratic variation, $[\omega]_t$, exists along $[0, \infty)$ where

$$\overline{[\omega]}_t := \limsup_{n\to\infty} [\omega]^n_t, \quad \underline{[\omega]}_t := \liminf_{n\to\infty} [\omega]^n_t.$$

It represents the integrated volatility of ω over the time period $[0, t]$. However, Definition 3.8 relies on the arbitrary choice $\mathbb{D}_n$ of sequence of grids, [46]. Moreover, price paths of typical ω are continuous functions. We now suspend the continuous model free approach and turn our attention to the càdlàg case.

The properties of continuous price paths for model free approach are well understood in the literature, see [34, 35, 44, 45, 46]. For example, typical continuous model free price paths reveals several properties of local martingales. More precisely, Vovk and Shafer in the recent book [50] defined the set $\mathscr{G}_\lambda$ of nonnegative simple capital processes, closed under the limit of the infimum, to be nonnegative supermartingales with respect to the filtration, see also [48]. When this set is closed under limit, the notion of supermartingales coincide with the notion of continuous martingales.

3.2 Model-Free Approach with Càdlàg Price Paths

Let $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ be the set of all integers. For a topological space $\mathscr{E}$ we will denote by $D([0,T], \mathscr{E})$ the space of all càdlàg functions $\omega : [0,T] \to \mathscr{E}$. Now let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be some non-decreasing function. We equip $\mathbb{R}^d$ with Euclidean norm $|\cdot|$ and consider the underlying space Ω which is a subset of the set $\Omega_\psi \subset D\left([0,T], \mathbb{R}^d\right)$ of càdlàg functions with mildly restricted jumps directed downwards, that is $\omega = (\omega^1, \dots, \omega^d) \in \Omega_\psi$ if for $i = 1, \dots, d$ it satisfies

$$\Delta\omega^i(t) := \omega^i(t) - \omega^i(t-) \geq -\psi\left(\sup_{s\in[0,t)} |\omega(s)|\right), \tag{3.2.1}$$

where $t \in (0,T]$ and $\omega^i(t-) := \lim_{s\to t, s<t} \omega(s)$.

The following sample spaces are examples of Ω:

i. $\Omega_c := C([0,T], \mathbb{R}^d)$, the space of all continuous functions $\omega\colon [0,T] \to \mathbb{R}^d$,

ii. $\Omega_+ := D([0,T], \mathbb{R}^d_+)$, the space of all non-negative càdlàg functions $\omega\colon [0,T] \to \mathbb{R}^d_+$ (here $\psi(x) = x$),

iii. $\tilde{\Omega}_\psi$ which is defined as the subset of all càdlàg functions $\omega\colon [0,T] \to \mathbb{R}^d$ such that

$$|\omega(t) - \omega(t-)| \leq \psi\left(\sup_{s\in[0,t)} |\omega(s)|\right), \quad t \in (0,T],$$

and $\psi\colon \mathbb{R}_+ \to (0,\infty)$ is a fixed non-decreasing function.

The choice of Ω rests on the investor's beliefs.

We say that an event happen *a.s.* if the trader has a strategy that multiplies his capital by an infinite factor if the event fails. Stopping times $\tau : \Omega \to [0,T] \cup \{\infty\}$ with respect to the filtration $(\mathscr{F}_t)_{t\in[0,T]}$ and the corresponding σ-algebra $\mathscr{F}_\tau$ are defined as usual, see Definition 2.1. In particular, we consider the sequence of partitions $0 \leq \tau_1 \leq \tau_2 \leq \cdots$. Then the corresponding n^{th} partition is:

$$\tau^n := 0 \leq \tau_1^n < \tau_2^n < \cdots \tau_{k_n-1}^n < T = \tau_{k_n}^n < +\infty = \tau_{k_n+1}^n = \tau_{k_n+2}^n = \cdots \tag{3.2.2}$$

for some $k = 1, 2, \cdots$ We also denote by

$$h_k : \Omega \to \mathbb{R}^d \tag{3.2.3}$$

the $\mathscr{F}_{\tau_k}$-measurable bounded functions. Recall that the first step to construct a classical stochastic integral is to define step-functions (also known as simple process).

Definition 3.10. *Let us consider the process $H : \Omega \times [0,T] \to \mathbb{R}^d$. Then H is called a simple strategy if there exist stopping times $0 = \tau_0 \leq \tau_1 \leq \dots$ and $\mathscr{F}_{\tau_k}$-measurable bounded functions $h_k : \Omega \to \mathbb{R}^d$, $k \in \mathbb{N}_0$, such that for each $\omega \in \Omega$, $\tau_K(\omega) = \tau_{K+1}(\omega) = \ldots \in [0,T] \cup \{\infty\}$ from some $K \in \mathbb{N}_0$ on, and*

$$H_t(\omega) = h_0(\omega)\, \mathbb{1}_{\{0\}}(t) + \sum_{k=0}^{+\infty} h_k(\omega)\, \mathbb{1}_{(\tau_k(\omega), \tau_{k+1}(\omega)]}(t).$$

Definition 3.11. *Let H be a simple trading strategy and $\lambda > 0$. Then the corresponding simple capital (integral) process with the initial capital λ is*

$$
\begin{aligned}
(H\cdot S)_t^{\lambda}(\omega) := &\ \lambda + \sum_{n=0}^{\infty} h_n(\omega)(S_{\tau_{n+1}\wedge t}(\omega) - S_{\tau_n\wedge t}(\omega)) \\
= &\sum_{n=0}^{\infty} h_k(\omega)\cdot S_{\tau_k\wedge t,\tau_{k+1}\wedge t}(\omega).
\end{aligned} \tag{3.2.4}
$$

This integral is well defined for all $\omega \in \Omega$ and for all $t \in [0,T]$. The scalar product in $\mathbb{R}^d$ is denoted by "$\cdot$" and $S_{u,v} := S_v - S_u$. For $\lambda = 0$, we will simply write

$$
(H\cdot S)^{\lambda} = (H\cdot S). \tag{3.2.5}
$$

We denote by $\mathscr{H}$ the family of simple trading strategies.

Definition 3.12. *Let H be a simply trading strategy and $\lambda \in \mathbb{R}$. Then H is called (strongly) λ-admissible if*

$$
(H\cdot S)_t^{\lambda}(\omega) \geq 0 \quad \text{for all } \omega \in \Omega \text{ and all } t \in [0,T]. \tag{3.2.6}
$$

The family of λ-admissible strategies is denoted by $\mathscr{H}_\lambda$.

Definition 3.13. *Vovk's outer measure $\bar{\mathbb{P}}$ of a set $\mathscr{A} \subseteq \Omega$ is defined as the minimal superhedging price for $\mathbb{1}_{\mathscr{A}}$, that is*

$$
\bar{\mathbb{P}}(\mathscr{A}) := \inf\{\lambda \in \mathbb{R} : \exists (H^n)_{n\in\mathbb{N}} \subset \mathscr{H}_\lambda \ s.t\ \forall \omega \in \Omega : \liminf_{n\to\infty} (H^n\cdot S)_T^{\lambda}(\omega) \geq \mathbb{1}_{\mathscr{A}}(\omega)\}.
$$

Definition 3.13 is a slightly modified Vovk's outer measure, see [46, Definition 2.3] . It denotes the minimal super-hedging price of a set of possible scenarios $\mathscr{A} \in \Omega$. Also, it dominates all local martingales measures on the sample space Ω [31]. Moreover, it is related to the notion of the no-arbitrage opportunities of the first kind (NA1). This precludes very large profit with small risk.

Definition 3.14. *A set $\mathscr{A} \in \Omega$ is null if $\bar{\mathbb{P}}(\mathscr{A}) = 0$.*

Naturally, $\mathscr{A}$ in Definition 3.14 maybe deemed an arbitrage opportunity scenarios in the classical mathematical finance, see [31, Proposition 2.6].

Definition 3.15. *A property of $\omega \in \Omega$ holds $\bar{\mathbb{P}}$-almost surely or for typical ω if the set $\mathscr{A}$ of ω where it fails is null.*

Definition 3.16. *A probability measure $\mathbb{P}$ on Ω admits no arbitrage opportunities of the first kind if*

$$
\lim_{c\to+\infty} \sup_{H\in\mathscr{H}^1} \mathbb{P}\Big((H\cdot S)_T^1 \geq c\Big) = 0. \tag{3.2.7}
$$

If the coordinate process on $(\Omega, (\mathscr{F}_t)_{0\leq t\leq T}, \mathbb{P})$ satisfies (NA1), and $\mathscr{A} \in \mathscr{F}_T$ is null, i.e. $(\bar{\mathbb{P}}(\mathscr{A}) = 0)$, then $\mathbb{P}(\mathscr{A}) = 0$.

Since we will need some continuity results of the model-free integrals, we will use measure of Definition 3.13, rather than original Vovk's outer measure. However, this measure is defined with the use of strongly λ-admissible strategies. A more useful version of weakly λ-admissible strategies in the proof of quadratic variation is defined below.

Definition 3.17. *Let $\lambda \geq 0$ and H be a simple trading strategy. Then H is called weakly λ-admissible strategy if for all $(t,\omega) \in [0,T] \times \Omega_\psi$, the following holds:*

$$(H \cdot S)_t(\omega) \geq -\lambda(1 + |S_{\rho\lambda(H)}(\omega)| \mathbb{1}_{[\rho\lambda(H)(\omega),T]}(t)), \tag{3.2.8}$$

where

$$\rho\lambda(H)(\omega) := \inf\{t \in [0,T] : (H \cdot S)_t(\omega) \leq -\lambda\}$$

and $H_t(\omega) = H_t(\omega)\mathbb{1}_{[0,\rho\lambda(H)(\omega)\wedge T]}(\omega)$.

Intuitively, using H as a strategy, one expects a payoff larger than $-\lambda$. However, there is a risk of losing all previous gains plus $\lambda(|S_t|)$ through one large jump. Now the class of weakly λ-admissible strategies is denoted by $\mathscr{G}_\lambda$. Let us now introduce the notion of nth Lebesque partition π^n, $n \in \mathbb{N}$, adapted to càdlàg setting which we will use in the sequel.

For $n \in \mathbb{N}$ we define

$$\mathbb{D}^n := \{k2^{-n} : k \in \mathbb{Z}\}.$$

For $f \in D\left([0,T],\mathbb{R}\right)$, $\pi^n(f)$ consists of points in time at which the path f crosses (in space) a dyadic number from $\mathbb{D}^n$ which is not the same as the dyadic number crossed (as the last number from $\mathbb{D}^n$) at the preceding time. This idea is made precise in the following definition.

Definition 3.18. *Let $n \in \mathbb{N}$ and let $\mathbb{D}^n$ be the n^{th} generation of dyadic numbers. For $f \in D\left([0,T],\mathbb{R}\right)$ its n^{th} Lebesgue partition $\pi^n(f) := \{\pi_k^n(f) : k \in \mathbb{N}_0\}$ is given by the sequence of times $(\pi_k^n(f))_{k\in\mathbb{N}_0}$ inductively defined by*

$$\pi_0^n(f) := 0 \quad \text{and} \quad D_0^n(f) := \max\{\mathbb{D}^n \cap (-\infty, S_0(f)]\},$$

and for every $k \in \mathbb{N}$ we further set

$$\pi_k^n(f) := \inf\{t \in [\pi_{k-1}^n(f),T] : [\![f\left(\pi_{k-1}^n(f)\right), f(t)]\!] \cap (\mathbb{D}^n \setminus \left\{D_{k-1}^n(f)\right\}) \neq \emptyset\}$$

$$D_k^n(f) := \operatorname*{arg\,min}_{D \in [\![f\left(\pi_{k-1}^n(f)\right), f\left(\pi_k^n(f)\right)]\!] \cap (\mathbb{D}^n \setminus \left\{D_{k-1}^n(f)\right\})} |D - f\left(\pi_k^n(f)\right)|$$

with the convention $\inf \emptyset = \infty$ and $[\![u,v]\!] := [\min(u,v), \max(u,v)]$.

Next we define the sequence of Lebesgue partitions for d-dimensional càdlàg function $\omega \in \Omega$.

Definition 3.19. *For $n \in \mathbb{N}$ and $\omega \in D\left([0,T],\mathbb{R}^d\right)$ its Lebesgue partition $\pi^n(\omega) := \{\pi^n_k(\omega) : k \in \mathbb{N}_0\}$ is iteratively defined by $\pi^n_0(\omega) := 0$ and*

$$\pi^n_k(\omega) := \min\left\{t > \pi^n_{k-1}(\omega) : t \in \bigcup_{i=1}^{d} \pi^n(\omega^i) \cup \bigcup_{i,j=1,i\neq j}^{d} \pi^n(\omega^i + \omega^j)\right\}, \quad k \in \mathbb{N},$$

where $\omega = (\omega^1,\dots,\omega^d)$ and $\pi^n(\omega^i)$ and $\pi^n(\omega^i + \omega^j)$ are the Lebesgue partitions of ω^i and $\omega^i + \omega^j$ as introduced in Definition 3.18, respectively.

Note that π^n defined in Definition 3.19 is indeed a stopping times, see [47, Lemma 3].

Definition 3.20. *Let $\omega \in \Omega$. Then the sequence of discrete quadratic (co)variations along the sequence of Lebesgue partitions is:*

$$\left[S^i,S^j\right]_t(\omega) := \sum_{k=1}^{\infty} S^i_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\omega) S^j_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\omega), \quad t \in [0,T]. \tag{3.2.9}$$

For typical price paths $\omega \in \Omega$, the sequence of quadratic variation (3.2.9) converges for $i,j = 1,2,\dots,d$ in the uniform topology to some (càdlàg) function, $[0,T] \ni t \mapsto [S^i,S^j]_t(\omega)$, which is proven in Subsection 3.2.2, which we will call the quadratic (co)variation of S^i and S^j.

We will use the following notation:

$$\left|[S]_T(\omega)\right| = \left(\sum_{i,j=1}^{d} [S^i,S^j]^2_T(\omega)\right)^{\frac{1}{2}}.$$

For $Z : \Omega \times [0,T] \to \mathbb{R}^r$ $(r = 1,2,\dots)$ let us define

$$\left\|Z(\omega)\right\|_\infty := \sup_{0\le t\le T} \left|Z_t(\omega)\right|,$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^r$. Following [31] we will identify two processes $X,Y : \Omega \times [0,T] \to \mathbb{R}^r$ if

$$\bar{\mathbb{P}}\left(\omega \in \Omega : \left\|X(\omega) - Y(\omega)\right\|_\infty > 0\right) = 0.$$

This defines an equivalence relation, and we will write $\overline{L}_0(\mathbb{R}^r)$ (or $\overline{L}_0$ in short) for the space of its equivalence classes. We equip the space $\overline{L}_0(\mathbb{R}^r)$ with the distance

$$d_\infty(X,Y) := \overline{E}[\|X-Y\|_\infty \wedge 1], \tag{3.2.10}$$

where $\overline{E}$ denotes an expectation operator defined for $Z\colon \Omega \to [0,\infty]$ by

$$\overline{E}[Z] := \inf\left\{\lambda > 0 : \exists (H^n)_{n\in\mathbb{N}} \subseteq \mathscr{H}_\lambda \text{ s.t. } \forall \omega \in \Omega_\psi : \liminf_{n\to\infty}(\lambda + (H^n\cdot S)_T(\omega)) \ge Z(\omega)\right\}.$$

It can be shown that $(\overline{L}_0(\mathbb{R}^r), d_\infty)$ is a complete metric space and $(\overline{\mathscr{D}}(\mathbb{R}^r), d_\infty)$ is a closed subspace, where $\overline{\mathscr{D}}(\mathbb{R}^r)$ are those processes in $\overline{L}_0(\mathbb{R}^r)$ which have a càdlàg representative.

Definition 3.21. *Let $\varepsilon > 0$. A sequence $X_n \in \overline{L}_0$ converges in the outer measure $\bar{\mathbb{P}}$ on a set $A \subset \Omega$ to $X \in \overline{L}_0$ if*

$$\lim_{n\to+\infty} \bar{\mathbb{P}}\Big(\omega \in A : \left\|X_n(\omega) - X(\omega)\right\|_\infty > \varepsilon\Big) = 0.$$

Definition 3.22. *Let $q, M > 0$. Then*

$$\Omega_{q,M} := \Big\{\omega \in \Omega : |[S]_T(\omega)| \le q, \left\|S(\omega)\right\|_\infty \le M\Big\}.$$

Note that if for any $q, M > 0$, X_n converges in outer measure $\bar{\mathbb{P}}$ on the set $\Omega_{q,M}$ to $X \in \overline{L}_0$ then X is unique. For fixed $q, M > 0$ let us now introduce the pseudo-distance $d_{\infty,q,M}$ on $\overline{L}_0$ which is given by

$$d_{\infty,q,M}(X,Y) := \overline{E}[\|X - Y\|_\infty \wedge \mathbb{1}_{\Omega_{q,M}}].$$

Now let us define for some fixed $\varepsilon \in (0,1)$ the following metric on $\overline{L}_0$

$$d^\varepsilon_{\infty,\psi}(X,Y) := \sum_{n,m=1}^{\infty} 2^{-(n/2+m)(1+\varepsilon)}(\psi(2^m) \vee 2^m)^{-1} d_{\infty,2^n,2^m}(X,Y) \tag{3.2.11}$$

One of the main results of [31] is the following Theorem.

Theorem 3.23. *There exists two metric spaces $(\overline{H}_1, d_{\overline{H}_1})$ and $(\overline{H}_2, d_{\overline{H}_2})$ such that the (equivalence classes of) step functions (simple strategies) are dense in $\overline{H}_1$, $\overline{H}_2$ embedds into $\overline{\mathscr{D}}(\mathbb{R}^d)$ and the integral map $I\colon H \mapsto (H\cdot S) = \left(\int_{(0,t]} H_s \mathrm{d}S_s\right)_{t\in[0,T]}$, defined for simple strategies in Equation (3.2.4), has a continuous extension that maps $(\overline{H}_1, d_{\overline{H}_1})$ to $(\overline{H}_2, d_{\overline{H}_2})$. Moreover, one has the following continuity estimate*

$$d^\varepsilon_{\infty,\psi}((F\cdot S),(G\cdot S)) \lesssim d_\infty(F,G)^{1/3}, \tag{3.2.12}$$

and one can take $d_{\overline{H}_1} = d_\infty$ and $d_{\overline{H}_2} = d^\varepsilon_{\infty,\psi}$.

The metric space $(\overline{H}_1, d_{\overline{H}_1})$ can be chosen to contain the left-continuous versions of adapted càdlàg processes, see [31, Remark 4.3]. If we replace the filtration $(\mathscr{F}_t)_{t\in[0,T]}$ by its right-continuous version, we can define $(\overline{H}_1, d_{\overline{H}_1})$ such that it contains at least the càdlàg adapted processes and, furthermore, such that if $(F_n)_{n\in\mathbb{N}} \subset \overline{H}_1$ is a sequence with $\sup_{\omega\in\Omega} \|F_n(\omega) - F(\omega)\|_\infty \to 0$ as $n \to +\infty$, then $F \in \overline{H}_1$ and there exists a subsequence (F_{n_k}) with

$$\lim_{k\to\infty} \|(F_{n_k}\cdot S)(\omega) - (F\cdot S)(\omega)\|_\infty = 0$$

for typical price paths $\omega \in \Omega$. By [31, Corollary 4.9] and Inequality (3.2.12) we also get

Corollary 3.24. *For $a, q, m, M > 0$ and any $H \in \overline{H}_1$ one has*

$$\bar{\mathbb{P}}\Big(\Big\{\|(H\cdot S)\|_\infty \geq a\Big\} \cap \Big\{\|H\|_\infty \leq m\Big\} \cap \Big\{|[S]_T| \leq q\Big\} \cap \Big\{\|S\|_\infty \leq M\Big\}\Big)$$
$$\leq (1 + 3dM + 2d\psi(M))\frac{6\sqrt{q} + 2 + 2M}{a} m.$$

3.2.1 Pathwise Quadratic Variation of Càdlàg Price Paths

In this subsection we show detailed proof of the existence of quadratic variation for typical càdlàg price paths. We follow closely [31]. Recall the definition of the quadratic (co)variation, see Equation (3.2.9). The corresponding one-dimensional quadratic variation is given by:

$$[\omega]_t^n := \sum_{k=1}^{\infty}\Big(S_{\tau_k^n \wedge t}(\omega) - S_{\tau_{k-1}^n \wedge t}(\omega)\Big)^2.$$

Thus we state the following theorem.

Theorem 3.25. *For typical price paths $\omega \in \Omega_\psi \subseteq D([0,T],\mathbb{R})$ the discrete quadratic variation*

$$[S]_t^n(\omega) := \sum_{k=1}^{\infty}\Big(S_{\tau_k^n \wedge t}(\omega) - S_{\tau_{k-1}^n \wedge t}(\omega)\Big)^2, \quad t \in [0,T],$$

along the Lebesgue partitions $(\pi_n(\omega))_{n\in\mathbb{N}}$ converges in the uniform metric to a function $[S](\omega) \in D([0,T],\mathbb{R}_+)$.

To prove the existence of this quantity, several auxiliary results are involved. First we define an auxiliary set of the class of trading strategies in the spirit of the minimal superhedging price.

Definition 3.26. *Let $\lambda > 0$ and the set $\mathscr{A} \in \Omega_\psi$. The set function $\bar{Q}$ is given by:*

$$\bar{Q}(\mathscr{A}) := \inf\Big\{\lambda > 0 : \exists (H^n)_{n\in\mathbb{N}} \subset \mathcal{G}_\lambda \, s.t. \forall \omega \in \Omega_\psi : \lim_{n\to\infty}\inf(\lambda + (H^n \cdot S)_T(\omega) + \lambda \mathbf{1}\cdot S_{\rho\lambda(H^n)}(\omega)\mathbb{1}_{\{\rho\lambda(H^n)<\infty\}}(\omega)) \geq \mathbb{1}_{\mathscr{A}}(\omega)\Big\}, \tag{3.2.13}$$

where $\mathbf{1} := (1, \cdots, 1) \in \mathbb{R}^d$.

Lemma 3.27. *If $\mathscr{A} \in \Omega_{\psi,K} := \{\omega \in \Omega_\psi : ||\omega||_\infty \leq K\}$ for $K \in \mathbb{R}_+$, then*

$$\bar{Q}(\mathscr{A}) \leq \bar{\mathbb{P}}(\mathscr{A}) \leq (1 + 3dK + 2d\psi(K))\bar{Q}(\mathscr{A}).$$

We refer to [31, Lemma 2.9] for the proof of this Lemma. Next we analyse the crossing behaviour of typical price paths with respect to the dyadic level $\mathbb{D}^n$. This process rest on the Doob's inequality.

Definition 3.28. *Let $f : [0,T] \to \mathbb{R}$ be a càdlàg function, $(a,b) \subset \mathbb{R}$ be an open non-empty interval and $t \in [0,T]$. The number $U_t^{(a,b)}(f)$ of upcrossings of the interval (a,b) by the function f during the time interval $[0,t]$ is given by*

$$U_t^{(a,b)}(f) := \sup_{n\in\mathbb{N}} \sup_{0\le s_1<t_1<\cdots<s_n<t_n\le t} \sum_{i=1}^{n} I(f(s_i), f(t_i)),$$

where

$$I(f(s_i), f(t_i)) := \begin{cases} 1 & \text{if } f(s_i) \le a \text{ and } f(t_i) \ge b, \\ 0 & \text{if otherwise.} \end{cases}$$

The number $D_t^{(a,b)}(f)$ of downcrossings is defined analogously. For $h > 0$ we also introduce the accumulated number of upcrossing respectively downcrossings by

$$U_t(f,h) := \sum_{k\in\mathbb{Z}} U_t^{(kh,(k+1)h)}(f) \quad \text{and} \quad D_t(f,h) := \sum_{k\in\mathbb{Z}} D_t^{(kh,(k+1)h)}(f).$$

The following stopping times

$$\gamma_K(\omega) := \inf\{t \in [0,T] : |S_t(\omega)| \ge K\} \tag{3.2.14}$$

for $\omega \in \Omega_\psi$ and $K \in \mathbb{N}$ play a vital role in the derivation of Doob's inequality.

Lemma 3.29. *Let $K > 0$. For each $n \in \mathbb{N}$, there exists a strongly 1-admissible simple strategy $H^n \in \mathscr{H}_1$ such that*

$$(H^n \cdot S)_t(\omega) \ge \frac{U_t(\omega, \frac{1}{2^n})}{2^{2n}\left[2K(2K+\psi(K))\right]} - 1 \tag{3.2.15}$$

for all $t \in [0,T]$ and every $\omega \in \{\omega \in \Omega_\psi : \|\omega\|_\infty < K\} \subseteq D([0,T],\mathbb{R})$.

The proof of this lemma follows a recursive algorithm on the open interval $(a,b) \subset \mathbb{R}$. The trader in this case buys one unit immediately when he realizes that the security price, $S_t(\omega)$, drops below the a and sell the security when the price is above b.

Proof. Let $U_t^{(a,b)}(\omega)$ be Doob's upcrossings over an interval $(a,b) \subseteq [-K,K]$. Follwing the algorithm:

i. Buy one unit immediately when $S_t(\omega) < a$.

ii. Sell when $S_t(\omega) > b$.

Carry out this process until the terminal time T or until we leave the interval $(-K,K)$, whatever occurs first, we obtain a simple strategy $H^{(a,b)} \in \mathscr{H}_{a+K+\psi(K)}$ with

$$a+K+\psi(K)+(H^{(a,b)}\cdot S)_{t\wedge\gamma_K}(\omega) \geq (b-a)U^{(a,b)}_{t\wedge\gamma_K}(\omega), \quad (t,\omega)\in[0,T]\times\Omega_\psi.$$

For a formal construction of $H^{(a,b)}$ we refer to [47, Lemma 4.5]. Note that we need the predictable bound of the jump size given by ψ to guarantee the strong admissibility of $H^{(a,b)}$. Set now

$$H^n := \frac{\sum_{k\in\mathbb{Z},\,(k+1)2^{-n}<K, k2^{-n}>-K} H^{(k2^{-n},(k+1)2^{-n})}}{K2^{n+1}(2K+\psi(K))}.$$

Since $H^{(k2^{-n},(k+1)2^{-n})} \in \mathscr{H}_{k2^{-n}+K+\psi(K)} \subseteq \mathscr{H}_{2K+\psi(K)}$ for all k with $(k+1)2^{-n}<K$ and $k2^{-n}>-K$, we have $H^n\in\mathscr{H}_1$, and

$$\begin{aligned}1+(H^n\cdot S)_{t(\omega)}(\omega) &\geq \frac{\sum_{\substack{k\in\mathbb{Z},\,(k+1)2^{-n}<K,\\ k2^{-n}>-K}} 2^{-n}U^{(k2^{-n},(k+1)2^{-n})}_{t(\omega)}(\omega)}{K2^{n+1}(2K+\psi(K))}\\ &= \frac{U_t(\omega,2^{-n})}{2^{2n}2K(2K+\psi(K))}\end{aligned}$$

for each $t\in[0,T]$ and all $\omega\in\{\omega\in\Omega_\psi : \|\omega\|_\infty<K\}$. □

The following corollary proves that both the upward crossing and downward crossing are bounded. The prove rest on the following lemma called Borel-Cantelli lemma.

Lemma 3.30. *Let $(A_j)_{j\in\mathbb{N}}\subseteq\Omega_\psi$ be a sequence of events. If $\sum_{j=1}^{\infty}\bar{\mathbb{P}}(A_j)<\infty$, then*

$$\bar{\mathbb{P}}\left(\bigcap_{i=1}^{\infty}\bigcup_{i=j}^{\infty}A_j\right) \leq \liminf_{i\to\infty}\bar{\mathbb{P}}\left(\bigcup_{j=i}^{\infty}A_j\right) \leq \liminf_{i\to\infty}\sum_{j=i}^{\infty}\bar{\mathbb{P}}(A_j)=0. \tag{3.2.16}$$

Thus, the pathwise version of Doob's upcrossing lemma enables us to control the number of level crossings of typical càdlàg price paths belonging to Ω_ψ.

Corollary 3.31. *For typical price paths $\omega\in\Omega_\psi\subseteq D([0,T],\mathbb{R})$ there exists an $N(\omega)\in\mathbb{N}$ such that*

$$U_T(\omega,2^{-n})\leq n^2 2^{2n} \quad \textit{and} \quad \mathbb{D}_T(\omega,2^{-n})\leq n^2 2^{2n}$$

for all $n\geq N(\omega)$.

Proof. Since for each $k \in \mathbb{Z}$, $U_t^{(k2^{-n},(k+1)2^{-n})}(\omega)$ and $\mathbb{D}_t^{(k2^{-n},(k+1)2^{-n})}(\omega)$ differ by no more than 1, we have $|U_T(\omega, 2^{-n}) - \mathbb{D}_T(\omega, 2^{-n})| \in [0, 2^{n+1}K]$ for all $n \in \mathbb{N}$ and for every $\omega \in \Omega_\psi$ with $\sup_{t\in[0,T]} |S_t(\omega)| < K$. So if we show that $\overline{\mathbb{P}}(B_K) = 0$ for all $K \in \mathbb{N}$, where

$$B_K := \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} A_{K,n}$$

with

$$A_{K,n} = \left\{ \omega \in \Omega_\psi : \sup_{t\in[0,T]} |S_t(\omega)| < K \text{ and } U_T(\omega, 2^{-n}) \geq \frac{n^2 2^{2n}}{2} \right\},$$

then our claim follows from the countable subadditivity of $\overline{\mathbb{P}}$. But using Lemma 3.29 we immediately obtain that

$$\overline{\mathbb{P}}(A_{K,n}) \leq n^{-2}[2K(2K + \psi(K))]$$

and applying the Borel-Cantelli Lemma 3.30, we obtain that $\overline{\mathbb{P}}(B_K) = 0$, since it is summable. □

To prove the convergence of the discrete quadratic variation processes $([S]^n)_{n\in\mathbb{N}}$, we shall show that the sequence $([S]^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the uniform metric on $D([0,T], \mathbb{R}_+)$. For this purpose, we define the auxiliary sequence $(Z^n)_{n\in\mathbb{N}}$ by

$$Z_t^n := [S]_t^n - [S]_t^{n-1}, \quad t \in [0,T].$$

Similarly as in Vovk [47], the proof of Theorem 3.25 is based on the sequence of integral processes $(\mathscr{K}^n)_{n\in\mathbb{N}}$ given by

$$\mathscr{K}_t^n := n^4 2^{-2n} + 2^{-n+5}(K + \psi(K))^2 + (Z_t^n)^2 - \sum_{k=1}^{\infty} (Z^n_{\tau_k^n \wedge t} - Z^n_{\tau_{k-1}^n \wedge t})^2, \quad t \in [0,T], \tag{3.2.17}$$

for $K \in \mathbb{N}$, and the stopping times

$$\sigma_K^n := \min\left\{ \tau_k^n : \sum_{i=1}^{k} \left(Z^n_{\tau_i^n} - Z^n_{\tau_{i-1}^n} \right)^2 > n^4 2^{-2n} \right\} \wedge \min\left\{ \tau_k^n : Z^n_{\tau_k^n} > K \right\}, \quad n \in \mathbb{N}. \tag{3.2.18}$$

The next lemma states that each $\mathscr{K}^n$ is indeed an integral process with respect to a weakly admissible simple strategy, cf. [47, Lemma 5].

Lemma 3.32. *For each $n \in \mathbb{N}$ and $K \in \mathbb{N}$, there exists a weakly admissible simple strategy $L^{K,n} \in \mathbb{G}_{n^4 2^{-2n} + 2^{-n+5}(K+\psi(K))^2}$ such that*

$$\mathscr{K}^n_{\gamma_K \wedge \sigma_K^n \wedge t} = n^4 2^{-2n} + 2^{-n+5}(K + \psi(K))^2 + (L^{K,n} \cdot S)_t, \quad t \in [0,T].$$

Proof. For each $K \in \mathbb{N}$ and each $n \in \mathbb{N}$ [47, Lemma 5] shows the equality for the strategy

$$L_t^{K,n} := \mathbb{1}_{(0,\gamma_K \wedge \sigma_K^n]}(t) \sum_k (-4) Z^n_{\tau_k^n} (S_{\tau_k^n} - S_{\chi^{n-1}(\tau_k^n)}) \mathbb{1}_{(\tau_k^n, \tau_{k+1}^n]}(t), \quad t \in [0,T],$$

where

$$\chi^{n-1}(t) := \max \left\{ \tau_{k'}^{n-1} \,:\, \tau_{k'}^{n-1} \leq t \right\}.$$

Since $L^{K,n}$ is obviously a simple strategy, it remains to prove that

$$L^{K,n} \in \mathbb{G}_{n^4 2^{-2n} + 2^{-n+5}(K+\psi(K))^2}. \tag{3.2.19}$$

First we observe up to time $\tilde{\tau}^n := \max\{\tau_k^n \,:\, \tau_k^n < \gamma_K \wedge \sigma_K^n\}$ that

$$\min_{t \in [0,\tilde{\tau}^n]} \mathscr{K}^n_{\gamma_K \wedge \sigma_K^n \wedge t} \geq 2^{-n+5}(K + \psi(K))^2, \tag{3.2.20}$$

which follows directly from the definition of $\mathscr{K}^n$ and Equation (3.2.18). For $t \in (\tilde{\tau}^n, \gamma_K \wedge \sigma_K^n]$ notice that

$$|S_{\tilde{\tau}^n} - S_{\chi^{n-1}(\tilde{\tau}^n)}| \leq 2^{-n+2}, \tag{3.2.21}$$

since we either have $\tilde{\tau}^n \in \pi_{n-1}$, which implies $\chi^{n-1}(\tilde{\tau}^n) = \tilde{\tau}^n$ and $|S_{\tilde{\tau}^n} - S_{\chi^{n-1}(\tilde{\tau}^n)}| = 0$, or we have $\tilde{\tau}^n \notin \pi_{n-1}$, which implies (3.2.21) as $\tilde{\tau}^n < \tau_{k'+1}^{n-1}$ and

$$|S_{\tilde{\tau}^n} - S_{\chi^{n-1}(\tilde{\tau}^n)}| \leq |S_{\tilde{\tau}^n} - D_{k'}^{n-1}| + |S_{\chi^{n-1}(\tilde{\tau}^n)} - D_{k'}^{n-1}| \leq 2^{-n+1} + 2^{-n+1},$$

where k' is such that $\chi^{n-1}(\tilde{\tau}^n) = \tau_{k'}^{n-1}$. Using Equation (3.2.21), $|Z^n_{\tilde{\tau}^n}| \leq K$ and $|S_{\tilde{\tau}^n}| \leq K$, we estimate

$$|4 Z^n_{\tilde{\tau}^n} (S_{\tilde{\tau}^n} - S_{\chi^{n-1}(\tilde{\tau}^n)})(S_t - S_{\tilde{\tau}^n})| \leq 4K 2^{-n+2}(|S_t| + K) = 2^{-n+4}(K|S_t| + K^2),$$

which together with (3.2.20) gives weak admissibility as claimed in (3.2.19). □

Corollary 3.33. *For typical price paths $\omega \in \Omega_\psi \subseteq D([0,T],\mathbb{R})$ there exist an $N(\omega) \in \mathbb{N}$ such that*

$$\mathscr{K}^n_{\gamma_K \wedge \sigma_K^n \wedge t}(\omega) < n^6 2^{-n}, \quad t \in [0,T],$$

for all $n \geq N(\omega)$.

Proof. Consider the events

$$A_{n,m} := \left\{ \omega \in \Omega_\psi \,:\, \exists t \in [0,T] \text{ s.t. } \mathscr{K}^n_{\gamma_K \wedge \sigma_K^n \wedge t}(\omega) \geq n^6 2^{-n} \text{ and } \sup_{t \in [0,T]} |S_t(\omega)| \leq m \right\}$$

for $n,m\in\mathbb{N}$. By the countable subadditivity of $\overline{\mathbb{P}}$ and the Borel-Cantelli lemma the claim follows once we have shown that $\sum_n \overline{\mathbb{P}}(A_{n,m})<\infty$ for every $m\in\mathbb{N}$. To that end, we define the stopping times

$$\rho^n := \inf\left\{t\in[0,T]\,:\,\mathscr{K}^n_{\gamma_K\wedge\sigma^n_K\wedge t}\geq n^6 2^{-n}\right\},\quad n\in\mathbb{N},$$

so that

$$A_{n,m}=\left\{\omega\in\Omega_\psi\,:\,n^{-6}2^n\mathscr{K}^n_{\gamma_K\wedge\sigma^n_K\wedge\rho^n\wedge T}(\omega)\geq 1 \text{ and } \sup_{t\in[0,T]}|S_t(\omega)|\leq m\right\}.$$

Now it follows directly from Lemma 3.32 that

$$\overline{Q}(A_{n,m})\leq n^{-6}2^n(n^4 2^{-2n}+2^{-n+5}(K+\psi(K))^2)=n^{-2}2^{-n}+n^{-6}2^5(K+\psi(K))^2,$$

which is summable. Since $\overline{\mathbb{P}}(A_{n,m})\leq(1+3m+2\psi(m))\overline{Q}(A_{n,m})$ by Lemma 3.27, the proof is complete. □

Finally, we have collected all necessary ingredients to prove the main result of this section, namely Theorem 3.25. More precisely, we shall show that $([S]^n-[S]^{n-1})_{n\in\mathbb{N}}$ is a Cauchy sequence. This implies Theorem 3.25 since the uniform metric on $D([0,T],\mathbb{R}_+)$ is complete.

Proof of Theorem 3.25. For $K\in\mathbb{N}$ let us define

$$A_K:=\left\{\omega\in\Omega_\psi\,:\,\sup_{t\in[0,T]}|S_t(\omega)|\leq K \text{ and } \sup_{t\in[0,T]}|Z^n_t(\omega)|\geq n^3 2^{-\frac{n}{2}} \text{ for infinitely many } n\in\mathbb{N}\right\}$$

and

$$B:=\Bigg\{\omega\in\Omega_\psi\,:\exists N(\omega)\in\mathbb{N}\text{ s.t. }\mathscr{K}^n_{\gamma_K\wedge\sigma^n_K\wedge t}(\omega)<n^6 2^{-n},\quad t\in[0,T],$$
$$U_T(\omega,2^{-n})\leq n^2 2^{2n}\text{ and }\mathbb{D}_T(\omega,2^{-n})\leq n^2 2^{2n},\quad n\geq N(\omega)\Bigg\}.$$

Thanks to the countable subadditivity of $\overline{P}$ it is sufficient to show that $\overline{P}(A_K)=0$ for every $K\in\mathbb{N}$. Moreover, again by the subadditivity of $\overline{P}$ we see

$$\overline{P}(A_K)\leq\overline{P}(A_K\cap B)+\overline{P}(A_K\cap B^c).$$

By Corollary 3.31 and Corollary 3.33 it is already known that $\overline{P}(A_K\cap B^c)=0$. In the following we show that $A_K\cap B=\emptyset$.

For this purpose, let us fix an $\omega\in B$ such that $\sup_{t\in[0,T]}|S_t(\omega)|\leq K$. Since $\omega\in B$ there exits an $N(\omega)\in\mathbb{N}$ such that for all $m\geq N(\omega)$:

(a) The number of stopping times in π_m does not exceed $2m^2 2^{2m} + 2 \leq 3m^2 2^{2m}$.

(b) The number of stopping times in π_m such that

$$\left|\Delta S_{\tau_k^m}(\omega)\right| := \left|S_{\tau_k^m}(\omega) - \lim_{s \to \tau_k^m, s < \tau_k^m} S_s(\omega)\right| \geq 2^{-m+1}, \quad \tau_k^m \in \pi_m,$$

is less or equal to $2m^2 2^{2m}$.

As $\sup_{t \in [0,T]} |S_t(\omega)| \leq K$, notice that $\gamma_K(\omega) = T$ and that for $t \in [0,T]$ we have

$$\begin{aligned}
Z^n_{\tau^n_{k+1} \wedge t}(\omega) - Z^n_{\tau^n_k \wedge t}(\omega) &= \left([S]^n_{\tau^n_{k+1} \wedge t}(\omega) - [S]^n_{\tau^n_k \wedge t}(\omega)\right) - \left([S]^{n-1}_{\tau^n_{k+1} \wedge t}(\omega) - [S]^{n-1}_{\tau^n_k \wedge t}(\omega)\right) \\
&= \left(S_{\tau^n_{k+1} \wedge t}(\omega) - S_{\tau^n_k \wedge t}(\omega)\right)^2 \\
&\quad - \left(\left(S_{\tau^n_{k+1} \wedge t}(\omega) - S_{\chi^{n-1}(\tau^n_k \wedge t)}(\omega)\right)^2 - \left(S_{\tau^n_k \wedge t}(\omega) - S_{\chi^{n-1}(\tau^n_k \wedge t)}(\omega)\right)^2\right) \\
&= -2\left(S_{\tau^n_k \wedge t}(\omega) - S_{\chi^{n-1}(\tau^n_k \wedge t)}(\omega)\right)\left(S_{\tau^n_{k+1} \wedge t}(\omega) - S_{\tau^n_k \wedge t}(\omega)\right),
\end{aligned}$$

where we recall that $\chi^{n-1}(t) := \max\{\tau^{n-1}_{k'} : \tau^{n-1}_{k'} \leq t\}$. Therefore, keeping (3.2.21) in mind, the infinite sum in (3.2.17) can be estimated by

$$\sum_{k=0}^{\infty}\left(Z^n_{\tau^n_{k+1} \wedge t}(\omega) - Z^n_{\tau^n_k \wedge t}(\omega)\right)^2 \tag{3.2.22}$$

$$= 4\sum_{k=0}^{\infty}\left(S_{\tau^n_k \wedge t}(\omega) - S_{\chi^{n-1}(\tau^n_k \wedge t)}(\omega)\right)^2\left(S_{\tau^n_{k+1} \wedge t}(\omega) - S_{\tau^n_k \wedge t}(\omega)\right)^2$$

$$\leq 2^{6-2n}\sum_{k=0}^{\infty}\left(S_{\tau^n_{k+1} \wedge t}(\omega) - S_{\tau^n_k \wedge t}(\omega)\right)^2. \tag{3.2.23}$$

For $n \geq N = N(\omega)$ and $t \in [0,T]$ we observe for the summands in (3.2.22) the following bounds, which are similar to the bounds (A)-(E) in the proof of [47, Theorem 1]:

1. If $\tau^n_{k+1} \notin \pi_{n-1}$, then one has $\chi^{n-1}(\tau^n_{k+1}) = \chi^{n-1}(\tau^n_k) = \tau^{n-1}_{k'}$ for some k' and thus

   $$\left|S_{\tau^n_{k+1} \wedge t}(\omega) - S_{\tau^n_k \wedge t}(\omega)\right| \leq |S_{\tau^n_{k+1} \wedge t}(\omega) - D^{n-1}_{k'}| + |S_{\tau^n_k \wedge t}(\omega) - D^{n-1}_{k'}| \leq 2^{2-n}.$$

   The number of such summands is at most $3n^2 2^{2n}$.

2. If $\tau^n_{k+1} \in \pi_{n-1}$ and $|\Delta S_{\tau^n_{k+1}}| \leq 2^{-n+1}$, then one has

   $$\left|S_{\tau^n_{k+1} \wedge t}(\omega) - S_{\tau^n_k \wedge t}(\omega)\right| \leq 2^{1-n} + 2^{-n+1} = 2^{2-n}$$

   and the number of such summands is at most $3n^2 2^{2n}$.

3. If $\tau^n_{k+1} \in \pi_{n-1}$ and $|\Delta S_{\tau^n_{k+1}}| \in [2^{-m+1}, 2^{-m+2})$, for some $m \in \{N, N+1, \dots, n\}$ than one has that

$$\left|S_{\tau^n_{k+1}\wedge t}(\omega) - S_{\tau^n_k \wedge t}(\omega)\right| \le 2^{1-n} + 2^{-m+2}.$$

and the number of such summands is at most $2m^2 2^{2m}$.

4. If $\tau^n_{k+1} \in \pi_{n-1}$ and $\Delta S_{\tau^n_{k+1}} \ge 2^{-N+2}$, then one has

$$\left|S_{\tau^n_{k+1}\wedge t}(\omega) - S_{\tau^n_k(\omega)\wedge t}\right| \le 2K$$

and the number of such summand is bounded by a constant $C = C(\omega, K)$ independent of n.

Using the bounds derived in (1)-(4), the estimate (3.2.22) can be continued by

$$\sum_{k=0}^{\infty}\left(Z^n_{\tau^n_{k+1}\wedge t}(\omega) - Z^n_{\tau^n_k\wedge t}(\omega)\right)^2 \le 2^{6-2n}\left(6n^2 2^{2n} 2^{4-2n} + \sum_{m=N}^{n} 2m^2 2^{2m}(2^{1-n} + 2^{-m+2})^2 + 4CK^2\right),$$

and thus there exists an $\tilde{N} = \tilde{N}(\omega) \in \mathbb{N}$ such that

$$\sum_{k=0}^{\infty}\left(Z^n_{\tau^n_{k+1}\wedge t}(\omega) - Z^n_{\tau^n_k\wedge t}(\omega)\right)^2 \le 2^{-2n} n^4, \quad t \in [0,T],$$

for all $n \ge \tilde{N}$. Combining the last estimate with the definition of $\mathscr{K}^n$ (cf. (3.2.17)), we obtain

$$\mathscr{K}^n_{\sigma^n_K \wedge t}(\omega) \ge \left(Z^n_{\sigma^n_K\wedge t}(\omega)\right)^2, \quad t \in [0,T],$$

for all $n \ge \tilde{N}$. Moreover, by assumption on ω one has

$$\mathscr{K}^n_{\sigma^n_K \wedge t}(\omega) < n^6 2^{-n}, \quad t \in [0,T],$$

for all $n \ge N \vee \tilde{N}$. In particular, we conclude that $\sup_{t\in[0,T]} \left|Z^n_{\sigma^n_K \wedge \gamma_K \wedge t}(\omega)\right| < K$ whenever n is large enough and thus

$$n^6 2^{-n} > \left(Z^n_t(\omega)\right)^2, \quad t \in [0,T],$$

for all sufficiently large n. Finally, we have $\sup_{t\in[0,T]} |Z^n_t(\omega)| < n^3 2^{-\frac{n}{2}}$ for all large n and therefore $\omega \notin A_K \cap B$. □

Remark 3.34. *As proven by [47, Proposition 3], the existence of quadratic variation in Theorem 3.25 is equivalent to the existence of quadratic variation in the sense of Föllmer. Thus, Theorem 3.25 allow us to use Föllmer's pathwise Itô formula [17] to typical price paths belonging to the sample space Ω_ψ and in particular to define the pathwise integral*

$$\int f'(S_s)dS_s$$

for $f \in C^2$ or for more general path-dependent functionals as in [2, 9, 23].

3.2.2 Extension to Multi-dimensional Price Paths

In order to extend the existence of quadratic variation from one-dimensional to multi-dimensional typical price paths, we consider now the sample space $\Omega_\psi \subseteq D([0,T],\mathbb{R}^d)$ and introduce a d-dimensional version of the Lebesgue partitions for $d \in \mathbb{N}$.

Definition 3.35. *For $n \in \mathbb{N}$ and a d-dimensional càdlàg function $\omega\colon [0,T] \to \mathbb{R}^d$ its* Lebesgue partition $\pi_n(\omega) := \{\tau_k^n(\omega) : k \geq 0\}$ *is iteratively defined by $\tau_0^n(\omega) := 0$ and*

$$\tau_k^n(\omega) := \min\left\{\tau > \tau_{k-1}^n(\omega) : \tau \in \bigcup_{i=1}^{d} \pi_n(\omega^i) \cup \bigcup_{i,j=1, i\neq j}^{d} \pi_n(\omega^i + \omega^j)\right\}, \quad k \in \mathbb{N},$$

where $\omega = (\omega^1, \dots, \omega^d)$ and $\pi_n(\omega^i)$ and $\pi_n(\omega^i + \omega^j)$ are the Lebesgue partitions of ω^i and $\omega^i + \omega^j$, respectively.

To state the existence of quadratic variation for typical price paths in Ω_ψ, we define the canonical projection on Ω_ψ by $S_t^i(\omega) := \omega^i(t)$ for $\omega = (\omega^1, \dots, \omega^d) \in \Omega_\psi$, $t \in [0,T]$ and $i = 1, \dots, d$.

Corollary 3.36. *Let $d \in \mathbb{N}$ and $1 \leq i, j \leq d$. For typical price paths $\omega \in \Omega_\psi$ the discrete quadratic variation*

$$\left[S^i, S^j\right]_t^n(\omega) := \sum_{k=1}^{\infty} \left(S^i_{\tau_k^n \wedge t}(\omega) - S^i_{\tau_{k-1}^n \wedge t}(\omega)\right)\left(S^j_{\tau_k^n \wedge t}(\omega) - S^j_{\tau_{k-1}^n \wedge t}(\omega)\right), \quad t \in [0,T],$$

converges along the Lebesgue partitions $(\pi_n(\omega))_{n\in\mathbb{N}}$ in the uniform metric to a function $[S^i, S^j](\omega) \in D([0,T],\mathbb{R})$.

Proof. To show the convergence of $\left[S^i, S^j\right]_\cdot^n(\omega)$ for a path $\omega \in \Omega_\psi$, we observe that

$$S^i_{s,t}(\omega)S^j_{s,t}(\omega) = \frac{1}{2}\left(\left((S^i_t(\omega) + S^j_t(\omega)) - (S^i_s(\omega) + S^j_s(\omega))\right)^2 - (S^i_{s,t}(\omega))^2 - (S^j_{s,t}(\omega))^2\right)$$

for $s,t \in [0,T]$ and thus it is sufficient to prove the existence of the quadratic variation of $S^i(\omega)$ and $S^i(\omega)+S^j(\omega)$ for $1 \leq i,j \leq d$ with $i \neq j$. For typical price paths this can be done precisely as in the proof of Theorem 3.25 with the only exception that the bounds (a)-(b) and (1)-(4) change by a multiplicative constant depending only on the dimension d. □

3.3 Föllmer's Pathwise Integral and Model-Free Quadratic Variation

In this section we present Föllmer's pathwise integral for càdlàg paths. We consider a sequence of partitions, $\pi = (\pi)^n, n = 1,2,\cdots,$ of the interval $[0,T]$,

$$\pi^n := \{0, = t_0^n < t_1^n < t_2^n < \cdots < t_{N_n}^n = T\}$$

such that the *mesh*

$$\max_{i=1,2,\cdots,N_n} (t_i^n - t_{i-1}^n)$$

tends to zero as $n \to \infty$.

Definition 3.37. *Let X be a locally compact Hausdorff space. A measure ξ is called a radon measure if it is a Borel measure satisfying the following*

1. *$\xi(K) < \infty$ for all compact $K \subset X$,*
2. *$\xi(V) = \sup\{\xi(K) : K \subset V$ compact$\}$ for all open $V \subset X$,*
3. *$\xi(B) = \inf\{\xi(V) : B \subset V, V \subset X$ open$\}$ for all Borel sets B.*

Let now $x : [0,T] \to \mathbb{R}$ be a càdlàg function and assume that the sequence of measures

$$\sum_{i=1}^{N_n} \left(x_{t_i^n} - x_{t_{i-1}^n}\right)^2 \delta_{t_{i-1}^n}$$

tends vaguely to Radon measure ξ on $[0,T]$ such that for any $t \in [0,T]$, $\xi(\{t\}) = (\Delta x_t)^2$. That is, for any continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$,

$$\int_{[0,T]} f d\xi_n \to \int_{[0,T]} f d\xi, \quad n \to \infty \tag{3.3.1}$$

where $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of Radon measures.

Definition 3.38. *Let π and x be as above. We say that x possesses a quadratic variation along the sequence of partition π and this quadratic variation is defined as*

$$[x]^{\pi}_t := \xi[0,t]. \tag{3.3.2}$$

By the assumption that for any $t \in [0,T]$, $\xi(\{t\}) = (\Delta x_t)^2$, the following decomposition is justified

$$[x]^{\pi}_t := [x]^{\pi,\text{cont.}}_t + \sum_{0<s\leq t} (\Delta x_s)^2 \tag{3.3.3}$$

where $[x]^{\pi,\text{cont.}}$ will be called the continuous part of $[x]^{\pi}$.

Theorem 7. *[17] If $x : [0,T] \to \mathbb{R}$ is càdlàg, possesses the quadratic variation along the sequence of partitions π and $f : \mathbb{R} \to \mathbb{R}$ is of the class C^2, then*

$$f(x_T) = f(x_0) + \int_{0+}^{T} f'(x_{s-})dx_s + \frac{1}{2}\int_{0+}^{T} f''(x_{s-})d[x]^{\pi,cont.}_s + \sum_{0<s\leq T} \{\Delta f(x_s) - f'(x_{s-})\Delta x_s\},$$

where the integral $\int_{0+}^{T} f'(x_{s-})dx_s$ is defined as the limit

$$\int_{0+}^{T} f'(x_{s-})dx_s = \lim_{n\to+\infty} \sum_{i=1}^{N_n} f'\left(x_{t^n_{i-1}}\right)\left(x(t^n_i) - x(t^n_{i-1})\right). \tag{3.3.4}$$

The following result appear in our paper [20].

Theorem 8. *Let $\omega, \tilde{\omega} : [0,T] \to \mathbb{R}$ be two càdlàg paths. Assume that $\tilde{\omega}$ has finite total variation and let us consider two integrals*

1. *the integral $\int_{(0,T]} \omega(t-)\,\mathrm{d}\tilde{\omega}(t)$ understood as the Lebesgue-Stieltjes integral (with respect to the measure $\mathrm{d}\tilde{\omega}$ given by $\mathrm{d}\tilde{\omega}(a,b] := \tilde{\omega}(b) - \tilde{\omega}(a)$);*

2. *the integral $(F)\int_{(0,T]} \omega(t-)\,\mathrm{d}\tilde{\omega}(t)$ understood as Föllmer's integral along the sequence of partitions $(\tau^n)_{n\in\mathbb{N}}$ such that for $n \in \mathbb{N}$, τ^n contains nth Lebesgue partition of the path ω, i.e.*

$$(F)\int_{(0,T]} \omega(t-)\,\mathrm{d}\tilde{\omega}(t) := \lim_{n\to\infty} \sum_{i=1}^{k_n} \omega\left(\tau^n_{i-1}\right)\left\{\tilde{\omega}\left(\tau^n_i\right) - \tilde{\omega}\left(\tau^n_{i-1}\right)\right\},$$

where $\tau^n := \left\{0 = \tau^n_0 < \tau^n_1 < \ldots < \tau^n_{k_n-1} < T = \tau^n_{k_n} < +\infty = \tau^n_{k_n+1} = \tau^n_{k_n+2} = \ldots\right\}$ and $\pi^n(\omega) \subset \tau^n$.

These two integrals coincide, provided that the latter exists.

Proof. Step 1. First, let us notice that for any $\varepsilon > 0$ we may uniformly approximate ω by a step function

$$\omega^{\varepsilon}(t) := \sum_{i=1}^{N} \omega(t_{i-1}) \mathbb{1}_{[t_{i-1},t_i)}(t) + \omega(T) \mathbb{1}_{\{T\}}(t),$$

where $0 = t_0 < t_1 < \ldots < t_N = T$, such that

$$\left\|\omega^{\varepsilon} - \omega\right\|_{\infty} \leq \varepsilon.$$

To obtain such ω^{ε} we simply define $t_0 := 0, t_i := \inf\left\{t > t_{i-1} : \left|\omega(t) - \omega(t_{i-1})\right| > \varepsilon\right\}$ for $i = 1, 2, \ldots$ such that $t_{i-1} < +\infty$ (we apply the convention that $\inf \emptyset = +\infty$) and $N := \max\left\{i \in \{1, 2, \ldots\} : t_{i-1} < T\right\}$.

Step 2. We have the estimate

$$\begin{aligned} &\left|\int_{(0,T]} \omega(t-) \mathrm{d}\tilde{\omega}(t) - \int_{(0,T]} \omega^{\varepsilon}(t-) \mathrm{d}\tilde{\omega}(t)\right| \\ &\leq \int_{(0,T]} \left|\omega(t-) - \omega^{\varepsilon}(t-)\right| \left|\mathrm{d}\tilde{\omega}(t)\right| \\ &\leq \varepsilon \int_{(0,T]} \left|\mathrm{d}\tilde{\omega}(t)\right| = \varepsilon \mathrm{TV}\left(\tilde{\omega}, [0,T]\right), \end{aligned} \tag{3.3.5}$$

where $\mathrm{TV}\left(\tilde{\omega}, [0,T]\right)$ denotes the total variation of $\tilde{\omega}$. Moreover

$$\int_{(0,T]} \omega^{\varepsilon}(t-) \mathrm{d}\tilde{\omega}(t) = \sum_{i=1}^{N} \omega^{\varepsilon}(t_{i-1}) \left(\tilde{\omega}(t_i) - \tilde{\omega}(t_{i-1})\right).$$

We also have

$$\begin{aligned} &\left|\sum_{i=1}^{k_n} \omega\left(\tau_{i-1}^n\right) \left\{\tilde{\omega}\left(\tau_i^n\right) - \tilde{\omega}\left(\tau_{i-1}^n\right)\right\} - \sum_{i=1}^{k_n} \omega^{\varepsilon}\left(\tau_{i-1}^n\right) \left\{\tilde{\omega}\left(\tau_i^n\right) - \tilde{\omega}\left(\tau_{i-1}^n\right)\right\}\right| \\ &\leq \sum_{i=1}^{k_n} \left|\omega\left(\tau_{i-1}^n\right) - \omega^{\varepsilon}\left(\tau_{i-1}^n\right)\right| \left|\tilde{\omega}\left(\tau_i^n\right) - \tilde{\omega}\left(\tau_{i-1}^n\right)\right| \\ &\leq \varepsilon \mathrm{TV}\left(\tilde{\omega}, [0,T]\right). \end{aligned} \tag{3.3.6}$$

Step 3. Now let us consider the difference

$$\begin{aligned} &\int_{(0,T]} \omega^{\varepsilon}(t-) \mathrm{d}\tilde{\omega}(t) - \sum_{i=1}^{k_n} \omega^{\varepsilon}\left(\tau_{i-1}^n\right) \left\{\tilde{\omega}\left(\tau_i^n\right) - \tilde{\omega}\left(\tau_{i-1}^n\right)\right\} \\ &= \sum_{i=1}^{N} \omega^{\varepsilon}(t_{i-1}) \left(\tilde{\omega}(t_i) - \tilde{\omega}(t_{i-1})\right) - \sum_{i=1}^{k_n} \omega^{\varepsilon}\left(\tau_{i-1}^n\right) \left\{\tilde{\omega}\left(\tau_i^n\right) - \tilde{\omega}\left(\tau_{i-1}^n\right)\right\}. \end{aligned} \tag{3.3.7}$$

Let $\tau^n(t)$ denotes the first point τ_i^n in the partition τ^n such that $\tau_i^n \geq t$. From the definition of the nth Lebesgue partition of ω it follows that for $t < T$

$$\limsup_{n\to+\infty} \tau^n(t) \leq \inf\{u > t : \omega(u) \neq \omega(t-)\}.$$

By the definition of times $t_1, t_2, \ldots, t_{N-1}$ we have that for any $t \in \{t_1, t_2, \ldots, t_{N-1}\}$, $\omega(t) \neq \omega(t-)$ or $\omega(t) = \omega(t-)$ but ω is not constant on any interval of the form $[t, u]$, $u \in (t, T]$ and $\lim_{n\to+\infty} \pi^n(t) = t$. Thus, for sufficiently large ns

$$t_i \leq \tau^n(t_i) < t_{i+1} \text{ for } i = 1, 2, \ldots, N-1.$$

Now, denoting by $k_n(t)$ such index that $\tau^n(t) = \tau^n_{k_n(t)}$, for sufficiently large ns and $i = 2, \ldots, N$ we have $t_{i-2} \leq \tau^n_{k_n(t_{i-1})-1} < t_{i-1}$. Thus $\omega^\varepsilon\left(\tau^n_{k_n(t_{i-1})-1}\right) = \omega^\varepsilon(t_{i-2})$ and since ω^ε is constant on $[t_{i-1}, t_i)$, $i = 1, 2, \ldots, N$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \omega^\varepsilon(t_{i-1})\left(\tilde{\omega}(t_i) - \tilde{\omega}(t_{i-1})\right) - \sum_{i=1}^{k_n} \omega^\varepsilon\left(\tau^n_{i-1}\right)\left\{\tilde{\omega}\left(\tau^n_i\right) - \tilde{\omega}\left(\tau^n_{i-1}\right)\right\} \\
&= \sum_{i=2}^{N} \left(\omega^\varepsilon(t_{i-1}) - \omega^\varepsilon(t_{i-2})\right)\left(\tilde{\omega}\left(\tau^n(t_{i-1})\right) - \tilde{\omega}(t_{i-1})\right). \qquad (3.3.8)
\end{aligned}$$

Since $\lim_{n\to+\infty} \tau^n(t) = t$ for $t \in \{t_1, t_2, \ldots, t_{N-1}\}$ and $\tilde{\omega}$ is càdlàg, we finally get that the difference (3.3.7) tends to 0 as $n \to +\infty$.

From this and (3.3.5), (3.3.6) (taking ε as close to 0 as we wish) we get the assertion. □

The multi-dimensional Föllmer's integration by parts formula is given by:

$$\begin{aligned}
S^i_T(\omega)S^j_T(\omega) - S^i_0(\omega)S^j_0(\omega) &= \int_{(0,T]} S^i_{t-}(\omega)dS^j_t(\omega) \\
&\quad + \int_{(0,T]} S^j_{t-}(\omega)dS^i_t(\omega) + [S^i, S^j](\omega), \qquad (3.3.9)
\end{aligned}$$

where $f(x_1, \cdots, x_d) = x_i x_j, \quad i, j \in \{1, 2, \cdots\}$ and the integral

$$\int_{(0,T]} S^i_{t-}(\omega)dS^j_t(\omega)$$

denotes Föllmer's pathwise integral, see [17]. The corresponding model-free version is given by:

$$\begin{aligned}
\omega^i(t)\,\omega^j(t) - \omega^i(0)\,\omega^j(0) &= (F)\int_{(0,t]} \omega^i(s-)\,\mathrm{d}\omega^j(s) + (F)\int_{(0,t]} \omega^j(s-)\,\mathrm{d}\omega^i(s) \\
&\quad + [S^i, S^j]_t(\omega), \qquad (3.3.10)
\end{aligned}$$

where $(F)\int_{(0,t]}\dots$ denotes Föllmer's pathwise integral (along the sequence of the Lebesgue partitions), $t\in[0,T]$ and $\omega\in D\left([0,T],\mathbb{R}^d\right)$ possesses quadratic variation along the same partitions. This means that the sequence of discrete quadratic (co)variation, see (3.2.9), converges for $i,j=1,2,\cdots,d$ in the uniform topology $[S^i,S^j](\omega)$.

Notice also that for typical price path ω, Föllmer's integral in (3.3.10) coincides with the model-free Itô integral $\int_{(0,t]} S^i_{s-}(\omega)\,\mathrm{d}S^j_s(\omega)$. The latter may be expressed as the limit of the sums of the form

$$\sum_{k=1}^{\infty} S^i_{\pi^n_{k-1}}(\omega)S^j_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\omega)$$

and $\sum_{k=1}^{\infty} S^i_{\pi^n_{k-1}}(\omega)\mathbb{1}_{(\pi^n_{k-1},\pi^n_k]}(t)$ converges uniformly to $S^i_{t-}(\omega)$ for $t\in(0,T]$. Using Theorem 3.23, in particular, continuity results (3.2.12), the distance $d^{\varepsilon}_{\infty,\psi}$ between Föllmer's integral and the model-free Itô integral is 0. This implies that for typical price paths the two integrals coincide. Thus (3.3.10) may be also viewed as the integration by parts formula for the model-free Itô integral.

Next we prove the multi-dimentional model-free Itò fourmula for càdlàg price paths.

Lemma 3.39. *Let $\tilde{S}:\Omega\times[0,T]\to\mathbb{R}^d$ be such that the process $\tilde{S}$ is adapted (to the filtration $(\mathscr{F}_t)_{t\in[0,T]}$), and for any $\omega\in\Omega$, $\tilde{\omega}:=\tilde{S}(\omega)$ is finite variation, càdlàg path on $[0,T]$. Then for typical path $\omega\in\Omega$ and $i,j=1,2,\dots,d$, $t\in[0,T]$, the following integration by parts formula holds*

$$\omega^i(t)\,\tilde{\omega}^j(t)-\omega^i(0)\,\tilde{\omega}^j(0)=\int_{(0,t]}\tilde{S}^j_{s-}(\omega)\,\mathrm{d}S^i_s(\omega)+\int_{(0,t]}\omega^i(s-)\,\mathrm{d}\tilde{\omega}^j(s) + \sum_{0<s\le t}\Delta\omega^i(s)\,\Delta\tilde{\omega}^j(s)\,, \tag{3.3.11}$$

where $\int_{(0,t]}\tilde{S}^j_{s-}(\omega)\,\mathrm{d}S^i_s(\omega)$ denotes the model-free Itô integral and $\int_{(0,t]}\omega^i(s-)\,\mathrm{d}\tilde{\omega}^j(s)$ denotes the Lebesgue-Stieltjes integral which coincides with the Föllmer integral.

Proof. Consider Föllmer's pathwise integration by parts formula (3.3.10). Using the fact that for typical price paths, model-free Itô integral

$$\int_{(0,t]}\tilde{S}^j_{s-}(\omega)\,\mathrm{d}S^i_s(\omega)$$

coincides with Föllmer's integral (along the sequence of the Lebesgue partitions $(\pi^n)_{n\in\mathbb{N}}$ of $(\omega,\tilde{\omega})\in\mathbb{R}^{2d}$) and that since (by Lemma 8), Föllmer's integral

$$(F)\int_{(0,t]}\omega^i(s-)\,\mathrm{d}\tilde{\omega}^j(s)$$

(along the same sequence of partitions) coincides with the classical Lebesgue-Stieltjes integral

$$\int_{(0,t]} \omega^i(s-)\,\mathrm{d}\tilde{\omega}^j(s),$$

to prove this Lemma, we need only to show that for $i, j = 1, 2, \ldots, d$, $t \in [0, T]$, the sequence of discrete quadratic (co)variation

$$\sum_{k=1}^{\infty} S^i_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\omega) S^j_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\tilde{\omega}) = \sum_{k=1}^{\infty} S^i_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\omega) \tilde{S}^j_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\omega)$$

converges to

$$\sum_{0<s\le t} \Delta\omega^i(s)\,\Delta\tilde{\omega}^j(s)$$

uniformly in t. The prove of this Lemma relies on the properties of Lebesgue partitions and the Schwartz inequality. Let $\varepsilon > 0$ be such that there is no jump of $\tilde{\omega}$ of size equal ε and let $I^{\varepsilon,n}$, $n \in \mathbb{N}$, be the sequence of all indices $k \in \mathbb{N}$ for which $\left|S^j_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\tilde{\omega})\right| > \varepsilon$. We have

$$\begin{aligned}
&\left|\sum_{k=1}^{\infty} S^i_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\omega) S^j_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\tilde{\omega}) - \sum_{k\in I^{\varepsilon,n}} S^i_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\omega) S^j_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\tilde{\omega})\right| \\
&= \left|\sum_{k\in\mathbb{N}\setminus I^{\varepsilon,n}} S^i_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\omega) S^j_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge T}(\tilde{\omega})\right| \\
&\le \sqrt{\sum_{k\in\mathbb{N}\setminus I^{\varepsilon,n}} S^i_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\omega)^2} \sqrt{\sum_{k\in\mathbb{N}\setminus I^{\varepsilon,n}} S^j_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\tilde{\omega})^2} \\
&\le \sqrt{\varepsilon} \sqrt{\sum_{k=1}^{\infty} \left|S^j_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\tilde{\omega})\right|} \sqrt{\sum_{k=1}^{\infty} S^i_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\omega)^2}.
\end{aligned}$$

Notice that $\sum_{k=1}^{\infty} \left|S^j_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\tilde{\omega})\right|$ is bounded by the total variation of $\tilde{\omega}$ while

$$\left(\sum_{k=1}^{\infty} S^i_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\omega)^2\right)_{t\in[0,T]}$$

converges to the quadratic variation of ω^i as $n \to +\infty$ (the convergence still holds though partitions $(\pi^n)_{n\in\mathbb{N}}$ may be finer than the Lebesgue partitions of ω, see the proofs of [31, Corollary 3.11], [47, Theorem 2]). Finally notice that

$$\lim_{n\to\infty} \sum_{k\in I^{\varepsilon,n}} S^i_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\omega) S^j_{\pi^n_{k-1}\wedge t,\pi^n_k\wedge t}(\tilde{\omega}) = \sum_{0<s\le t, \left|\Delta\tilde{\omega}(s)\right|>\varepsilon} \Delta\omega^i(s)\,\Delta\tilde{\omega}^j(s).$$

(recall that there is no jump of $\tilde{\omega}$ of size equal ε). Since there is only finite number of jumps of $\tilde{\omega}$ of size greater than ε and $\varepsilon > 0$ may be as close to 0 as we wish (since $\tilde{\omega}$ has only countable number of jumps), the result follows. □

3.4 Quadratic Variation along the Partitions Obtained from Stopping Times

Let $\tau = \{0 = \tau_0 \le \tau_1 \le \ldots\}$ be a (possibly random) partition of the interval $[0,T]$, by which we will mean that τ_i, $i = 0,1,\ldots,$ is infinite, non-decreasing sequence of elements of $[0,T] \cup \{+\infty\}$ such that from some $i \in \{1,2,\ldots\}$ on, $\tau_i = \tau_{i+1} = \ldots = +\infty$. To avoid redundancies, we may also assume (as in [13], though this will not change the reasoning) that for all $i = 0,1,\ldots,$ $\tau_i < \tau_{i+1}$ whenever $\tau_i < +\infty$. Similarly as Davis, Obłój and Siorpaes in [13], for any $\omega \in \Omega$ we will denote

$$O_T(\omega,\tau) := \sup\left\{\left|\omega(t) - \omega(s)\right| : s,t \in [\tau_{i-1},\tau_i) \cap [0,T], i \in \{1,2,\ldots\}\right\}.$$

Moreover, similarly to (3.2.9), for $\omega \in \Omega$, $i,j = 1,2,\ldots,d$ and $t \in [0,T]$ let us define

$$\left[S^i,S^j\right]_t^\tau(\omega) := \sum_{k=1}^{\infty} S^i_{\tau_{k-1}\wedge t,\tau_k\wedge t}(\omega) S^j_{\tau_{k-1}\wedge t,\tau_k\wedge t}(\omega). \tag{3.4.1}$$

Definition 3.40. *We will say that the partition $\tau = \{0 = \tau_0 \le \tau_1 \le \ldots\}$ is an* optional *one (with respect to a given filtration) if all $\tau_1 \le \tau_2 \le \ldots$ are stopping times (with respect to this filtration).*

Let us recall also the definition of the convergence in outer measure on a given set (Definition 3.21). Now we are ready to state

Theorem 3.41. *Let τ^n, $n = 1,2\ldots,$ be a sequence of optional partitions (with respect to the filtration $(\mathscr{F}_t)_{t\in[0,T]}$) of the interval $[0,T]$ such that for all $q,M > 0$, $O_T(\cdot,\tau^n)$ converges in outer measure to 0 (as $n \to 0$) on $\Omega_{q,M}$. Then for any $q,M > 0$, $i,j = 1,2,\ldots,d$, the sequence $\left(\left[S^i,S^j\right]^{\tau^n}\right)_{n\in\mathbb{N}}$ converges in outer measure $\overline{\mathbb{P}}$ on $\Omega_{q,M}$ to $\left[S^i,S^j\right]$, i.e. the quadratic (co)variation obtained along the sequence of the Lebesgue partitions.*

Proof. First we will prove the thesis for $i = j \in \{1,2,\ldots,d\}$. Let $\tau^n = \left\{0 = \tau^n_0 \le \tau^n_1 \le \ldots\right\}$ and take

$$F^n_t(\omega) := \sum_{i=1}^{+\infty} S^i_{\tau^n_{i-1}}(\omega)\, \mathbb{1}_{[\tau^n_{i-1},\tau^n_i)}(t).$$

By the definition of $O_T(\omega,\tau^n)$ we have

$$\left\|F^n(\omega) - S^i(\omega)\right\|_\infty \le O_T(\omega,\tau^n).$$

Now, using the integration by parts formula (3.3.10) for any $t \in [0,T]$ we calculate

$$\left[S^i,S^j\right]_t^{\tau^n}(\omega) - \left[S^i,S^i\right]_t(\omega) = 2\int_{(0,t]} \left(S^i_{s-}(\omega) - F^n_{s-}(\omega)\right) \mathrm{d}S^i_s(\omega), \tag{3.4.2}$$

where $\int_{(0,t]} \left(S^i_{s-}(\omega) - F^n_{s-}(\omega)\right) \mathrm{d}S^i_s(\omega)$ denotes the model-free Itô integral. (Note that this integral is well defined since the partition τ^n is optional one.) Now, fixing $\varepsilon, \delta > 0$, and applying Corollary 3.24 we get for some constant $C_{q,M}$ depending on q and M (and ψ) only

$$
\begin{aligned}
&\overline{\mathbb{P}}\left(\omega \in \Omega_{q,M} : \left\| \left[S^i, S^j\right]^{\tau^n}_t (\omega) - \left[S^i, S^i\right]_t (\omega) \right\|_\infty > \varepsilon \right) \\
&= \overline{\mathbb{P}}\left(\omega \in \Omega_{q,M} : \left\| \left(\int_{(0,t]} \left(S^i_{s-}(\omega) - F^n_{s-}(\omega)\right) \mathrm{d}S^i_s(\omega) \right)_{t\in[0,T]} \right\|_\infty > \varepsilon/2 \right) \\
&\leq \overline{\mathbb{P}}\left(\omega \in \Omega_{q,M} : O_T(\omega, \tau^n) > \delta\right) \\
&\quad + \overline{\mathbb{P}}\left(\omega \in \Omega_{q,M} : O_T(\omega, \tau^n) \leq \delta, \left\| \left(\int_{(0,t]} \left(S^i_{s-}(\omega) - F^n_{s-}(\omega)\right) \mathrm{d}S^i_s(\omega) \right)_{t\in[0,T]} \right\|_\infty > \varepsilon/2 \right) \\
&\leq \overline{\mathbb{P}}\left(\omega \in \Omega_{q,M} : O_T(\omega, \tau^n) > \delta\right) + C_{q,M}\frac{\delta}{\varepsilon}.
\end{aligned}
$$

Since $\overline{\mathbb{P}}\left(\omega \in \Omega_{q,M} : O_T(\omega, \tau^n) > \delta\right) \to 0$ as $n \to +\infty$ and δ/ε may be chosen arbitrary close to 0, we get the convergence result.

To obtain the convergence for $i, j \in \{1, 2, \ldots, d\}$, $i \neq j$, we will use polarization. More precisely, we define

$$
H^n_t(\omega) := \sum_{i=1}^{+\infty} \left(S^i_{\tau^n_{i-1}}(\omega) + S^j_{\tau^n_{i-1}}(\omega)\right) \mathbb{1}_{[\tau^n_{i-1}, \tau^n_i)}(t),
$$

and using integration by parts we obtain that the process

$$
R^{i,j,\tau^n}_t(\omega) := \sum_{k=1}^{\infty} \left(S^i_{\tau^n_{k-1}\wedge t, \tau^n_k \wedge t}(\omega) + S^j_{\tau^n_{k-1}\wedge t, \tau^n_k \wedge t}(\omega)\right)^2
$$

converges in outer measure $\overline{\mathbb{P}}$ on $\Omega_{q,M}$ to $\left[S^i + S^j, S^i + S^j\right] = \left[S^i, S^i\right] + 2\left[S^i, S^j\right] + \left[S^j, S^j\right]$ since

$$
R^{i,j,\tau^n}_t(\omega) - \left[S^i + S^j, S^i + S^j\right]_t(\omega) = 2\int_{(0,t]} \left(S^i_{s-}(\omega) + S^j_{s-}(\omega) - H^n_{s-}(\omega)\right) \mathrm{d}\left(S^i_s(\omega) + S^j_s(\omega)\right).
$$

Similarly, one proves that the process

$$
T^{i,j,\tau^n}_t(\omega) = \sum_{k=1}^{\infty} \left(S^i_{\tau^n_{k-1}\wedge t, \tau^n_k \wedge t}(\omega) - S^j_{\tau^n_{k-1}\wedge t, \tau^n_k \wedge t}(\omega)\right)^2
$$

converges in outer measure $\overline{\mathbb{P}}$ on $\Omega_{q,M}$ to $\left[S^i - S^j, S^i - S^j\right] = \left[S^i, S^i\right] - 2\left[S^i, S^j\right] + \left[S^j, S^j\right]$. Thus we obtain that the difference of the processes R^{i,j,τ^n} and T^{i,j,τ^n} which is equal $4\left[S^i, S^j\right]^{\tau^n}$ converges in outer measure $\overline{\mathbb{P}}$ on $\Omega_{q,M}$ to $4\left[S^i, S^j\right]$. □

When we assume some stronger mode of convergence of $O_T(\cdot,\tau^n)$ then, naturally, we may expect to obtain a stronger mode of convergence of $\left[S^i,S^j\right]^{\tau^n}$. Let us recall the definition of distances d_∞ and $d^{\varepsilon}_{\infty,\psi}$ (defined in (3.2.10) and (3.2.11)). We have the following result.

Theorem 3.42. *Let τ^n, $n=1,2\ldots$, be a sequence of optional partitions of the interval $[0,T]$, $i\in\{1,2,\ldots,d\}$ and*

$$F^n_t(\omega) := \sum_{i=1}^{+\infty} S^i_{\tau^n_{i-1}}(\omega)\,\mathbb{1}_{\left[\tau^n_{i-1},\tau^n_i\right)}(t).$$

Assume that $\lim_{n\to+\infty} d_\infty\left(F^n,S^i\right)=0$ then for any $\varepsilon\in(0,1)$,

$$d^{\varepsilon}_{\infty,\psi}\left(\left[S^i,S^j\right]^{\tau^n},\left[S^i,S^i\right]\right)\to 0 \quad \text{as } n\to 0. \tag{3.4.3}$$

Similarly, for $i,j\in\{1,2,\ldots,d\}$ define

$$H^n_t(\omega) := \sum_{i=1}^{+\infty}\left(S^i_{\tau^n_{i-1}}(\omega)+S^j_{\tau^n_{i-1}}(\omega)\right)\mathbb{1}_{\left[\tau^n_{i-1},\tau^n_i\right)}(t)$$

and

$$J^n_t(\omega) := \sum_{i=1}^{+\infty}\left(S^i_{\tau^n_{i-1}}(\omega)-S^j_{\tau^n_{i-1}}(\omega)\right)\mathbb{1}_{\left[\tau^n_{i-1},\tau^n_i\right)}(t).$$

Assume that $\lim_{n\to+\infty} d_\infty\left(H^n,S^i+S^j\right)=0$ and $\lim_{n\to+\infty} d_\infty\left(J^n,S^i-S^j\right)=0$ then for any $\varepsilon\in(0,1)$,

$$d^{\varepsilon}_{\infty,\psi}\left(\left[S^i,S^j\right]^{\tau^n},\left[S^i,S^j\right]\right)\to 0 \quad \text{as } n\to 0. \tag{3.4.4}$$

Proof. Convergence (3.4.3) follows immediately from (3.4.2) and Theorem 3.23 applied to $F_t=\left(0,0,\ldots,F^n_{t-},\ldots,0\right)$ (ith component of F_t is equal F^n_{t-} and all other components are equal 0), and $G_t=\left(0,0,\ldots,S^i_{t-},\ldots,0\right)$ (ith component of G_t is equal S^i_{t-} and all other components are equal 0). Convergence (3.4.4) follows in a similar way from polarization, integration by parts and Theorem 3.23. □

3.5 Quadratic Variation Expressed in Terms of the Truncated Variation

In this section we present results about quadratic variation expressed in terms of truncated variations. Moreover, we will relax the dependency on choice of partition for the definition of quadratic variation. In particular, we obtain partition-independent formula for the continuous part of quadratic variation denoted by $[\omega]^{cont}_T$, where

$$[\omega]_T^{cont} = [\omega]_T - \sum_{0<s\le T} (\Delta\omega(t))^2.$$

The truncated variation of a càdlàg path $\omega : [0,T] \to \mathbb{R}$ is defined as

$$TV^c(\omega,[0,T]) := \sup_n \sup_{0\le t_0<t_1<\ldots<t_n\le T} \sum_{i=1}^n \max\left\{\left|\omega(t_i) - \omega(t_{i-1})\right| - c, 0\right\}.$$

Theorem 9. *Let $q, M > 0$. For typical càdlàg non-negative price path or càdlàg price path with mildly restricted jumps the following convergence holds*

$$c \cdot TV^c(\omega,[0,T]) \to^{\bar{\mathbb{P}}}_{c\to 0} [\omega]_T^{cont},$$

where $[\omega]_T^{cont}$ denotes the quadratic variation defined along the sequence of Lebesgue partitions and $\to^{\bar{\mathbb{P}}}_{c\to 0} [\omega]_T^{cont}$ denotes the convergence in the outer measure $\bar{\mathbb{P}}$ as $c \to 0_+$ on the set $\Omega_{q,M}$.

Proof. Using construction in [27, Sect. 2] or in [28, p. 11] we know that for any $c > 0$ there exists a càdlàg path $\omega^c : [0,T] \to \mathbb{R}$ such that

1. ω^c has finite total variation;
2. $\omega^c(0) = \omega(0)$;
3. for every $t \in [0,T]$, $\left|\omega(t) - \omega^c(t)\right| \le c$;
4. for every $t \in [0,T]$, $\left|\Delta\omega^c(t)\right| := \left|\omega^c(t) - \omega^c(t-)\right| \le \left|\Delta\omega(t)\right| := \left|\omega(t) - \omega(t-)\right|$.

Moreover (see [27, Lemma 5.1] and [28, p. 11]), we have

$$\mathrm{TV}^{2c}\left(\omega,[0,T]\right) \le \mathrm{TV}\left(\omega^c,[0,T]\right) \le \mathrm{TV}^{2c}\left(\omega,[0,T]\right) + 2c \tag{3.5.1}$$

and

$$c \cdot \mathrm{TV}\left(\omega^c,[0,T]\right) = \int_0^T (\omega - \omega^c)\,\mathrm{d}\omega^c, \tag{3.5.2}$$

where $\int_0^T (\omega - \omega^c)\,\mathrm{d}\omega^c$ denotes the standard Riemann-Stieltjes integral (recall that ω^c has finite total variation).

Now, we calculate

$$\begin{aligned}\int_0^T (\omega - \omega^c)\,\mathrm{d}\omega^c &= \int_{(0,T]} \Big(\omega(t-) - \omega^c(t-) + \Delta\left(\omega(t) - \omega^c(t)\right)\Big)\,\mathrm{d}\omega^c(t) \\ &= \int_{(0,T]} \omega(t-)\,\mathrm{d}\omega^c(t) - \int_{(0,T]} \omega^c(t-)\,\mathrm{d}\omega^c(t) \\ &\quad + \sum_{0<s\le T} \Delta\left(\omega(t) - \omega^c(t)\right)\Delta\omega(t). \end{aligned} \tag{3.5.3}$$

By the change of variable formula, see Lemma (3.39), we have

$$\int_{(0,T]} \omega(t-)\,\mathrm{d}\omega^c(t) = \omega^c(T)\,\omega(T) - \omega^c(0)\,\omega(0) - \int_{(0,T]} \omega^c(t-)\,\mathrm{d}\omega(t) - \sum_{0<s\le T} \Delta\omega^c(t)\,\Delta\omega(t). \quad (3.5.4)$$

Now, by the continuity result for the integral $\int_{(0,T]} \omega^c(t-)\,\mathrm{d}\omega(t)$ (see [31, Corollary 4.6 and Corollary 4.8]), the fact that $|\omega - \omega^c| \le c$ and (3.5.4) we get

$$\int_{(0,T]} \omega(t-)\,\mathrm{d}\omega^c(t) \to_{c\to 0+} (\omega(T))^2 - (\omega(0))^2 - \int_{(0,T]} \omega(t-)\,\mathrm{d}\omega(t) - \sum_{0<s\le T} (\Delta\omega(t))^2.$$

Next (recall that $\omega^c(t)$ has finite total variation, thus the Riemann-Stieltjes integral rules apply and $\int_{[0,T]} \omega^c(t)\,\mathrm{d}\omega^c(t) = \frac{1}{2}\left((\omega^c(T))^2 - (\omega^c(0))^2\right)$)

$$\begin{aligned}\int_{(0,T]} \omega^c(t-)\,\mathrm{d}\omega^c(t) &= \int_{[0,T]} \omega^c(t)\,\mathrm{d}\omega^c(t) - \frac{1}{2}\sum_{0<s\le T} (\Delta\omega^c(t))^2 \\ &= \frac{1}{2}\left((\omega^c(T))^2 - (\omega^c(0))^2\right) - \frac{1}{2}\sum_{0<s\le T} (\Delta\omega^c(t))^2 \\ &\to_{c\to 0+} \frac{1}{2}\left((\omega(T))^2 - (\omega(0))^2\right) - \frac{1}{2}\sum_{0<s\le T} (\Delta\omega(t))^2\end{aligned}$$

and

$$\sum_{0<s\le T} \Delta\left(\omega(t) - \omega^c(t)\right)\Delta\omega(t) \to_{c\to 0+} 0,$$

this follows from the fact that $\left|\Delta\left(\omega(t) - \omega^c(t)\right)\right| \le \min\left\{\left|\Delta\omega(t)\right| + \left|\Delta\omega^c(t)\right|, 2c\right\} \le 2\min\left\{\left|\Delta\omega(t)\right|, c\right\}$ and the Schwarz inequality.

In the next step, from (3.5.3) and the last three convergences we get

$$\begin{aligned}\int_0^T (\omega - \omega^c)\,\mathrm{d}\omega^c \to_{c\to 0+}\ & (\omega(T))^2 - (\omega(0))^2 - \int_{(0,T]} \omega(t-)\,\mathrm{d}\omega(t) - \sum_{0<s\le T} (\Delta\omega(t))^2 \\ & - \frac{1}{2}\left((\omega(T))^2 - (\omega(0))^2\right) + \frac{1}{2}\sum_{0<s\le T} (\Delta\omega(t))^2. \quad (3.5.5)\end{aligned}$$

By the Ito formula applied to $\int_{(0,T]} \omega(t-)\,\mathrm{d}\omega(t)$ we get

$$\int_{(0,T]} \omega(t-)\,\mathrm{d}\omega(t) = \frac{1}{2}\left((\omega(T))^2 - (\omega(0))^2\right) - \frac{1}{2}[\omega]_T, \quad (3.5.6)$$

where $[\omega]_T$ denotes the quadratic variation of ω, i.e. $[\omega]_T = [\omega]_T^{cont} + \sum_{0<s\le T} \left(\Delta\omega(t)\right)^2$.

Finally, from (3.5.5) and (3.5.6) we have

$$\int_0^T (\omega - \omega^c)\,\mathrm{d}\omega^c \to_{c\to 0+} \frac{1}{2}[\omega]_T^{cont}$$

and from (3.5.1) and (3.5.2) we get

$$2c\cdot \mathrm{TV}^{2c}\left(\omega,[0,T]\right) \to_{c\to 0+} [\omega]_T^{cont}.$$

□

Chapter 4

Model-Free Stochastic Differential Equations

Although the space of continuous model-free price paths is well understood, it still depicts some interesting research trend. In this Chapter we present the proof of the existence and uniqueness of the solutions of model-free stochastic differential equations (SDEs), see [19]. The proof uses the Lipschitz condition and the Burkholder-Davis-Gundy (BDG) inequality for integrals driven by model-free continuous price paths.

An alternative proof for this problem already exists in the literature, see Bartl, Kupper and Neufeld [4]. However, their approach uses Hilbert spaced-valued processes under the assumptions that one can also trade the difference

$$||S||^2 - < S >$$

and the measure $d < S >$ is majorized by the *Lebesgue measure dt* multiplied by constant. $|| \cdot ||$ denotes in this case the norm in the Hilbert space and $< S >$ denotes the quadratic variation process of the coordinate process S but defined in a different way than the usual tensor quadratic variation of a Hilbert spaced-valued semimartingale.

4.1 Settings

We consider the space of continuous price paths Ω and introduce the outer expectation $\overline{\mathbb{E}}Z$ of a process $Z : [0,T] \times \Omega \to [0,+\infty]$. The outer expectation $\overline{\mathbb{E}}Z$ may be interpreted as the superhedging cost of any value Z_τ, where $\tau \in [0,T]$ is the stopping time. We will consider the following differential equation (or rather integral) driven by continuous price paths $\omega \in \Omega$:

$$X_t(\omega) - X_0(\omega) = \int_0^t K\left(s, X(\omega), \omega\right) \mathrm{d}A_s + \int_0^t F\left(s, X(\omega), \omega\right) \mathrm{d}S_s(\omega), \quad (4.1.1)$$

where $A:[0,T]\times\Omega\to\mathbb{R}$ is a continuous, finite-variation process, $S:[0,T]\times\Omega\to\mathbb{R}^d$ is the coordinate process, $S_t(\omega)=\omega(t)$, and $(\mathscr{F}_t)_{t\in[0,T]}$ is the natural filtration of S. We will assume the following:

1. X_0 is such that the process $X=(X_t)_{t\in[0,T]}$ defined by $X_t=X_0$, $t\in[0,T]$, satisfies $X\in\mathscr{M}$ (the space $\mathscr{M}$ will be described in detail in Section 4.4);

2. $A=A^u_t-A^v_t$ and $A^u_t,A^v_t:[0,T]\times\Omega\to\mathbb{R}$ are continuous, non-decreasing, adapted processes, starting from 0 such that for all $\omega\in\Omega$, $A^u_T(\omega)+A^v_T(\omega)\le M$, where M is a deterministic constant and $A^u_t(\omega):=A^u(t,\omega)$;

3. $K:[0,T]\times(\mathbb{R}^d)^{[0,T]}\times\Omega\to\mathbb{R}^d$ and $F:[0,T]\times(\mathbb{R}^d)^{[0,T]}\times\Omega\to\mathbb{R}^d\times\mathbb{R}^d$ are non-anticipating, by which we mean that for any adapted processes $X,Y:[0,T]\times\Omega\to\mathbb{R}^d$, $K\left(t,X(\omega),\omega\right)=K\left(t,Y(\omega),\omega\right)$ and $F\left(t,X(\omega),\omega\right)=F\left(t,Y(\omega),\omega\right)$ whenever $X_s(\omega)=Y_s(\omega)$ for all $s\in[0,t]$, and the processes $K_t(\omega)=K\left(t,X(\omega),\omega\right)$, $F_t(\omega)=F\left(t,X(\omega),\omega\right)$ are adapted (see also [9, Sect. 1]);

4. $$\int_0^{\cdot}K(s,0,\omega)\,\mathrm{d}A^u_s,\ \int_0^{\cdot}K(s,0,\omega)\,\mathrm{d}A^v_s,\ \int_0^{\cdot}F(s,0,\omega)\,\mathrm{d}S_s(\omega)\in\mathscr{M};\tag{4.1.2}$$

5. K and F satisfy the following condition
$$\left|K(t,x,\omega)-K(t,y,\omega)\right|+\left|F(t,x,\omega)-F(t,y,\omega)\right|\le L\sup_{s\in[0,t]}\left|x(s)-y(s)\right|,\tag{4.1.3}$$
where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$.

Instead of $\int_0^t K\left(s,X(\omega),\omega\right)\mathrm{d}A_s$, $\int_0^t F\left(s,X(\omega),\omega\right)\mathrm{d}S_s(\omega)$ we will often write $\int_0^t K(s,X)\,\mathrm{d}A_s$ and $\int_0^t F(s,X)\,\mathrm{d}S_s$ respectively. Next, we define the sequence of *Lebesgue partitions* generated by $\omega\in\Omega$.

Definition 4.1. *Let $\omega\in\Omega$ and $n=0,1,2,\ldots$. The nth Lebesgue partition $\pi^n(\omega^i)$ of $[0,T]$ generated by ω^i for each $i=1,\ldots,d$ as: $\pi^n_0\left(\omega^i\right)=0$ and for $k=0,1,\ldots,$ is*

$$\pi^n_{k+1}(\omega^i)=\begin{cases}\inf\left\{t\in\left[\pi^n_k(\omega^i),T\right]:\left|\omega^i(t)-\omega^i\left(\pi^n_k(\omega^i)\right)\right|=2^{-n}\right\} & \text{if } \pi^n_k(\omega^i)\le T,\\ +\infty & \text{if } \pi^n_k(\omega^i)=+\infty.\end{cases}$$

By convention $\inf\emptyset=+\infty$. Similarly, replacing ω^i by $\omega^i+\omega^j$ we define the Lebesgue partitions $\pi^n(\omega^i+\omega^j)$ generated by $\omega^i+\omega^j$ for each $i,j=1,\ldots,d$ as follows.

Definition 4.2. *Let $\omega \in \Omega$ and $k = 0, 1, \ldots,$. Then the sequence of Lebesgue partitions generated by ω is:*

$$\pi_{k+1}^n(\omega) := \min\left\{ t > \pi_k^n(\omega) : t \in \bigcup_{i=1}^d \pi^n(\omega^i) \cup \bigcup_{i,j=1, i\neq j}^d \pi^n(\omega^i + \omega^j) \right\}.$$

In Vovk's paper [47], it is proven that for $t \in [0,T]$, typical continuous price path $\omega \in \Omega$ possess the continuous limit

$$[S^i, S^j]_t(\omega) := \lim_{n\to+\infty} \sum_{k=1}^{\infty} S^i_{\pi_k^n\wedge t, \pi_{k+1}^n\wedge t}(\omega) S^j_{\pi_k^n\wedge t, \pi_{k+1}^n\wedge t}(\omega), \tag{4.1.4}$$

and this convergence is *uniform* in $[0,T]$. We will use the following notation:

$$[S]_t := \left([S^i, S^j]_t\right)_{i,j=1}^d \quad \text{and} \quad \left|[S]\right|_t := \sum_{i=1}^d [S^i, S^i]_t. \tag{4.1.5}$$

Definition 4.3. *Let $G \in \mathscr{G}$ be given by (3.1.2). The quadratic variation process of the real integral process $(G \cdot S)$ is defined as*

$$\begin{aligned}
\left[(G\cdot S)\right]_t(\omega) :&= \sum_{l=0}^{\infty}\sum_{i,j=1}^d g_l^i(\omega) g_l^j(\omega)\cdot\left(\left[S^i, S^j\right]_{\tau_{l+1}\wedge t}(\omega) - \left[S^i, S^j\right]_{\tau_l\wedge t}(\omega)\right) \\
&= \sum_{l=0}^{\infty}\sum_{i,j=1}^d g_l^i(\omega) g_l^j(\omega)\left[S^i, S^j\right]_{\tau_l\wedge t, \tau_{l+1}\wedge t}(\omega) \\
&= \sum_{i,j=1}^d \int_0^t G_s^i G_s^j(\omega)\,\mathrm{d}\left[S^i, S^j\right]_s(\omega) \\
&= \int_0^t G_s^{\otimes 2}\mathrm{d}\,[S]_s(\omega).
\end{aligned} \tag{4.1.6}$$

For any process $G : [0,T]\times\Omega \to \mathbb{R}^m$ $(m = 1, 2, \ldots)$ let us define

$$G_t^* := \sup_{s\in[0,t]} |G_s|$$

(where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^m$). We have the following estimate.

Lemma 4.4. *Let $G \in \mathscr{G}$, $Q \geq 0$ and $G^Q : [0,T]\times\Omega \to \mathbb{R}^d$ be defined as*

$$G_t^Q := G_t \mathbb{1}_{[0,Q]}\left(\left|[S]\right|_t\right).$$

Then $G^Q \in \mathscr{G}$ and for any $t \in [0,T]$

$$\left[\left(G^Q \cdot S\right)\right]_t \leq d\,(G_t^*)^2 Q.$$

Proof. $G^Q \in \mathscr{G}$ since $\inf\left\{t \geq 0 : \left|[S]\right|_t > Q\right\}$ is a stopping time since the filtration $\mathscr{F}$ is right continuous. Using the inequality $\left|\mathrm{d}[S^i, S^j]\right| \leq \frac{1}{2}\mathrm{d}[S^i, S^i] + \frac{1}{2}\mathrm{d}[S^j, S^j]$ (which follows from $\mathrm{d}[S^i - S^j, S^i - S^j] \geq 0$, $\mathrm{d}[S^i + S^j, S^i + S^j] \geq 0$) and $[(G^Q \cdot S)]_t = \sum_{i,j=1}^{d} \int_0^t G_s^i G_s^j \mathbb{1}_{[0,Q]}\left(\left|[S]\right|_t\right) \mathrm{d}[S^i, S^j]_s$, we obtain

$$\begin{aligned}
\left[\left(G^Q \cdot S\right)\right]_t &\leq \sum_{i,j=1}^{d} \int_0^t (G_t^*)^2 \mathbb{1}_{[0,Q]}\left(\left|[S]\right|_s\right) \frac{1}{2}\left(\mathrm{d}[S^i, S^i]_s + \mathrm{d}[S^j, S^j]_s\right) \\
&= (G_t^*)^2 \int_0^t \mathbb{1}_{[0,Q]}\left(\left|[S]\right|_s\right) \sum_{i,j=1}^{d} \frac{1}{2}\left(\mathrm{d}[S^i, S^i]_s + \mathrm{d}[S^j, S^j]_s\right) \\
&= (G_t^*)^2 \int_0^t \mathbb{1}_{[0,Q]}\left(\left|[S]\right|_s\right) d\mathrm{d}\left|[S]\right|_s \\
&= d(G_t^*)^2 \left(Q \wedge \left|[S]\right|_t\right) \leq d(G_t^*)^2 Q.
\end{aligned}$$

□

In order to prove the existence and uniqueness of solution of the Integral (4.1.1), we need a model-free version of the BDG inequality, see [37]. Unlike in [4], our approach requires the outer expectation $\overline{\mathbb{E}}$ defined below.

Definition 4.5. *Let $\mathscr{T}[0,T]$ be the family of stopping times τ such that $0 \leq \tau \leq T$. For any process $Z : [0,T] \times \Omega \to [0,+\infty]$ we define*

$$\overline{\mathbb{E}}Z = \inf_{\tilde{\Omega}} \inf\left\{\lambda > 0 : \exists H^n \in \mathscr{G}_\lambda \text{ s.t. } \forall \omega \in \tilde{\Omega} \quad \forall \tau \in \mathscr{T}[0,T] \quad \liminf_{n\to+\infty}\left(\lambda + (H^n \cdot S)_\tau\right)(\omega) \geq Z_\tau(\omega)\right\} \quad (4.1.7)$$

where the first infimum is over all subsets $\tilde{\Omega} \subset \Omega$ of typical price paths, that is all $\tilde{\Omega}$ such that $\bar{\mathbb{P}}\left(\Omega \setminus \tilde{\Omega}\right) = 0$.

The outer expectation (4.1.7) is *countably subadditive*, *monotone* and *positively homogeneous*. By $\mathbb{H}$ we denote the family of processes $G \in \mathscr{G}$ such that

$$\overline{\mathbb{E}}\sqrt{[(G \cdot S)]} < +\infty.$$

Remark 4.6. *The outer expectation (4.1.7) differs with Vovk's outer expectation introduced in [46, Definition 6.1]. The latter focusses only at the terminal value of a non negative process $\lambda + (H^n \cdot S)$.*

4.2 Model-Free Version of the BDG Inequality

In this section we establish the following model-free version of the BDG inequality:

Theorem 4.7. *For any $G \in \mathbb{H}$*

$$\overline{\mathbb{E}}\,(G\cdot S)^* \le c_1 \overline{\mathbb{E}}\sqrt{[(G\cdot S)]}, \tag{4.2.1}$$

where $c_1 \le 6$.

Proof. Let us recall the pathwise BDG inequalities of Beiglböck and Siorpaes [6]: if for real numbers $x_0, x_1, \ldots, e_0, e_1, \ldots$ and $m = 0, 1, \ldots$ we define

$$x_m^* := \max_{0\le k\le m} |x_k|, \quad [x]_m := x_0^2 + \sum_{k=0}^{m-1}(x_{k+1}-x_k)^2, \quad (e\cdot x)_m := \sum_{k=0}^{m-1} e_k\,(x_{k+1}-x_k)$$

then for any $p \ge 1$ there exist positive constant $c_p < +\infty$ and numbers $f_0^p, f_1^p, \ldots$ such that f_k^p, $k = 0, 1, \ldots$ depends only on $x_0, x_1, \ldots, x_k$,

$$f_k^p = f_k^p\,(x_0, x_1, \ldots, x_k),$$

and such that for any $N = 0, 1, \ldots$ one has

$$\left(x_N^*\right)^p \le c_p \sqrt{[x]_N^p} + (f^p \cdot x)_N. \tag{4.2.2}$$

Moreover, for $p = 1$ one has $c_1 \le 6$, $\sup_{k\ge 0}\left|f_k^1\right| \le 2$ and the following estimate also holds

$$x_N^* \ge \sqrt{[x]_N} + \left(f^1\cdot x\right)_N. \tag{4.2.3}$$

Let now $G \in \mathbb{H}$ and (3.1.2) be its representation. Let $\left(\sigma_m^n\right)_{m\ge 0}$ be a non-decreasing rearrangement of $\left(\pi_k^n\right)_{k\ge 0} \cup (\tau_l)_{l\ge 0}$, where $\left(\pi_k^n\right)_{k\ge 0}$ is the nth Lebesgue partition ($n = 1, 2, \ldots$). For $n = 1, 2, \ldots$ and $\omega \in \Omega$ we define $x_0^n = 0$ and for $m = 0, 1, \ldots$

$$x_{m+1}^n = x_m^n + G_{\sigma_m^n\wedge T}\,(\omega)\cdot S_{\sigma_m^n\wedge T, \sigma_{m+1}^n\wedge T}(\omega)$$

(to ease notation we write $\sigma_m^n, \sigma_{m+1}^n$ instead of $\sigma_m^n(\omega), \sigma_{m+1}^n(\omega)$). Let us notice that for $\omega \in \Omega$ we have

$$G_t\,(\omega) = G_0\,(\omega)\,\mathbb{1}_{\{0\}}\,(t) + \sum_{m=0}^{+\infty} G_{\sigma_m^n(\omega)\wedge T}\,(\omega)\,\mathbb{1}_{\left(\sigma_m^n\wedge T, \sigma_{m+1}^n\wedge T\right]}\,(t)$$

and for $\omega \in \tilde{\Omega}$

$$\lim_{n\to+\infty}\sum_{m=0}^{\infty}\sum_{i,j=1}^{d} G^i_{\sigma_m^n\wedge T}\,(\omega)\,G^j_{\sigma_m^n\wedge T}(\omega) S^i_{\sigma_m^n\wedge t, \sigma_{m+1}^n\wedge t}(\omega) S^j_{\sigma_m^n\wedge t, \sigma_{m+1}^n\wedge t}(\omega) = [G\cdot S]_t\,(\omega), \tag{4.2.4}$$

where $\tilde{\Omega}$ is the set of typical paths, for which the quadratic variation along the sequence of Lebesgue partitions exist and the convergence in (4.1.4) is uniform.

Moreover, by the definition of σ_m^n,

$$\left|S_{\sigma_{m+1}^n\wedge t}(\omega)-S_{\sigma_m^n\wedge t}(\omega)\right|\leq\sqrt{d}2^{-n}.$$

Let us now define the simple strategy Φ^n which just after time $\sigma_m^n\wedge T$ attains the position

$$\Phi_m^n:=f_m^1\left(x_0^n,x_1^n,\ldots,x_m^n\right)G_{\sigma_m^n\wedge T},$$

i.e.

$$\Phi_t^n(\omega)=\Phi_0^n(\omega)\mathbb{1}_{\{0\}}(t)+\sum_{m=0}^{+\infty}\Phi_m^n(\omega)\mathbb{1}_{\left(\sigma_m^n\wedge T,\sigma_{m+1}^n\wedge T\right]}(t).$$

For $t\in[0,T]$ by $m^n(t)$ we denote the unique $m=0,1,\ldots$ such that

$$\sigma_m^n<t\leq\sigma_{m+1}^n.$$

We estimate

$$\begin{aligned}\left|(\Phi^n\cdot S)_t(\omega)-\left(f^1\cdot x^n\right)_{m^n(t)}\right| &= \left|f^1_{m^n(t)}G_{\sigma^n_{m^n(t)}}(\omega)\right|\left|S_t(\omega)-S_{\sigma^n_{m^n(t)}}(\omega)\right|\\ &\leq\sqrt{d}\cdot2^{-n}2\sup_{s\in[0,t]}\left|G_s(\omega)\right|, \qquad (4.2.5)\end{aligned}$$

where we used the fact that $\sup_{k\geq0}\left|f_k^1\right|\leq2$.

Let now λ_0,λ_1, λ_2 and λ_3 be finite reals such that

$$\lambda_0>\lambda_1>\lambda_2>\lambda_3>\overline{\mathbb{E}}\sqrt{[(G\cdot S)]}$$

and let H^n be a sequence of λ_3-admissible strategies such that

$$\forall\omega\in\bar{\Omega}\quad\forall\tau\in\mathscr{T}[0,T]\quad\liminf_{n\to+\infty}\left(\lambda_3+(H^n\cdot S)_\tau\right)\geq\sqrt{[(G\cdot S)]_\tau}, \qquad (4.2.6)$$

where $\bar{\Omega}$ is the closure of the set Ω. Now let us define

$$\eta^n:=\inf\left\{t\in[0,T]:\left|\left(\left[(G\cdot S)\right]_t\right)^{\frac12}-\left(\sum_{m=0}^{\infty}\sum_{i,j=1}^{d}G^i_{\sigma_m^n\wedge T}G^j_{\sigma_m^n\wedge T}S^i_{\sigma_m^n\wedge t,\sigma_{m+1}^n\wedge t}S^j_{\sigma_m^n\wedge t,\sigma_{m+1}^n\wedge t}\right)^{\frac12}\right|\right.$$
$$\left.\geq\lambda_1-\lambda_2\right\},$$

$$\rho^n:=\inf\left\{t\in[0,T]:c_1\lambda_0+\left(c_1\left(H^n\cdot S\right)_t+\left(\Phi^n\cdot S\right)_t\right)\leq0\right\}$$

and let us consider the strategy

$$\Psi_t^n:=\left(c_1H_t^n+\Phi_t^n\right)\cdot\mathbb{1}_{[0,\eta^n\wedge\rho^n]}(t).$$

Directly from the definition it follows that Ψ^n is $c_1\lambda_0$-admissible. Moreover, for $\omega \in \tilde{\Omega}$ the convergence in (4.2.4) is uniform in $[0,T]$. We have $\bar{\mathbb{P}}\left(\Omega \setminus \tilde{\Omega}\right) = 0$ and for each $\omega \in \tilde{\Omega}$, $\eta^n(\omega) \to +\infty$ as $n \to +\infty$. Thus for each $\omega \in \tilde{\Omega}$ and $\tau \in \mathscr{T}[0,T]$ by (4.2.6), by the estimate $\left[(G\cdot S)\right]_\tau \geq \left[(G\cdot S)\right]_{\sigma^n_{m^n(\tau)}}$, (4.2.5), the definition of η^n and by (4.2.2), for sufficiently large n we have

$$
\begin{aligned}
& c_1\lambda_0 + (c_1 H^n\cdot S)_{\tau\wedge\eta^n}(\omega) + (\Phi^n\cdot S)_{\tau\wedge\eta^n}(\omega) \\
& = c_1(\lambda_0-\lambda_2) + c_1\lambda_2 + (c_1H^n\cdot S)_\tau(\omega) + (\Phi^n\cdot S)_\tau(\omega) \\
& \geq c_1(\lambda_0-\lambda_2) + c_1\left(\left[(G\cdot S)\right]_\tau(\omega)\right)^{\frac{1}{2}} + (\Phi^n\cdot S)_\tau(\omega) \\
& \geq c_1(\lambda_0-\lambda_2) + c_1\left(\left[(G\cdot S)\right]_{\sigma^n_{m^n(\tau)}}(\omega)\right)^{\frac{1}{2}} + (\Phi^n\cdot S)_{\sigma^n_{m^n(\tau)}}(\omega) - 2\cdot 2^{-n}\sqrt{d}\sup_{s\in[0,\tau]}\left|G_s(\omega)\right| \\
& \geq c_1(\lambda_0-\lambda_2) + c_1\left(\sum_{m=0}^{m^n(\tau)-1}\sum_{i,j=1}^{d} G^i_{\sigma^n_m}G^j_{\sigma^n_m}S^i_{\sigma^n_m,\sigma^n_{m+1}}S^j_{\sigma^n_m,\sigma^n_{m+1}}\right)^{\frac{1}{2}} - c_1(\lambda_1-\lambda_2) \\
& \quad + (\Phi^n\cdot S)_{\sigma^n_{m^n(\tau)}}(\omega) - 2\cdot 2^{-n}\sqrt{d}\sup_{s\in[0,\tau]}\left|G_s(\omega)\right| \\
& = c_1(\lambda_0-\lambda_1) + c_1\left([x^n]_{m^n(\tau)}\right)^{\frac{1}{2}} + \left(f^1\cdot x^n\right)_{m^n(\tau)} - 2\cdot 2^{-n}\sqrt{d}\sup_{s\in[0,\tau]}\left|G_s(\omega)\right| \\
& \geq c_1(\lambda_0-\lambda_1) + (x^n)^*_{m^n(\tau)} - 2\cdot 2^{-n}\sqrt{d}\sup_{s\in[0,\tau]}\left|G_s(\omega)\right| \\
& \geq (G\cdot S)^*_\tau(\omega) + c_1(\lambda_0-\lambda_1) - 3\cdot 2^{-n}\sqrt{d}\sup_{s\in[0,\tau]}\left|G_s(\omega)\right| \\
& > (G\cdot S)^*_\tau(\omega).
\end{aligned}
$$

As a result we get that for $\omega \in \bar{\Omega}\cap\tilde{\Omega}$, $\rho^n(\omega) \to +\infty$ as $n \to +\infty$ and

$$\liminf_{n\to+\infty}\left(c_1\lambda_0 + (\Psi^n\cdot S)(\omega)\right)_\tau \geq (G\cdot S)^*_\tau(\omega).$$

Since λ_0 may be as close to $\bar{\mathbb{E}}\sqrt{\left[(G\cdot S)\right]}$ as we please, we obtain (4.2.1). □

Remark 4.8. *The proof of Theorem 4.7 relies on the fact that having sequence of strategies $(H^n)_n$ which dominate (in the sense of (4.1.7)) $\sqrt{\left[(G\cdot S)\right]}$ we are able to construct strategies $\Psi^n_t := \left(c_1H^n_t + \Phi^n_t\right)\cdot\mathbb{1}_{[0,\eta^n\wedge\rho^n]}(t)$ which dominate $(G\cdot S)^*$.*

The constant $c_1 = 6$ is not optimal since it is possible to construct strategies which give $c_1 = 4$, see [33]. Using (4.2.3) and proceeding in a similar way as in the proof of Theorem 4.7 we also get (for $G \in \mathbb{H}$) the estimate:

$$\bar{\mathbb{E}}(G\cdot S)^* \geq \bar{\mathbb{E}}\sqrt{\left[(G\cdot S)\right]}. \tag{4.2.7}$$

4.3 Multidimensional Version of the Model-free BDG Inequality

In this section we will prove the model-free BDG inequality in the case when G is a matrix-valued, simple process, i.e. $G : [0,T] \times \Omega \to \mathbb{R}^d \times \mathbb{R}^d = \mathbb{R}^{2d}$ and $G = \left(G^1, G^2, \ldots, G^d\right)$ where $G^i \in \mathbb{H}$, $i = 1, 2, \ldots, d$. The family of such processes will be denoted by $\mathbb{H}^d$. For $G \in \mathbb{H}^d$ we define the integral $(G \cdot S)$ as the vector of integrals

$$(G \cdot S) = \left((G^1 \cdot S), (G^2 \cdot S), \ldots, (G^d \cdot S)\right).$$

Also, similarly to (4.1.5) we define

$$\left|[(G \cdot S)]\right|_t := \sum_{i=1}^{d} \left[\left(G^i \cdot S\right)\right]_t. \tag{4.3.1}$$

Now we have the following generalisation of (4.2.1).

Proposition 4.9. *For any* $G \in \mathbb{H}^d$

$$\overline{\mathbb{E}}\,(G \cdot S)^* \leq c_1 d \overline{\mathbb{E}} \sqrt{\left|[(G \cdot S)]\right|}, \tag{4.3.2}$$

where $c_1 \leq 6$.

Proof. Using the inequality $\sqrt{\sum_{i=1}^{d} a_i^2} \leq \sum_{i=1}^{d} |a_i|$ valid for any real $a_1, a_2, \ldots, a_d$ we estimate for $t \in [0,T]$

$$(G \cdot S)_t^* \leq \sup_{0 \leq s \leq t} \sum_{i=1}^{d} \left|\left(G^i \cdot S\right)_s\right| \leq \sum_{i=1}^{d} \sup_{0 \leq s \leq t} \left|\left(G^i \cdot S\right)_s\right| = \sum_{i=1}^{d} \left(G^i \cdot S\right)_t^*.$$

Next, using subadditivity of $\overline{\mathbb{E}}$, (4.2.1) and the monotonicity of $\overline{\mathbb{E}}$ we obtain

$$\begin{aligned}\overline{\mathbb{E}} \sum_{i=1}^{d} \left(G^i \cdot S\right)^* &\leq \sum_{i=1}^{d} \overline{\mathbb{E}} \left(G^i \cdot S\right)_t^* \leq c_1 \sum_{i=1}^{d} \overline{\mathbb{E}} \sqrt{[(G^i \cdot S)]} \\ &\leq c_1 \sum_{i=1}^{d} \overline{\mathbb{E}} \sqrt{\left|[(G \cdot S)]\right|} = c_1 d \overline{\mathbb{E}} \sqrt{\left|[(G \cdot S)]\right|}.\end{aligned}$$

□

4.4 Spaces $\mathscr{M}$, $\mathscr{M}^d$, $loc\mathscr{M}$ and $loc\mathscr{M}^d$

Now, we introduce the spaces of (equivalence classes of) adapted processes $G : [0,T]\times\Omega\to\mathbb{R}^d$ (resp. $G:[0,T]\times\Omega\to\mathbb{R}^d\times\mathbb{R}^d$) such that $\overline{\mathbb{E}}G^* < +\infty$ (G is equivalent with H if $\overline{\mathbb{E}}(G-H)^* = 0$). Using standard arguments (see for example [35, proof of Lemma 2.11]) we see that these spaces equipped with the metric

$$d(G,H) := \overline{\mathbb{E}}(G-H)^*$$

are complete metric spaces.

Remark 4.10. *Let us notice that the fact $d(G,H) = 0$ implies that for a typical price path $\omega\in\Omega$*

$$G_t(\omega) = H_t(\omega) \text{ for all } t\in[0,T].$$

Let $\mathscr{M}$ (resp. $\mathscr{M}^d$) denote the closures of the sets of simple processes $\mathbb{H}$ (resp. $\mathbb{H}^d$) in the defined spaces. Thus the simple processes are dense in $\mathscr{M}$ (resp. $\mathscr{M}^d$).

Let us notice that for a simple process X and $Q\geq 0$ the process X^Q defined as

$$X_t^Q = X_t \mathbb{1}_{[0,Q]}\left(\left|[S]\right|_t\right), \quad t\in[0,T],$$

is also a simple process and if $X\in\mathscr{M}$ then using Lemma 4.4 we get

$$\overline{\mathbb{E}}\sqrt{[(X^Q\cdot S)]} \leq \sqrt{dQ}\overline{\mathbb{E}}X^* < +\infty.$$

Similarly, if $X\in\mathscr{M}^d$ is a simple process we get

$$\overline{\mathbb{E}}\sqrt{|[(X^Q\cdot S)]|} \leq d\sqrt{Q}\overline{\mathbb{E}}X^* < +\infty.$$

Using this, Theorem 4.7 or Proposition 4.9 and completeness of the space $\mathscr{M}$ (resp. $\mathscr{M}^d$) we see that for any $X\in\mathscr{M}$ (resp. $X\in\mathscr{M}^d$), any sequence of simple processes $X^n\in\mathscr{M}$ (resp. $X^n\in\mathscr{M}^d$) such that $X^n\to X$ in $\mathscr{M}$ (resp. $\mathscr{M}^d$) (i.e. $\lim_{n\to+\infty} d(X^n,X) = 0$) and any $Q\geq 0$, the sequence of integrals $\left((X^n)^Q\cdot S\right)$ converges in $\mathscr{M}$ (resp. in $\mathscr{M}^d$) to the integral process $\left(X^Q\cdot S\right)$.

In analogy to (4.1.6) for $G\in\mathscr{M}$ we define

$$\left[(G\cdot S)\right]_t := \int_0^t G_s^{\otimes 2}\mathrm{d}[S]_s = \sum_{i,j=1}^d \int_0^t G_s^i G_s^j \mathrm{d}\left[S^i,S^j\right]_s \tag{4.4.1}$$

and in analogy to (4.3.1) for $G\in\mathscr{M}^d$ we define

$$\left|[(G\cdot S)]\right|_t := \sum_{i=1}^d \left[\left(G^i\cdot S\right)\right]_t.$$

Finally, let us introduce the space $loc\mathscr{M}$ (resp. $loc\mathscr{M}^d$) of (adapted) processes $X : [0,T] \times \Omega :\to \mathbb{R}^d$ $\left(\text{resp. } \mathbb{R}^d \times \mathbb{R}^d\right)$ such that for any $Q \geq 0$, $X^Q \in \mathscr{M}$ (resp. $X \in \mathscr{M}^d$).

The following analog of Proposition 4.9 holds:

Proposition 4.11. *If $X \in loc\mathscr{M}^d$ and $Q \geq 0$ then*

$$\overline{\mathbb{E}}\left(X^Q \cdot S\right)^* \leq c_1 d \overline{\mathbb{E}} \sqrt{\left|[(X^Q \cdot S)]\right|} \tag{4.4.2}$$

where $c_1 \leq 6$.

4.5 Existence and Uniqueness Theorem

In this section we prove the existence and uniqueness of the solution of SDE (4.1.1) with Lipschitz coefficient driven by continuous model-free price paths. Now we will use the just obtained model-free version of the BDG inequality and Picard's iterations to prove the following theorem.

Theorem 4.12. *Under the assumptions stated above, integral equation (4.1.1) has unique solution in the space $loc\mathscr{M}$.*

Remark 4.13. *The assumption that $A_T^u(\omega) + A_T^v(\omega) \leq M$, where M is a deterministic constant seems to be important in the sense that when we allow M to be random then we can not prove that $X \in loc\mathscr{M}$.*

Remark 4.14. *Theorem 4.12 implies the existence of the solution of (4.1.1) in the space $loc\mathscr{M}$. More precisely, it implies the existence of a process $X : [0,T] \times \Omega \to \mathbb{R}^d$ which is a uniform limit of simple processes (till the volatility measured by $\left|[S]\right|$ is not too high) and such that for typical $\omega \in \Omega$,*

$$X_t(\omega) = X_0(\omega) + \int_0^t K\left(s, X(\omega), \omega\right) \mathrm{d}A_s + \int_0^t F\left(s, X(\omega), \omega\right) \mathrm{d}S_s(\omega).$$

Naturally, for many equations, like for example the one-dimensional Black-Scholes equation

$$X_t = x_0 + \int_0^t X_s \mathrm{d}A_s + \sigma \int_0^t X_s \mathrm{d}S_s \tag{4.5.1}$$

(x_0 - deterministic) we can write the solution explicitly

$$X_t = x_0 \exp\left(A_t - \frac{1}{2}\sigma^2 [S]_t + \sigma (S_t - S_0)\right)$$

and verify that it satisfies (4.5.1) using the (model-free) Itô formula (see [47]). However, for more general equations we often have no explicit solutions and the existence of a solution is not obvious.

4.6 Proof of Theorem 4.12

4.6.1 Existence

Let us set $q = 1/(4c_1^2L^2d^4)$, $r = 1/(3L)$,

$$\sigma_0 := \inf\left\{t \in [0,T] : \left|[S]_t\right| \geq q\right\}, \quad \vartheta_0 := \inf\{t \in [0,T] : A_t^u + A_t^v \geq r\},$$

$$\theta_0 = \sigma_0 \wedge \vartheta_0$$

and define $T^0 : \mathscr{M} \to \mathscr{M}$ such that for $G \in \mathscr{M}$,

$$\left(T^0G\right)_t = X_0 + \int_0^{t\wedge\theta_0} K(s,G)\,\mathrm{d}A_s + \int_0^{t\wedge\theta_0} F(s,G)\,\mathrm{d}S_s, \text{ for } t \in [0,T].$$

(By the assumption on X_0, (4.1.3), (4.1.2) and the calculation below (i.e. (4.6.1)) this definition is correct.) Now, by the Lipschitz property, the BDG inequality (4.4.2) and similar reasoning as in the proof of Lemma 4.4 we estimate

$$\begin{aligned}
&\overline{\mathbb{E}}\left(T^0G^1 - T^0G^2\right)^* \\
&\leq \overline{\mathbb{E}}\left(\int_0^{t\wedge\theta_0}\left\{K\left(s,G^1\right) - K\left(s,G^2\right)\right\}\mathrm{d}A_s\right)^*_{t\in[0,T]} \\
&\quad + \overline{\mathbb{E}}\left(\int_0^{t\wedge\theta_0}\left\{F\left(s,G^1\right) - F\left(s,G^2\right)\right\}\mathrm{d}S_s\right)^*_{t\in[0,T]} \\
&\leq \overline{\mathbb{E}}\left(\int_0^{t\wedge\vartheta_0} L\left(G^1 - G^2\right)^*_s \mathrm{d}A_s\right)^*_{t\in[0,T]} \\
&\quad + c_1 d\overline{\mathbb{E}}\left(\sum_{i=1}^d\left[\left(\left(\left(F^i\left(\cdot,G^1\right) - F^i\left(\cdot,G^2\right)\right)\mathbb{1}_{[0,q]}\left(\left|[S]\right|_t\right)\right)\cdot S\right)\right]\right)^{\frac{1}{2}} \\
&\leq \overline{\mathbb{E}}\left(L\left(G^1 - G^2\right)^*_{t\wedge\vartheta_0} r\right)^*_{t\in[0,T]} + c_1 d\overline{\mathbb{E}}\sqrt{d\cdot dL^2\left(\left(G^1 - G^2\right)^*\right)^2 q} \\
&\leq \frac{1}{3}\overline{\mathbb{E}}\left(G^1 - G^2\right)^* + \frac{1}{2}\overline{\mathbb{E}}\left(G^1 - G^2\right)^* \\
&= \frac{5}{6}\overline{\mathbb{E}}\left(G^1 - G^2\right)^*.
\end{aligned} \tag{4.6.1}$$

Thus T^0 is a contraction on $\mathscr{M}$ and it has unique fixed point X^0. Next, we define

$$\sigma_1 := \inf\left\{t \in [\theta_0, T] : \left|[S]\right|_t - \left|[S]\right|_{\theta_0} \geq q\right\},$$

$$\vartheta_1 := \inf\{t \in [\theta_0, T] : A^u_t - A^u_{\theta_0} + A^v_t - A^v_{\theta_0} \geq r\},$$
$$\theta_1 := \sigma_1 \wedge \vartheta_1,$$

(we apply the convention that $\sigma_1 = \theta_1 = +\infty$ if $\theta_0 \geq T$), and introduce the following operator $T^1 : \mathscr{M} \to \mathscr{M}$,

$$\left(T^1 G\right)_t := X^0_t \mathbb{1}_{[0,\theta_0]}(t) + \int_{t\wedge\theta_0}^{t\wedge\theta_1} K(s,G)\,\mathrm{d}A_s + \int_{t\wedge\theta_0}^{t\wedge\theta_1} F(s,G)\,\mathrm{d}S_s.$$

Similarly as before, we prove that T^1 is a contraction and has a fixed point $X^1 \in \mathscr{M}$. Moreover, X^0 and X^1 agree on the interval

$$[0, \theta_0 \wedge T].$$

Similarly, having defined σ_n, θ_n, $T^n : \mathscr{M} \to \mathscr{M}$, and its fixed point X^n, $n = 0, 1, \ldots$, by induction, we define

$$\sigma_{n+1} := \inf\left\{t \in [\theta_n, T] : \left|[S]\right|_t - \left|[S]\right|_{\theta_n} \geq q\right\},$$

$$\vartheta_{n+1} := \inf\{t \in [\theta_n, T] : A^u_t - A^u_{\theta_n} + A^v_t - A^v_{\theta_n} \geq r\},$$
$$\theta_{n+1} := \sigma_{n+1} \wedge \vartheta_{n+1},$$

and introduce the following operator $T^{n+1} : \mathscr{M} \to \mathscr{M}$,

$$\left(T^{n+1} G\right)_t := X^n_t \mathbb{1}_{[0,\theta_n]}(t) + \int_{t\wedge\theta_n}^{t\wedge\theta_{n+1}} K(s,G)\,\mathrm{d}A_s + \int_{t\wedge\theta_n}^{t\wedge\theta_{n+1}} F(s,G)\,\mathrm{d}S_s,$$

and its fixed point X^{n+1} which agrees with X^n on the interval

$$[0, \theta_n \wedge T].$$

Finally, setting

$$X := \lim_{n\to+\infty} X^n$$

we get that X satisfies

$$X_t := X_0 + \int_0^t K(s,X)\,\mathrm{d}A_s + \int_0^t F(s,X)\,\mathrm{d}S_s. \tag{4.6.2}$$

This will follow from the following lemma.

Lemma 4.15. *Let $n = 0, 1, 2, \ldots$. Assume that $\theta_n < +\infty$ and the following inequalities hold:*

$$A^u_{\theta_n} + A^v_{\theta_n} \leq k \cdot r \text{ and } \left|[S]\right|_{\theta_n} \leq l \cdot q$$

for some non-negative integers k and l. Then

$$k + l \geq n + 1. \tag{4.6.3}$$

Proof. The proof follows by easy induction. For $n=0$ from $\theta_0 < +\infty$ and $A^u_{\theta_0} + A^v_{\theta_0} < r$ it follows that $\theta_0 = \sigma_0$ and $\left|[S]\right|_{\sigma_0} = q$ thus inequality (4.6.3) holds. Assume that the thesis holds for some $n = 0,1,2,\ldots$. Similarly as for $n=0$, from the inequality $\theta_{n+1} < +\infty$ and $\theta_{n+1} = \vartheta_{n+1} \wedge \sigma_{n+1} < \vartheta_{n+1}$ it follows that $\theta_{n+1} = \sigma_{n+1}$ and $\left|[S]\right|_{\theta_{n+1}} - \left|[S]\right|_{\theta_n} = q$. Thus, from

$$A^u_{\theta_{n+1}} + A^v_{\theta_{n+1}} \le k \cdot r \text{ and } \left|[S]\right|_{\theta_{n+1}} \le l \cdot q$$

it follows that

$$A^u_{\theta_n} + A^v_{\theta_n} \le (k-1) \cdot r \text{ and } \left|[S]\right|_{\theta_n} \le l \cdot q$$

or

$$A^u_{\theta_n} + A^v_{\theta_n} \le k \cdot r \text{ and } \left|[S]\right|_{\theta_n} \le (l-1) \cdot q$$

In both cases, from the induction hypothesis,

$$k + l - 1 \ge n + 1$$

thus

$$k + l \ge n + 2.$$

□

For any $n = 0,1,2,\ldots$ and any $t \in [0, \theta_n \wedge T]$, X satisfies (4.6.2). Thus, if for some even n, $n \cdot r \ge 2\,(A^u_T + A^v_T)$ and $n \cdot q \ge 2\left|[S]\right|_T$ then $\theta_n > T$ (in fact $\theta_n = +\infty$) and X satisfies (4.6.2) for all $t \in [0,T]$. Otherwise, if $\theta_n \le T$, by Lemma 4.15 we would have

$$A^u_{\theta_n} + A^v_{\theta_n} \le A^u_T + A^v_T \le \frac{n}{2} \cdot r \text{ and } \left|[S]\right|_{\theta_n} \le \left|[S]\right|_T \le \frac{n}{2} \cdot q$$

which would yield

$$\frac{n}{2} + \frac{n}{2} \ge n + 1,$$

which is a contradiction.

4.6.2 Uniqueness

In general, we can not guarantee that $X \in \mathscr{M}$ but we will prove that $X \in loc\mathscr{M}$. Moreover is the unique solution of (4.6.2) in $loc\mathscr{M}$.

First, we will prove that $X \in loc\mathscr{M}$. We know that $X \cdot \mathbb{1}_{[0,\theta_n]}(t) = X^n \in \mathscr{M}$ for $n = 0,1,\ldots$. Now, from Lemma 4.15 it follows that for any fixed $Q > 0$ and $n = \lfloor M/r \rfloor + \lfloor Q/q \rfloor + 2$ we have that either $A^u_{\theta_n} + A^v_{\theta_n} > A^u_T + A^v_T$ (which implies $\theta_n \ge T$) or $\left|[S]\right|_{\theta_n} > Q$ (otherwise we would have

$$A^u_{\theta_n} + A^v_{\theta_n} \le A^u_T + A^v_T \le M < \left(\lfloor M/r \rfloor + 1\right) r$$

and

$$\left|[S]\right|_{\theta_n} \leq Q < \left(\lfloor Q/q \rfloor + 1\right) q$$

which by Lemma 4.15 would yield

$$\lfloor M/r \rfloor + 1 + \lfloor Q/q \rfloor + 1 \geq n + 1 = \lfloor M/r \rfloor + \lfloor Q/q \rfloor + 3$$

which is a contradiction). Thus for $t \in [0,T]$ we have

$$\mathbb{1}_{[0,Q]}\left(\left|[S]\right|_{\theta_n}\right) \leq \mathbb{1}_{[0,\theta_n]}(t)$$

and

$$X^Q = \left(X_t \cdot \mathbb{1}_{[0,Q]}\left(\left|[S]\right|_t\right)\right)_{t\in[0,T]} \in \mathscr{M}^d.$$

This proves that $X \in loc\mathscr{M}$.

To prove the uniqueness notice that if X was not unique in $loc\mathscr{M}$ then there would exist two processes $X \in loc\mathscr{M}$ and $Y \in loc\mathscr{M}$ satisfying (4.6.2) and such that

$$\overline{\mathbb{E}}\left((X_t - Y_t)^* \mathbb{1}_{[0,Q]}\left(\left|[S]\right|_t\right)\right)_{t\in[0,T]} > 0 \tag{4.6.4}$$

for some $Q > 0$. However, using the same reasoning as in (4.6.1) and the fact that X and Y solve (4.6.2) we can prove that

$$\begin{aligned}\overline{\mathbb{E}}\left((X-Y)\cdot\mathbb{1}_{[0,\theta_0]}(t)\right)^* &= \overline{\mathbb{E}}\left(\left(T^0X - T^0Y\right)\cdot\mathbb{1}_{[0,\theta_0]}(t)\right)^* \\ &\leq \frac{5}{6}\overline{\mathbb{E}}\left((X-Y)\,\mathbb{1}_{[0,\theta_0]}(t)\right)^*\end{aligned}$$

thus

$$\overline{\mathbb{E}}\left((X-Y)\cdot\mathbb{1}_{[0,\theta_0]}(t)\right)^* = 0.$$

Similarly, by induction (and subadditivity of $\overline{\mathbb{E}}$), we prove that for $n = 1, 2, \ldots$

$$\overline{\mathbb{E}}\left((X-Y)\cdot\mathbb{1}_{[0,\theta_n]}(t)\right)^* = 0.$$

Now, for any fixed $Q > 0$ and $n = \lfloor M/r \rfloor + \lfloor Q/q \rfloor + 2$ for $t \in [0,T]$ we have

$$\mathbb{1}_{[0,Q]}\left(\left|[S]\right|_t\right) \leq \mathbb{1}_{[0,\theta_n]}(t)$$

and

$$|X_t - Y_t| \cdot \mathbb{1}_{[0,Q]}\left(\left|[S]\right|_t\right) \leq |X_t - Y_t| \cdot \mathbb{1}_{[0,\theta_n]}(t)$$

thus

$$\overline{\mathbb{E}}\left((X_t - Y_t)^* \mathbb{1}_{[0,Q]}\left(\left|[S]\right|_t\right)\right)_{t\in[0,T]} = 0$$

which contradicts (4.6.4).

Chapter 5

Conclusion

In this thesis we reviewed the classical construction of quadratic variation of a semimartingale and its application in the stochastic calculus. Turning our attention to the basic fundamentals of game theory (model-free) to mathematical finance, various properties and definitions of model-free price paths have been discussed. In particular, various constructions of quadratic variation have been discussed and we have established integrals driven by such paths. We focussed mainly on càdlàg price paths rather than continuous price paths. The càdlàg price paths is the natural settings for processes with jumps.

In the càdlàg price paths setting, we proved that the model-free typical (in the sense of Vovk) càdlàg price paths with mildly restricted downward jumps, possess quadratic variation which does not depend on any sequence of partitions as long as those partitions are obtained from stopping times such that the oscillations of the path on the consecutive intervals of these partitions tend to zero. The results we obtained justify our claim that we prove the independence of the quadratic variation of model-free price path as long as the partitions are obtained from stopping times. We also showed that the quadratic variation exists for typical price paths with jumps that are bounded below in a weak sense, in particular, for typical non-negative price paths.

We also demonstrated the invariance of the definition which says that the limit in outer measure $\bar{\mathbb{P}}$ will stay the same if we replace the Lebesgue sequence of partitions by another sequence whose vertical mesh converges to zero in $\bar{\mathbb{P}}$. In addition, we proved another partition independent formula for the quadratic variation with jumps in terms of the truncated variation. Using integration by parts formula perturbed by an adapted finite variation path, we defined deterministic quadratic variation in the sense of Föllmer. Further, we discussed definition of sequences of quasi-explicit, partition-independent quantities (normalized truncated variations) that tend to the quadratic variation.

Using the assumptions as in the book by Revuz and Yor's [37] and the BDG inequality for integrals, we were able to prove the existence and uniqueness of solution of stochastic differential equations with Lipschitz coefficients. The BDG inequality made it possible to define a stochastic integral along càdlàg price paths with mildly restricted downward jumps for a wide class of integrands.

Future work will entail the use of the notion "ucqa" (uniformly on compacts quasi-always). The impression is that it should be relatively easy to obtain ucqa type results from the results obtained in Chapter 3, even over the time horizon $[0,\infty)$ instead of $[0,T]$. We have shown how to construct a sequence of admissible simple strategies to superhedge 1_A starting from $\lambda > 0$, where A is the event that quadratic variation does not exists over $[0,T]$. Another direction of future work will be to construct a sequence of admissible simple strategies that superhedge $\infty 1_A$ starting from 1, for the same A. One can then combine these sequences into a single sequence over rational $T \in (0,\infty)$. The resulting sequence of admissible strategies should bring infinite wealth immediately after quadratic variation ceases to exist (if it ever does).

Further extensions include considering the concepts of instant enforcement where a trader can instantly enforces some of the best-known properties of Brownian motion such as the existence of quadratic variation. Another trend would be to investigate the case when the jumps are not restricted, that is, for typical price paths ω in Ω_ξ, where $\xi := \infty$ (only non-negative ω) and establish integrals associated with such paths. Lastly, one can investigate the existence and uniqueness of solution of stochastic differential equations driven by Lécy processes with Lipschitz coefficients by using similar assumptions as in the book by Revuz and Yor's [37] and the BDG inequality for integrals.

Appendices

Appendix A

Codes in R

```
# Code generating example of continuous path

library("sm")
set.seed(123)

t <- seq(0, 10, by=0.01)
X<- sin(t)*cos(t)+ cos(sin(2*t))
plot(t,X,"l",col="blue")

# Code generating example of cadlag path
library("sm")
set.seed(123)

x <- seq(0, 10, by=0.01)
y<- -sin(x)-(2/3)*sin(2*x)-(2/3)^2*sin(2^2*x)
-(2/3)^3*sin(2^3*x)-(2/3)^4*sin(2^4*x)-(2/3)^5*sin(2^5*x)
z<- -(2/3)^3*sin(2^3*x)-(2/3)^4*sin(2^4*x)-(2/3)^5*sin(2^5*x)
w<- -(2/3)^2*sin(2^2*x)-(2/3)^3*sin(2^3*x)-(2/3)^4*sin(2^4*x)

plot(x,y,"l",lwd=2, col= 0)
lines(x[1:200],y[1:200], pch=19, col="blue")
lines(x[201],y[159]+0.05,  pch=19, type = "b", col="red")

lines(x[201:400],z[201:400], pch=19, col="blue")
lines(x[200],y[200]+0.9, pch=19, type = "b", col="blue")
lines(x[401],y[400]-0.25,  pch=19, type = "b", col="red")

lines(x[401:600],w[401:600]+0.7, pch=19, col="blue")
lines(x[401],y[400]+0.65, pch=19, type = "b", col="blue")
lines(x[601],y[600]-0.04,  pch=19, type = "b", col="red")

lines(x[601:800],z[601:800], pch=19, col="blue")
lines(x[601],y[6.01]+0.7, pch=19, type = "b", col="blue")
```

```
lines(x[801],y[8.00]+0.49,  pch=19, type = "b", col="red")

lines(x[801:1000],w[801:1000]+0.7, pch=19, col="blue")
lines(x[801],y[10.0]+0.9, pch=19, type = "b", col="blue")
lines(x[1000],y[10.0]+1.45,  pch=19, type = "b", col="red")

# Code generating example of approximation of cadlag path
library("sm")
set.seed(123)

x <- seq(0, 10, by=0.01)
y<- -sin(x)-(2/3)*sin(2*x)-(2/3)^2*sin(2^2*x)-(2/3)^3*sin(2^3*x)
-(2/3)^4*sin(2^4*x)-(2/3)^5*sin(2^5*x)
z<- -(2/3)^3*sin(2^3*x)-(2/3)^4*sin(2^4*x)-(2/3)^5*sin(2^5*x)
w<- -(2/3)^2*sin(2^2*x)-(2/3)^3*sin(2^3*x)-(2/3)^4*sin(2^4*x)

# First function plot and its approximation
plot(x,y,"l",lwd=2, col= 0)
lines(x[1:200],y[1:200], pch=19, col="blue")
lines(x[201],y[159]+0.05, pch=19,type = "b", col="red")

#Approximation with Step Functions
lines(x[1:150],rep(-0.52,150), pch=19, col="red")
lines(x[150],y[150], pch=19, type = "b", col="red")

lines(x[150:200],rep(-1.0,51), pch=19, col="red")
lines(x[150],y[150]-0.5, pch=19, type = "b", col="blue")

# -----------Second piece of the function---------------
lines(x[201:400],z[201:400], pch=19, col="blue")
lines(x[200],y[200]+0.9, pch=19, type = "b", col="blue")
lines(x[401],y[400]-0.25, pch=19, type = "b", col="red")

# Its approximation with step function
lines(x[200:235],rep(-0.09,36), pch=19, col="red")
lines(x[235],y[235]-0.14, pch=19, type = "b", col="red")

lines(x[235],y[235]+0.35, pch=19, type = "b", col="blue")
lines(x[235:310],rep(0.44,76), pch=19, col="red")
lines(x[310],y[310]-0.09, pch=19, type = "b", col="red")

lines(x[310],y[310]-0.95, pch=19, type = "b", col="blue")
lines(x[310:400],rep(-0.44,91), pch=19, col="red")

#-----------Middle function plot commands---------------
lines(x[401:600],w[401:600]+0.7, pch=19, col="blue")
lines(x[401],y[400]+0.65, pch=19, type = "b", col="blue")
```

```
lines(x[601],y[600]-0.04, pch=19, type = "b", col="red")

# Its approximation with step functions
lines(x[400:480],rep(0.44,81), pch=19, col="red")
lines(x[480],y[400]+0.64, pch=19, type = "b", col="red")
lines(x[480],y[400]+1.4, pch=19, type = "b", col="blue")
lines(x[480:600],rep(1.2,121), pch=19, col="red")
lines(x[600],y[600]+0.08, pch=19, type = "b", col="red")

# ------------Second Last Function commands ------------
lines(x[601:800],z[601:800], pch=19, col="blue")
lines(x[601],y[6.01]+0.7, pch=19, type = "b", col="blue")
lines(x[801],y[8.00]+0.49, pch=19, type = "b", col="red")

# Its approximation with step functions
lines(x[601:630],rep(0.15,30), pch=19, col="red")
lines(x[630],y[400]+0.38, pch=19, type = "b", col="red")
lines(x[630],y[400]+0.6, pch=19, type = "b", col="blue")
lines(x[631:700],rep(0.40,70), pch=19, col="red")
lines(x[700],y[400]+0.64, pch=19, type = "b", col="red")

lines(x[708],y[400]-0.28, pch=19, type = "b", col="blue")
lines(x[708:800],rep(-0.52,93), pch=19, col="red")
lines(x[800],y[400]-0.26, pch=19, type = "b", col="red")

# lines(x[785],y[400]+0.05, pch=19, type = "b", col="blue")
# lines(x[785:800],rep(-0.17,16), pch=19, col="red")

#-----------Last piece function plot commands------------
lines(x[801:1000],w[801:1000]+0.7, pch=19, col="blue")
lines(x[801],y[10.0]+0.9, pch=19, type = "b", col="blue")
lines(x[1000],y[10.0]+1.45, pch=19, type = "b", col="red")

#Its approximation with step function
lines(x[801:950],rep(0.09,150), pch=19, col="red")
lines(x[950],y[1000]+0.2, pch=19, type = "b", col="red")
lines(x[951:1000],rep(0.65,50), pch=19, col="red")
lines(x[950],y[1000]+0.75, pch=19, type = "b", col="blue")

# code generating example of ladlag path
library("sm")
set.seed(123)

x <- seq(0, 10, by=0.01)
y<- -sin(x)-(2/3)*sin(2*x)-(2/3)^2*sin(2^2*x)-(2/3)^3*sin(2^3*x)
-(2/3)^4*sin(2^4*x)-(2/3)^5*sin(2^5*x)
z<- -(2/3)^3*sin(2^3*x)-(2/3)^4*sin(2^4*x)-(2/3)^5*sin(2^5*x)
```

```
w<- -(2/3)^2*sin(2^2*x)-(2/3)^3*sin(2^3*x)-(2/3)^4*sin(2^4*x)

plot(x,y,"l",lwd=2, col= 0)
lines(x[1],y[1]+0.05, pch=19, type = "b", col="red")
lines(x[1:200],y[1:200], pch=19, col="blue")
lines(x[201],y[159]+0.05, pch=19, type = "b", col="blue")

lines(x[201:400],z[201:400], pch=19, col="blue")
lines(x[200],y[200]+0.75, pch=19,  type = "b", col="red")
lines(x[401],y[400]-0.25, pch=19, type = "b", col="blue")

lines(x[401:600],w[401:600]+0.7, pch=19, col="blue")
lines(x[401],y[400]+0.63,  pch=19, type = "b", col="red")
lines(x[601],y[600]-0.04, pch=19, type = "b", col="blue")

lines(x[601:800],z[601:800], pch=19, col="blue")
lines(x[601],y[6.01]+0.62,   pch=19, type = "b", col="red")
lines(x[801],y[8.00]+0.49,pch=19, type = "b", col="blue")

lines(x[801:1000],w[801:1000]+0.7, pch=19, col="blue")
lines(x[801],y[10.0]+0.76,  pch=19,  type = "b", col="red")
lines(x[1000],y[10.0]+1.45,pch=19, type = "b", col="blue")

#Typical finite variation process vs Brownian motion

library("sm")

N <- 10000
K <- 2
mu <- 0
c <- .3

set.seed(123)

## generowanie w ##
x <- rnorm(K*N)
b <- cumsum(x)/sqrt(N)
t <- seq(0,1,length = K*N)

###################
####### DTV ########
###################

## inicjalizacja ##
w <- -b - mu*t

Wsup <- 0
```

```
Winf <- 0
Winff <- 0
du <- 0
dtv <- 0
renewals <- 0

DTV <- c()

## petla wewnetrzna ###

for (i in 1:(K*N)){

if(w[i]>Wsup)

{
Wsup <- w[i]
Winff <- Winf
du1 <- Wsup - Winff
if (du >= c){
dtv <- dtv + du1 - du}
du <- du1
DTV <- c(DTV, dtv)
}#if(w[i]>Wsup)

else {

if(Wsup - w[i] >=c)
{
renewals <- renewals +1
Wsup <- w[i]
Winf <- w[i]
Winff <- w[i]
du <- 0
}#if(Wsup - w[i] >=c)
else
if(w[i]<Winf)
{
Winf <- w[i]
}#if(w[i]<Winf)

DTV <- c(DTV, dtv)

}#else


}#for(i in 1:N)
```

```
####################
####### UTV ########
####################

## inicjalizacja ##
w <- b + mu*t

Wsup <- 0
Winf <- 0
Winff <- 0
du <- 0
utv <- 0
renewals <- 0

UTV <- c()

## petla wewnetrzna ###

for (i in 1:(K*N)){

if(w[i]>Wsup)

{
Wsup <- w[i]
Winff <- Winf
du1 <- Wsup - Winff
if (du >= c){
utv <- utv + du1 - du}
du <- du1
UTV <- c(UTV, utv)
}#if(w[i]>Wsup)

else {

if(Wsup - w[i] >=c)
{
renewals <- renewals +1
Wsup <- w[i]
Winf <- w[i]
Winff <- w[i]
du <- 0
}#if(Wsup - w[i] >=c)
else
if(w[i]<Winf)
```

```
{
Winf <- w[i]
}#if(w[i]<Winf)

UTV <- c(UTV, utv)

}#else


}#for(i in 1:N)


plot(t, UTV-DTV-0.3, type = "l", col = 4, lwd = 2, main = "Typical
Brownian␣Motion␣[0,1],␣c=␣0.3", ylim = c(-1.2,0.8), cex=0.2)
#pause()
lines(t, w, type = "l", col = 3)

legend(0.6,0.98, legend = c("Brownian␣Motion", "Finite␣Variation"),
col=c("green", "blue"), box.lty=0, lty=1:1)
```

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