Pathwise functional Itô calculus and its applications to mathematical finance.

by

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DISSERTATION
Submitted in fulfilment of the
requirements for the degree of

Master of Science
in
Applied Mathematics

in the

FACULTY OF SCIENCE AND AGRICULTURE
(School of Mathematical and Computer Sciences)

at the

UNIVERSITY OF LIMPOPO
Supervisor: Dr. F J Mhlanga

2019

Declaration

I declare that the dissertation hereby submitted to the University of Limpopo, for the degree of Master of Science in Applied Mathematics has not previously been submitted by me for a degree at this or any other university; that it is my work in design and in execution, and that all material contained herein has been duly acknowledged.

Mr Nkosi S C August 2019

Acknowledgements

First and foremost I would like to thank my God who continually strengthened me throughout the journey and made it possible for me to prevail. Greatest gratitude also goes to the NRF and AIMS-SA for their financial support throughout my studies. I also appreciate the opportunity given to further my studies by the University of Limpopo, Department of Mathematics and Applied Mathematics.

Greatest appreciation is to Dr. Farai J. Mhlanga, for his supervision of the thesis and the provision of relevant materials to the topic. His suggestions and inputs are beyond measure. I also want to acknowledge my family, the Nkosi, the Masina and the Hlatshwayo family, and also my friends, Lesiba C. Galane, Shadrack M. Kgomo, Thembelihle P. Hlatshwayo and Richard K. Mahlakwana for their prayers and unwavering support. Thank you guys.

Dedication

This work is dedicated to the Nkosi, Masina and Hlatshwayo families, your prayers and spiritual support kept me going, even through the darkest moments. May the Almighty God richly bless you.

Abstract

Functional Itô calculus is based on an extension of the classical Itô calculus to functionals depending on the entire past evolution of the underlying paths and not only on its current value. The calculus builds on Föllmer's deterministic proof of the Itô formula Föllmer (1981) and a notion of pathwise functional derivative recently proposed by Dupire (2019). There are no smoothness assumptions required on the functionals, however, they are required to possess certain directional derivatives which may be computed pathwise, see Cont and Fournié (2013); Schied and Voloshchenko (2016a); Cont (2012).

In this project we revise the functional Itô calculus together with the notion of quadratic variation. We compute the pathwise change of variable formula utilizing the functional Itô calculus and the quadratic variation notion. We study the martingale representation for the case of weak derivatives, we allow the vertical operator, ∇_X, to operate on continuous functionals on the space of square-integrable $\mathcal{F}_t$-martingales with zero initial value. We approximate the hedging strategy, H, for the case of path-dependent functionals, with Lipschitz continuous coefficients. We study some hedging strategies on the class of discounted market models satisfying the quadratic variation and the non-degeneracy properties. In the classical case of the Black-Scholes, Greeks are an important part of risk-management so we compute Greeks of the price given by path-dependent functionals. Lastly we show that they relate to the classical case in the form of examples.

Keywords: stochastic calculus; functional calculus; horizontal derivative; vertical derivative; martingale representation; Euler approximation; path-dependence; Greeks.

Contents

Chapter 1
Introduction

Many problems in stochastic analysis and its applications to mathematical finance involves the study of path-dependent functionals of stochastic processes. It is common practice that portfolios are constructed from the entire past data, not just from the current market price or capitalisations. Therefore it is natural to ask whether there is a possibility to develop a calculus for portfolios that are generated by functionals of the entire past evolution of the market portfolio and other factors.

In his paper, using the concept of quadratic variation along a sequence of partitions, Föllmer (1981) provided a pathwise proof of the Itô formula. This result finds too much attention in the pathwise approach to stochastic analysis, and it is flexible enough to be extended to path-dependent functionals. In this regard, Föllmer's ideas gave birth to the works of Dupire (2019); Cont and Fournié (2010, 2013), who introduced a new type of stochastic calculus known as functional Itô calculus. Its basis are on the extension of the classical Itô formula to functionals depending on the entire past evolution of the underlying path, and not only its current value. While Dupire (2019) uses probabilistic arguments and Itô calculus on the representation, one can in fact do entirely without such arguments and derive results in a purely analytic framework without any reference to probability. This leads to the pathwise functional Itô calculus for non-anticipative functionals, which identifies the set of paths (cádlág) to which the calculus is applicable.

The representation of martingales as stochastic integrals Protter (2004) is an important result in stochastic analysis with many applications in mathematical finance. One of the challenges in this regard has been to obtain an explicit version of such representations. In this study we develop the functional Itô calculus, this calculus can be used to study portfolios that are generated by functionals of the entire past evolution of the market and also other factors.

We also derive approximations to the martingale representation formulae using pathwise functional Itô calculus. In order to make precise our goal, we introduce some mathematical notations.

Let W be a standard d-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_t)_{t\in[0,T]}$ is its natural filtration. Then for any square-integrable $\mathcal{F}_t$-measurable random variable H, there exists a unique $\mathcal{F}_t$-predictable process ϕ with $\mathbb{E}[\int_0^T (\phi(u))^t(\phi(u))\mathrm{d}u] < \infty$ such that

$$H = Y(T) = \mathbb{E}[H] + \int_0^T \phi \mathrm{d}W. \tag{1.0.1}$$

In many applications such as mathematical finance, one is interested in an explicit expression for ϕ which represents a hedging strategy. Expression for the integrand ϕ have been derived using a variety of methods and assumptions, using Markovian techniques, see Davis (1980), integration by parts, see Chen and Glasserman (2007), or Malliavin calculus, see Fournié et al. (1999). Some of these methods are limited to the case where Y is a Markov process, others requiring differentiability of H in Fretchet or Malliavin sense or an explicit form for the density. A systematic approach to obtaining martingale representation formula has been proposed in Cont and Fournié (2013) using the functional Itô calculus, see also Cont and Fournie (2010); Cont

(2012); Dupire (2019). It was shown in Cont and Fournié (2013) that for any square-integrable $(\mathcal{F}_t)$-martingale Y;

$$Y(t) = Y(0) + \int_0^t \nabla_X Y dW, \quad \mathbb{P} - a.s. \tag{1.0.2}$$

where $\nabla_X Y$ is the weak vertical derivative(to be defined later) of Y with respect to W, constructed as an L^2- limit of pathwise directional derivatives. The approach does not rely on any Markov property nor on the Gaussian structure of the Wiener space and is applicable to functionals of large class of Itô processes. This study builds on this approach to propose a general framework for computing explicit expression for the hedging strategy in a general setting in which H is allowed to be a functional of the solution of a stochastic differential equation with path-dependent coefficients:

$$dX(t) = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X(0) = x_0 \in \mathbb{R}^d \tag{1.0.3}$$

where $X_t = X(t \wedge \cdot)$ denotes the path stopped at t and $b : [0,T] \times D([0,T], \mathbb{R}^d) \to \mathbb{R}^d$, $\sigma : [0,T] \times D([0,T], \mathbb{R}^d) \to \mathbb{M}_d(\mathbb{R})$ are continuous non-anticipative functionals.

It is very difficult to predict the price of an option or a position involving multiple options as the market changes. This is due to the fact that the price does not always move in correspondence to the price of the underlying asset. There are other factors that contributes to the price of an option, and it is of high interest that we understand these factors and their effects. Now, the Greeks provides means of measuring the sensitivity of an option price by quantifying the factors. They are referred to as the Greeks because most of them are denoted by Greek letters, such as the delta, gamma, etc. However, some are not denoted by Greek letters, e.g. vega. In risk management the Greeks are vital tools in measuring the sensitivity of the value of the portfolio relative to a small change in an underlying parameter.

The computation of Greeks (Sensitivities of the price due to the underlying assets) is one of the most common applications in mathematical finances. Jazaerli and Saporito (2017), considered the functional Lie bracket to be the measure of path-dependence. In the Malliavin set up, the consideration of more general dynamics of a path-dependent volatility were not achieved. The functional Itô calculus and the measure of path-dependence proves to be crucial and able in achieving this and ultimately obtaining the Greeks.

Greeks are very important to derivatives traders who look to hedge their portfolios from adverse market conditions changes. We are going to dwell on the first-order derivatives, delta and vega. We will also look at the second-order derivative, gamma. We only concentrate on volatility models and we also give their Black-Scholes counterparts as examples.

1.1 Literature review

Itô stochastic calculus has proven to be a powerful and useful tool in the analysis of phenomenon involving random, irregular evolution in time Alos et al. (2001); Applebaum (2009). The central tool used is the Itô formula. The formula allows one to represent quantities in terms of stochastic integrals. We note that in many applications such as statistics of processes, physics or in mathematical finance, one is led to consider path-dependent functionals of stochastic processes. There has been a huge interest to extend the framework of stochastic calculus to such path-dependent functionals.

The Malliavin calculus, which is a weak differential calculus for functionals on the Wiener space, has been used to investigate various properties of Brownian functionals with path-dependent instruments. The Malliavin calculus naturally leads to representations of functionals in terms of anticipative processes Bally (2003); Watanabe et al. (1984). However, the Malliavin derivatives involve perturbations affecting the whole path (both past and future) of the process. This notion of perturbation is not readily interpretable in applications where the quantities are required to be non-anticipative.

Inspired by methods used by practitioners for the sensitivity analysis of path-dependent derivatives Dupire (2019) introduced a new notion of functional derivatives and used it to extend the Itô formula to the path-dependence case. Following Dupire's work, Cont and Fournié (2010, 2013); Cont and Fournie (2010) developed a rigorous mathematical framework for a path-dependent extension of the Itô calculus, the functional Itô calculus. The authors proved the pathwise nature of some of the results obtained in the probabilistic framework.

Schied and Voloshchenko (2016a) used the functional Itô calculus to establish the associativity property of pathwise Itô integral in a functional setting for continuous integrators. In addition, the authors derived the existence and uniqueness results for the Itô differential equations. Schied et al. (2016) used the same calculus to prove a pathwise version of the master formula in Fernholz' stochastic theory for non-anticipative functions. This is the portfolio-generating function which depends on the entire history of the asset trajectories and additional continuous trajectory of bounded variation.

The pathwise functional Itô calculus has only been particularly popular in Mathematical Finance and economics recently Bender et al. (2008); Davis et al. (2014); Schied and Voloshchenko (2016b). This is due to the fact that the results derived with the help of the pathwise functional Itô calculus are robust with respect to model risk that might stem from a misspecification of probabilistic dynamics. The calculus has also been used to obtain representation of martingales as stochastic integrals Cont and Fournié (2013); Cont and Lu (2016). The martingale representation formula allows an integration by parts formula for Itô stochastic integrals which in turn enables the definition of a weak functional derivative for a class of square-integrable functions.

In the context of Cont and Fournié (2013), the present study focuses on pathwise functional Itô calculus. Specifically, we will develop the calculus and explore its applications to mathematical finance. The calculus will be used to derive explicit martingale representations which are useful in obtaining hedging strategies.

Jazaerli and Saporito (2017) uses this calculus to introduce a measure of path-dependence of functionals within the functional Itô calculus framework, and they are exposed to an alternative approach to computation of Greeks (Sensitivity of the price due to the underlying assets) than the classical way, which may include the Malliavin calculus, this is a calculus that looks ahead, we may view it more like an anticipative functional.

1.2 Structure of the thesis

We study the extension of Itô calculus to a path-dependent functional setting, which is more like a calculus that uses past data to decide the present(non-anticipative) unlike the other settings like Malliavin calculus, which looks into the future, by means of perturbing the path at both past and future(anticipative).

In the introduction, we introduced our problem and how we propose to deal with it. We also look at the literature. Lastly, we provide the structure of the thesis.

In Chapter 2, we study the pathwise calculus for non-anticipative functionals, this is based on Föllmer's approach to the Itô calculus. We give the basic concepts and some preliminary definitions and results, which enables us to understand the calculus better. The cornerstone of this calculus comes in the form of the Dupire derivative, also known as the vertical derivative. This derivative is on the space of right-continuous paths with left limits, i.e. cádlág paths. We establish the fact that the horizontal and vertical derivative do not commute, and we define the functional Lie bracket, a result that is useful for this study, which plays a crucial role in the derivation of Greeks.

In Chapter 3, we establish the finite quadratic variation along a sequence of partitions for paths in the space of right-continuous paths with left limits(cádlág). We use the pathwise calculus and the notion of finite quadratic variation to derive the change of variable formula for functionals of continuous paths. We then establish that the representation is not unique, in fact, the definition of the vertical derivative $\nabla_\omega F$ which involves evaluating the functional F on cádlág paths, seems to be dependent on the choice of the representation of our functional.

In Chapter 4, we then extend the Dupire derivative ∇_ω to the space of square-integrable martingales, we obtain a weaker version of the Dupire derivative, ∇_X. This allows us to extend the pathwise functional Itô calculus to a weak calculus for non-anticipative functionals whose domain of applicability involves all square-integrable semimartingales adapted to the filtration generated by X. We then establish the martingale representation formula for predictable processes. We also establish an important result, the integration by parts formula, which is a useful result in later chapters. We consider the class of discounted market models satisfying the quadratic variation and the non-degeneracy properties and some examples of finding hedges for path-dependent options are presented. We then study the approximation of some hedging strategies using the Euler approximation.

In Chapter 5, we start by looking at stochastic differential equations (SDEs) with coefficients that are path-dependent, establish conditions at which strong solutions exists. We then compute the Greeks (sensitivity of the price of derivatives such as options to a change in underlying parameters on which the value of an instrument or portfolio of financial instruments is dependent) in the setting of pathwise functional Itô calculus, inspired by Jazaerli and Saporito (2017). We also look at their Black-Scholes counterparts in the form of examples.

In the last chapter, we discuss shortfalls and strengths of this calculus, and also future works that may span from this study.

Chapter 2
Preliminaries and Concepts

In this chapter, we provide definitions and basic concepts that forms the basis of the details of this thesis. We also outline the notations used.

2.1 Notations

We adopt the notation in Cont (2012); Cont and Lu (2016). Let $T > 0$. We denote by $C^0([0,T],\mathbb{R}^d)$ the space of continuous functions on $[0,T]$ with values in $\mathbb{R}^d$ and we denote by $D([0,T],\mathbb{R}^d)$ the space of functions defined on $[0,T]$ with values in $\mathbb{R}^d$ which are right-continuous with left limits (cádlág). Let $C_{x_0} = \{X \in C^0([0,T],\mathbb{R}^d) \mid X(0) = x_0\}$ be a space of continuous martingales starting at x_0. We denote by S_d^+ the set of positive symmetric $d \times d$ real-valued matrices and denote by $\mathbb{M}_d(\mathbb{R}) = \mathbb{M}_{d,d}(\mathbb{R})$ the set of all $d \times d$ matrices with real coefficients. For a path $\omega \in D([0,T],\mathbb{R}^d)$ and $t \in [0,T]$, we denote by:

- $\omega(t)$ the value ω at time t,
- $\omega(t-) = \lim_{s\to t, s<t} \omega(s)$ its left limit at t,
- $\omega_t = \omega(t \wedge \cdot)$ the path of ω stopped at t,
- $\omega_{t-} = \omega \mathbb{1}_{[0,t)} + \omega(t-)\mathbb{1}_{[t,T]}$,
- $\|\omega\|_\infty = \sup\{|\omega(t)|, t \in [0,T]\}$ the supremum norm.

We note that $\omega_t,\ \omega_{t-} \in D([0,T],\mathbb{R}^d)$. For a cádlág stochastic process X, similarly, we denote by

- $X(t)$ its value,
- $X_t = (X(u \wedge t),\ 0 \le u \le T)$ the process stopped at t,
- $X_{t-}(u) = X(u)\mathbb{1}_{[0,t)}(u) + X(t-)\mathbb{1}_{[t,T]}(u)$.

Table 2.1: Some notations and their descriptions.

Notation	**Description**
$T > 0$	The Maturity time.
$C^0([0,T],\mathbb{R}^d)$	The space of continuous functions.
$D([0,T],\mathbb{R}^d)$	The space of cádlág functions.
Λ_T	The space of stopped paths.
$\mathcal{W}_T$	The space of continuous stopped paths.
$V(\Lambda_T)$	The space of vertical $1-$forms integrands.
$\mathbb{S}(\Lambda_T)$	The space of simple predictable cylindrical functionals.
L^2	The space of square-integrable functionals.
$\mathcal{M}^2(X)$	The space of square-integrable $\mathcal{F}_t-$martingales.
$\mathcal{L}^2$	The Hilbert space of $\mathcal{F}_t-$predictable processes.
C_{x_0}	The space of continuous martingales starting at x_0.
$\mathcal{M}_\sigma$	The class of discounted market models satisfying the quadratic variation and the non-degeneracy properties.
$D(X)$	The space of square-integrable stochastic integrals with respect to X which are $\mathcal{F}_t-$adapted processes which admits a $\mathbb{C}^{1,2}-$representation.
$\omega_t = \omega(t \wedge \cdot)$	The path stopped at t.
$X_t = (X(u \wedge t),\ 0 \leq u \leq T)$	The process stopped at t.
S_d^+	The set of positive symmetric $d \times d$ matrices.
$\mathbb{M}_d(\mathbb{R})$	The set of $d \times d$ matrices.
$\mathbb{B}$	The set of boundedness-preserving functionals.
$\mathbb{C}_l^{0,0}$	The set of left-continuous non-anticipative functionals.
$\mathbb{C}_r^{0,0}$	The set of right-continuous non-anticipative functionals.
$\mathbb{C}^{0,0}$	The set of continuous non-anticipative functionals.
$\mathbb{C}_b^{1,k}$	The set of horizontal and up to k^{th} vertical derivatives are boundedness-preserving.
$Q^\pi([0,T],\mathbb{R}^d)$	The set of $\mathbb{R}^d-$valued cádlág paths with finite quadratic variation.
$BV([0,T])$	The set of functions with bounded variation over the interval $[0,T]$.
C_{σ,x_0}	The set of positive valued functions starting at x_0.
d_∞	The distance structure on the space of stopped paths.
$\mathfrak{L}F$	The functional Lie bracket of F.
π_n	The partition of the interval $[0,T]$ into small equal intervals.
$\lvert\pi_n\rvert$	The mesh size of the partition π_n
$[x]$	The quadratic variation of x along a sequence of partitions.
$\langle A,B\rangle = \text{tr}\,({}^tA \cdot B)$	The Hilbert-Schmidt scalar product of two real $d \times d$ matrices.
∇_X	Weak vertical derivative.
τ	Stopping time.
$\sigma(X_t)$	Volatility.
$\lVert\cdot\rVert_\infty$	Supremum norm.
ODE	Ordinary Differential Equation.
PDE	Partial Differential Equation.
SDE	Stochastic Differential Equation.
LHS	Left hand side.
RHS	Right hand side.
$:=$	LHS is by definition equal to RHS.
$=:$	RHS is by definition equal to LHS.

2.2 Basic concepts and results

We now state some basic concepts and results, following Cont (2012). Let X be the canonical process on $\Omega = D([0,T], \mathbb{R}^d)$, and $(\mathcal{F}_t^0)_{t\in[0,T]}$ be the filtration generated by X. We focus on non-anticipative functionals, that is, $F : [0,T] \times D([0,T], \mathbb{R}^d) \to \mathbb{R}$ such that

$$\forall \omega \in \Omega,\ F(t,\omega) = F(t,\omega_t). \tag{2.2.1}$$

The process $t \mapsto F(t,\omega_t)$ then only depends on the path ω up to time t and is $(\mathcal{F}_t^0)$-adapted. We notice in Cont (2012) that it is convenient to define such functionals on the space of stopped paths. A stopped path is an equivalence class in $[0,T]\times D([0,T], \mathbb{R}^d)$ for the following equivalence relation:

$$(t,\omega) \sim (t',\omega') \Leftrightarrow (t = t' \text{ and } \omega_t = \omega'_{t'}). \tag{2.2.2}$$

The space of stopped paths is defined as the quotient of $[0,T] \times D([0,T], \mathbb{R}^d)$ by the equivalence relation (2.2.2); that is,

$$\Lambda_T := \{(t,\omega_t), (t,\omega) \in [0,T] \times D([0,T], \mathbb{R}^d)\} = [0,T] \times D([0,T], \mathbb{R}^d)/\sim .$$

We denote by $\mathcal{W}_T$ the subset of Λ_T consisting of continuous stopped paths, that is,

$$\mathcal{W}_T := \{(t,\omega) \in \Lambda_T,\ \omega \in C^0([0,T], \mathbb{R}^d)\}.$$

We equip Λ_T with a metric space structure by defining the distance,

$$d_\infty((t,\omega),(t',\omega')) = \sup_{u\in[0,T]} |\omega(u\wedge t) - \omega'(u \wedge t')| + |t-t'| = \|\omega_t - \omega'_{t'}\|_\infty + |t-t'|. \tag{2.2.3}$$

(Λ_T, d_∞) is then a complete metric space. It is noted that any functional verifying the non-anticipative condition (2.2.1) can be equivalently be viewed as a functional on the space Λ_T of stopped paths.

Definition 2.2.1. *A non-anticipative functional on $D([0,T], \mathbb{R}^d)$ is a measurable map $F : (\Lambda_T, d_\infty) \to \mathbb{R}$ on the space (Λ_T, d_∞) of stopped paths.*

Definition 2.2.2. *(Predictable functionals) A non-anticipative functional $F : (\Lambda_T, d_\infty) \to \mathbb{R}$ is called predictable if*

$$\forall (t,\omega) \in \Lambda_T,\ F(t,\omega) = F(t,\omega_{t-}). \tag{2.2.4}$$

Now that we have equipped the space of stopped paths with the metric d_∞, we now can define some notions of continuity for non-anticipative functionals, see Cont and Fournié (2010); Cont (2012); Cont and Lu (2016).

Definition 2.2.3. *A non-anticipative functional F is said to be :*

1. *continuous at fixed times if for any $t \in [0,T]$, $F(t,\cdot) : D([0,T], \|\cdot\|_\infty) \to \mathbb{R}$ is continuous, i.e. $\forall \omega \in D([0,T], \mathbb{R}^d)$, $\forall \epsilon > 0$, $\exists \eta > 0$, $\forall \omega' \in D([0,T], \mathbb{R}^d)$,*

$$\|\omega_t - \omega'_t\|_\infty < \eta \ \Rightarrow\ |F(t,\omega) - F(t,\omega')| < \epsilon,$$

2. *continuous if* $\forall (t,\omega) \in \Lambda_T,\ \forall \epsilon > 0,\ \exists \eta > 0$ *such that* $\forall (t',\omega') \in \Lambda_T$,

$$d_\infty((t,\omega),(t',\omega')) < \eta \ \Rightarrow\ |F(t,\omega) - F(t',\omega')| < \epsilon,$$

3. *left-continuous if* $\forall (t,\omega) \in \Lambda_T,\ \forall \epsilon > 0,\ \exists \eta > 0$ *such that* $\forall (t',\omega') \in \Lambda_T$,

$$(t' < t \text{ and } d_\infty((t,\omega),(t',\omega')) < \eta) \ \Rightarrow\ |F(t,\omega) - F(t',\omega')| < \epsilon,$$

4. *right-continuous if* $\forall (t,\omega) \in \Lambda_T,\ \forall \epsilon > 0,\ \exists \eta > 0$ *such that* $\forall (t',\omega') \in \Lambda_T$,

$$(t < t' \text{ and } d_\infty((t,\omega),(t',\omega')) < \eta) \ \Rightarrow\ |F(t,\omega) - F(t',\omega')| < \epsilon.$$

Remark 2.2.4. The set of left-continuous non-anticipative functionals is denoted by $\mathbb{C}_l^{0,0}(\Lambda_T)$. Similarly, the set of right-continuous non-anticipative functionals is denoted by $\mathbb{C}_r^{0,0}(\Lambda_T)$. We denote by $\mathbb{C}^{0,0}(\Lambda_T) = \mathbb{C}_l^{0,0}(\Lambda_T) \cap \mathbb{C}_r^{0,0}(\Lambda_T)$ the set of continuous non-anticipative functionals.

The following definition gives the notion of local boundedness for non-anticipative functionals Ananova and Cont (2017); Cont and Fournié (2010); Cont (2012); Cont and Lu (2016); Lindensjö (2016).

Definition 2.2.5. *A non-anticipative functional F is said to be boundedness-preserving if for every compact subset K of $\mathbb{R}^d$, $\forall t_0 \in [0,T],\ \exists C(K,t_0) > 0$ such that*

$$\forall t \in [0,t_0],\ \forall (t,\omega) \in \Lambda_T,\ \omega([0,t]) \subset K \ \Rightarrow\ |F(t,\omega)| < C(K,t_0). \tag{2.2.5}$$

We denote by $\mathbb{B}(\Lambda_T)$ the set of boundedness-preserving functionals

Now follows some useful pathwise regularities Cont and Fournié (2010):

Lemma 2.2.6. *(Pathwise regularity).*

1. *If $F \in \mathbb{C}_l^{0,0}(\Lambda_T)$ then for any $\omega \in D([0,T],\mathbb{R}^d)$, the path $t \mapsto F(t,\omega_{t-})$ is left-continuous.*
2. *If $F \in \mathbb{C}_r^{0,0}(\Lambda_T)$ then for any $\omega \in D([0,T],\mathbb{R}^d)$, the path $t \mapsto F(t,\omega_t)$ is right-continuous.*
3. *If $F \in \mathbb{C}^{0,0}(\Lambda_T)$ then for any $\omega \in D([0,T],\mathbb{R}^d)$, the path $t \mapsto F(t,\omega_t)$ is cádlág and continuous at all points where ω is continuous.*
4. *If $F \in \mathbb{B}(\Lambda_T)$ then for any $\omega \in D([0,T],\mathbb{R}^d)$, the path $t \mapsto F(t,\omega_t)$ is bounded.*

Proof. 1. Suppose that $F \in \mathbb{C}_l^{0,0}(\Lambda_T)$ and $t \in [0,T)$. For $h > 0$ small enough,

$$d_\infty((t-h,\omega_{t-h}),(t,\omega_{t-})) = d_\infty((t',\omega_{t'}),(t,\omega_{t-}))$$

where $t' = t - h$. Now,

$$d_\infty((t',\omega_{t'}),(t,\omega_{t-})) = \|\omega_{t'} - \omega_{t-}\|_\infty + |t' - t|.$$

Since ω is cádlág, this quantity converges to 0 as $t \to t'^+$, so

$$F(t',\omega_{t'}) - F(t,\omega_{t-}) \to 0.$$

Hence $t \mapsto F(t,\omega_{t-})$ is left-continuous.

2. Suppose that $F \in \mathbb{C}_r^{0,0}(\Lambda_T)$ and $t \in [0,T)$. For $h > 0$ small enough,
$$d_\infty((t+h,\omega_{t+h}),(t,\omega_t)) = d_\infty((t',\omega_{t'}),(t,\omega_t))$$
where $t' = t+h$. Now,
$$d_\infty((t',\omega_{t'}),(t,\omega_t)) = \|\omega_{t'} - \omega_t\|_\infty + |t'-t|.$$
Since ω is cádlág, this quantity converges to 0 as $t' \to t^+$, so
$$F(t',\omega_{t'}) - F(t,\omega_t) \to 0.$$
Hence $t \mapsto F(t,\omega_t)$ is right-continuous.

3. Suppose that $F \in \mathbb{C}^{0,0}(\Lambda_T)$ and $t \in (0,T]$. We denote by $\Delta\omega(t)$ the jump of ω at t. Then
$$d_\infty((t-h,\omega_{t-h}),(t,\omega_t^{-\Delta\omega(t)})) = d_\infty((t',\omega_{t'}),(t,\omega_t^{-\Delta\omega(t)}))$$
where $t' = t-h$. Now,
$$d_\infty((t',\omega_{t'}),(t,\omega_t^{-\Delta\omega(t)})) = \sup_{u\in[t',t)} |\omega(u\wedge t') - \omega(u\wedge t) + \Delta\omega(t)\mathbb{1}_{[t',t)}| + |t'-t|.$$
Since ω has left limit, this quantity converges to 0 as $t \to t'^+$. Hence the path has left limit $F(t,\omega_t^{-\Delta\omega(t)})$ at t. Similarly, it has right limit $F(t,\omega_t)$.

4. If $F \in \mathbb{B}(\Lambda_T)$ then by definition, for $t \in [0,T]$ and $\omega \in D([0,T],\mathbb{R}^d)$, $|F(t,\omega_t)| < C(K,t)$ for some constant C. Hence, F is bounded.

□

2.3 Horizontal and vertical derivatives

We now recall some notions of differentiability for non-anticipative functionals following Cont and Fournié (2010); Cont (2012); Cont and Fournié (2013); Cont and Lu (2016); Dupire (2019). We first consider the case of a non-anticipative functional F which is applied to a piecewise constant path
$$\omega = \sum_{k=1}^{n} x_k \mathbb{1}[t_k,t_{k+1}) \in D([0,T],\mathbb{R}^d).$$
The piecewise-constant path ω is obtained by a finite sequence of operations consisting of

1. "horizontal stretching" of a path from t_k to t_{k+1} followed by
2. the addition of a jump at each discontinuity point.

In respect of the stopped path (t,ω), these two operations correspond to, respectively,

1. incrementing the first component:
$$(t_k,\omega_{t_k}) \to (t_{k+1},\omega_{t_k})$$

2. shifting the path by $(x_{k+1} - x_k)\mathbb{1}_{[t_{k+1},T]}$:

$$\omega_{t_{k+1}} = \omega_{t_k} + (x_{k+1} - x_k)\mathbb{1}_{[t_{k+1},T]}.$$

We note that the variation of a non-anticipative functional along ω can also be decomposed into the corresponding "horizontal" and "vertical" increments:

$$F(t_{k+1}, \omega_{t_{k+1}}) - F(t_k, \omega_{t_k}) = \underbrace{F(t_{k+1}, \omega_{t_{k+1}}) - F(t_{k+1}, \omega_{t_k})}_{\text{vertical increment}} + \overbrace{F(t_{k+1}, \omega_{t_k}) - F(t_k, \omega_{t_k})}^{\text{horizontal increment}}.$$

Therefore, if one control the behaviour of F under these two types of path perturbations, then one can compute the variation of F along any piecewise-constant path ω.

We now recall the notion of horizontal extensions, consider a path $\omega \in D([0,T], \mathbb{R}^d)$ and denote by $\omega_t \in D([0,T], \mathbb{R}^d)$ its restriction to $[0,t]$ for $t < T$. For $h \geq 0$, the horizontal extension of a stopped path (t, ω_t) to $[0, t+h]$ is the stopped path $(t+h, \omega_t)$.

Definition 2.3.1. *(Horizontal derivative). Cont and Fournié (2010) A non-anticipative functional $F : \Lambda_T \to \mathbb{R}$ is said to be horizontally differentiable at $(t, \omega) \in \Lambda_T$ if the limit*

$$\lim_{h \to 0^+} \frac{F(t+h, \omega_t) - F(t, \omega_t)}{h} =: \mathcal{D}F(t, \omega) \tag{2.3.1}$$

exists. If $\mathcal{D}F(t,\omega)$ exists for all $(t,\omega) \in \Lambda_T$, then Equation (2.3.1) defines a non-anticipative function $\mathcal{D}F$, called the horizontal derivative of F at (t, ω).

Consider $e \in \mathbb{R}^d$ and $\omega \in D([0,T], \mathbb{R}^d)$, we define the vertical perturbation ω_t^e of ω_t as a cádlág path obtained by shifting the path ω by e at time t,

$$\omega_t^e = \omega_t + e\mathbb{1}_{[t,T]}$$

where $\mathbb{1}_{[t,T]}$ is the indicator function.

Definition 2.3.2. *(Vertical derivative). Dupire (2019) The vertical derivative of a non-anticipative functional F is defined as*

$$\nabla_\omega F(t, \omega) = (\partial_i F(t, \omega),\ i = 1, \ldots, d)$$

with

$$\partial_i F(t, \omega) = \lim_{h \to 0} \frac{F(t, \omega_t + he_i \mathbb{1}_{[t,T]}) - F(t, \omega_t)}{h},$$

where $\{e_i,\ i = 1, \ldots, d\}$ is the canonical basis of $\mathbb{R}^d$.

Remark 2.3.3. The first term in the numerator in Equation (2.3.1) depends on $\omega_t = \omega(t \wedge \cdot)$ not ω_{t+h}. If F is horizontally differentiable at all $(t, \omega) \in \Lambda_T$ then the map $\mathcal{D}F : (t, \omega) \to \mathcal{D}F(t, \omega)$ defines a non-anticipative functional which is $\mathcal{F}_t^0-$measurable.

Remark 2.3.4. $\partial_i F(t,\omega)$ is simply the directional derivative of $F(t,\omega)$ in the direction $\mathbb{1}_{[t,T]}e_i$. We also note that this involves examining cádlág perturbations of the path ω, even if ω is continuous. In addition, it is computed for t fixed and involves perturbing the endpoint of the path at t.

Remark 2.3.5. We note that $\mathcal{D}F(t,\omega)$ is not the usual partial derivative in t, that is,

$$\mathcal{D}F(t,\omega) \neq \partial_t F(t,\omega) = \lim_{h\to 0} \frac{F(t+h,\omega) - F(t,\omega)}{h}.$$

$\partial_t F(t,\omega)$ corresponds to a Lagrangian derivative which follows the increment of F along the path from t to $t+h$, whereas in Equation (2.3.1) the path is stopped at t and the increment is taken along the stopped path.

Remark 2.3.6. The Malliavin derivative applies shocks at any point of a full path. However, the non-anticipative functional derivatives (horizontal and vertical derivatives), for a given current path $X_t \in \Lambda_T$, our derivatives correspond to changes in the current value of the process and in the current time. This is different to the Malliavin derivative case.

Remark 2.3.7. The increment in the vertical derivative definition can be either positive or negative, whereas the increment in the horizontal derivative is only positive. Therefore, it is a right-derivative.

Example 2.3.8. If $F(t,X_t) = g(t,x_t)$, then

$$\nabla_\omega F = \frac{\partial g}{\partial x} \text{ and } \mathcal{D}F = \frac{\partial g}{\partial t}.$$

The horizontal and vertical satisfy the classical properties for derivatives: linearity, product and chain rule.

Remark 2.3.9. Let f be a real-valued function with $f \in C^{1,1}([0,T],\mathbb{R}^d)$ and $F(t,\omega) = f(t,\omega(t))$, then the horizontal and vertical derivatives reduce to the standard partial derivatives:

$$\mathcal{D}F(t,\omega_t) = \partial_t f(t,\omega(t)),\ \nabla_\omega F(t,\omega_t) = \nabla_\omega f(t,\omega(t)).$$

The horizontal derivative is an extension of the notion of partial derivative in time for non-anticipative functionals:

Example 2.3.10. Define

$$F(t,\omega) = \int_0^t g(\omega(u))\rho(u)\mathrm{d}u, \tag{2.3.2}$$

where $g \in C^0(\mathbb{R}^d)$ and $\rho : \mathbb{R}_+ \to \mathbb{R}$ are bounded and measurable. Then $F \in \mathbb{C}_b^{1,\infty}(\Lambda_T)$, with

$$\mathcal{D}F(t,\omega) = g(\omega(t))\rho(t) \text{ and } \nabla_\omega F(t,\omega) = 0.$$

Example 2.3.11. For $g \in C^0(\mathbb{R}^{m\times d}),\ h \in C^k(\mathbb{R}^d)$ with $h(0) = 0$, let

$$F(t,\omega) = h(\omega(t) - \omega(t_m-))\mathbb{1}_{t\geq t_m} g(\omega(t_1-),\omega(t_2-),\ldots,\omega(t_m-)). \tag{2.3.3}$$

Then

$$F \in \mathbb{C}_b^{1,k}(\Lambda_T) \text{ and } \mathcal{D}F(t,\omega) = 0,$$

and $\forall j = 1, 2, \ldots, k$

$$\nabla_\omega^j F(t,\omega) = \nabla_\omega^j h(\omega(t) - \omega(t_m-))\mathbb{1}_{t \geq t_m} g(\omega(t_1-), \omega(t_2-), \ldots, \omega(t_m-)).$$

If F admits a vertical derivative $\nabla_\omega F$ we may iterate the vertical derivative operation described above and define higher order vertical derivatives. One may repeat the operation on $\nabla_\omega F$ and define ∇_ω^k, for $k = 2, 3, \ldots$. For example, ∇_ω^2 is the gradient at 0 of the map $e \in \mathbb{R}^d \mapsto \nabla_\omega F(t, \omega + e\mathbb{1}_{[t,T]})$, provided it exists. We now define a special case of non-anticipative functionals which possess some already defined properties Cont (2012):

Definition 2.3.12. *(Class of $\mathbb{C}_b^{1,2}$ functionals). Define $\mathbb{C}_b^{1,2}(\Lambda_T)$ as the set of left-continuous functionals $F \in \mathbb{C}_l^{0,0}(\Lambda_T)$ such that*

1. *F admits a horizontal derivative $\mathcal{D}F(t,\omega)$ for all $(t,\omega) \in \Lambda_T$ and $\mathcal{D}F(t,\cdot) : (D([0,T],\mathbb{R}^d), \|\cdot\|_\infty) \to \mathbb{R}$ is continuous for each $t \in [0,T)$.*
2. *$\nabla_\omega F,\ \nabla_\omega^2 F \in \mathbb{C}_l^{0,0}(\Lambda_T)$.*
3. *$\mathcal{D}F,\ \nabla_\omega F,\ \nabla_\omega^2 F \in \mathbb{B}(\Lambda_T)$.*

Similarly, we can define the class $\mathbb{C}_b^{1,k}(\Lambda_T)$.
A function $\tau : D([0,T],\mathbb{R}^d) \to [0,\infty)$ such that for each $t \geq 0$, $\{\omega \in D([0,T],\mathbb{R}^d) : \tau(\omega) \leq t\} \in \mathcal{F}_t^0$ is called a stopping time for the filtration.

Example 2.3.13. Let F be defined by

$$F(t, X_t) = x^2(t) - \int_0^t \sigma^2(s, B_s)\mathrm{d}s.$$

Then

$$\mathcal{D}F(t,X_t) = -\sigma^2(t,B_t) \qquad \nabla_\omega F(t,X_t) = 2x(t)$$
$$\nabla_\omega^2 F(t,X_t) = 2 \qquad \nabla_\omega^j F(t,X_t) = 0,\ j \geq 3$$

Example 2.3.14. Let F be defined by

$$F(t,X_t) = e^{x(t) - \frac{1}{2}\int_0^t \sigma^2(s,B_s)\mathrm{d}s}.$$

Then

$$\begin{aligned}\mathcal{D}F(t,X_t) &= -\frac{1}{2}\sigma^2(t,B_t)e^{x(t)-\frac{1}{2}\int_0^t \sigma^2(s,B_s)\mathrm{d}s} \\ &= -\frac{1}{2}\sigma^2(t,B_t)F(t,X_t) \\ \nabla_\omega^j F(t,X_t) &= F(t,X_t) \qquad \forall j = 1,\ldots,k.\end{aligned}$$

Many examples of functionals may fail to be globally smooth, but their horizontal and vertical derivatives may still be well behaved except for at certain stopping times, which motivates the following definition Cont (2012).

Definition 2.3.15. *(Locally regular functionals $\mathbb{C}^{1,2}_{\mathrm{loc}}(\Lambda_T)$). $F \in \mathbb{C}^{0,0}_b(\Lambda_T)$ is said to be locally regular if there exists an increasing sequence $(\tau_k)_{k\geq 0}$ of stopping times with $\tau_0,\ \tau_k \uparrow \infty$ and a sequence of functionals $F^k \in \mathbb{C}^{1,2}_b(\Lambda_T)$ such that*

$$F(t,\omega) = \sum_{k\geq 0} F^k(t,\omega)\mathbb{1}_{[\tau_k(\omega),\tau_{k+1}(\omega))}(t), \quad \forall (t,\omega) \in \Lambda_T.$$

We note that $\mathbb{C}^{1,2}_b(\Lambda_T) \subset \mathbb{C}^{1,2}_{\mathrm{loc}}(\Lambda_T)$ but the notion of local regularity allows discontinuities or explosions at times described by $(\tau_k,\ k \geq 1)$.

Definition 2.3.16 (Cylindrical non-anticipative functionals)**.** *Let $\pi_n = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be a partition of $[0,T]$. Let also $\omega \in D([0,T],\mathbb{R}^d)$ and $g \in C^0(\mathbb{R}^{n\times d})$. Then the functional*

$$F(t,\omega) = \sum_{i=0}^{n} g_i(\omega(t_0-),\omega(t_1-),\ldots,\omega(t_n-))\mathbb{1}_{[t_i,t_{i+1})}(t) \tag{2.3.4}$$

is called a cylindrical functional.

We denote by $\mathbb{S}(\pi_n,\Lambda_T)$ the space of cylindrical non-anticipative piecewise-constant functionals along π_n.

Definition 2.3.17 (Cylindrical integrands)**.** *Let $\pi_n = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be a fixed partition of $[0,T]$. Define a cylindrical functional*

$$F(t,\omega) = \sum_{i=0}^{n} g_i(\omega(t_0-),\omega(t_1-),\ldots,\omega(t_n-))\mathbb{1}_{[t_i,t_{i+1})}(t)(\omega(t)-\omega(t_i)). \tag{2.3.5}$$

The vertical derivative

$$\nabla_\omega F(t,\omega) = \sum_{i=0}^{n} g_i(\omega(t_0-),\omega(t_1-),\ldots,\omega(t_n-))\mathbb{1}_{[t_i,t_{i+1})}(t) \in \mathbb{C}^{0,0}_l \cap \mathbb{B}$$

is called a cylindrical integrand.

2.4 Functional Lie bracket

If $F \in \mathbb{C}^{1,2}_b(\Lambda_T)$, then for any $(t,\omega) \in \Lambda_T$ with

$$\begin{aligned} \mathbb{R}^d &\to \mathbb{R} \\ e &\mapsto F(t,\omega + e\mathbb{1}_{[t,T]}), \end{aligned}$$

is twice continuously differentiable on a neighborhood of the origin and

$$\nabla_\omega F(t,\omega) = \partial_i F(t,\omega) \text{ and } \nabla_\omega^2 F(t,\omega) = \partial_i^2 F(t,\omega).$$

It can be shown that a second-order Taylor expansion of the map at the origin yields that any $F \in \mathbb{C}_b^{1,2}(\Lambda_T)$ admits a second-order Taylor expansion with respect to a vertical perturbation:

$$F(t,\omega + e\mathbb{1}_{[t,T]}) = F(t,\omega) + \nabla_\omega F(t,\omega)\cdot e + \langle e, \nabla_\omega^2 F(t,\omega)\cdot e\rangle + o\left(\|e\|^2\right).$$

Unlike the usual partial derivatives in finite dimensions, the horizontal and vertical derivatives do not commute. In general

$$\mathcal{D}(\nabla_\omega F) \neq \nabla_\omega(\mathcal{D}F).$$

This arises from the fact that the elementary operators of horizontal extension and vertical perturbation defined earlier on do not commute, that is, a horizontal extension of the stopped path (t,ω) to $t+h$ followed by a vertical perturbation yields the path $\omega_t + e\mathbb{1}_{[t+h,T]}$ while a vertical perturbation at t followed by a horizontal extension to $t+h$ yields

$$\omega_t + e\mathbb{1}_{[t,T]} \neq \omega_t + e\mathbb{1}_{[t+h,T]}.$$

We note that these two paths have the same value at $t+h$, so only functionals which are truly path-dependent will be affected by this lack of commutativity.

Example 2.4.1. Set

$$I(Y_t) = \int_0^t y_u \mathrm{d}u.$$

Then,

$$\begin{aligned} \mathcal{D}I(Y_t) &= y_t \\ \nabla_\omega I(Y_t) &= 0 \end{aligned}$$

and hence

$$\nabla_\omega(\mathcal{D}I)(Y_t) = 1 \neq 0 = \mathcal{D}(\nabla_\omega I)(Y_t).$$

The following definition can be used to quantify the path-dependency of F.

Definition 2.4.2. *The Lie bracket of the operators $\mathcal{D}$ and ∇_ω is defined by*

$$\mathfrak{L}F(t,\omega_t) := [\mathcal{D},\nabla_\omega]F(t,\omega_t) = \nabla_\omega(\mathcal{D}F)(t,\omega_t) - \mathcal{D}(\nabla_\omega F)(t,\omega_t),$$

where F is such that all the derivatives above exists.

Proposition 2.4.3. Suppose that the functional $F : \Lambda_T \to \mathbb{R}$ is given by $f(t,\omega_t) = \phi(t, f_1(\omega_1), f_2(\omega_2), \ldots, f_k(\omega_t))$, where $\phi : \mathbb{R}_+ \times \mathbb{R}^k \to \mathbb{R}$ has all the first and second order partial derivatives and the Lie bracket of f_i exists $\forall i = 1, 2, \ldots, k$. Then

$$\mathfrak{L}f(t,\omega_t) = \sum_{i=1}^k \frac{\partial \phi}{\partial \omega_i}(t, f_1(\omega_1), f_2(\omega_2), \ldots, f_k(\omega_t))\mathfrak{L}f_i(\omega_t).$$

Proof. We first note that

$$\nabla_\omega f(t,\omega_t) = \sum_{i=1}^{k} \frac{\partial \phi}{\partial \omega_i} \nabla_\omega f_i(\omega_t),$$
$$\mathcal{D} f(t,\omega_t) = \frac{\partial \phi}{\partial t} + \sum_{i=1}^{k} \frac{\partial \phi}{\partial \omega_i} \mathcal{D} f_i(\omega_t).$$

Then,

$$\mathcal{D}(\nabla_\omega f(t,\omega_t)) = \sum_{i=1}^{k} \frac{\partial^2 \phi}{\partial \omega_i \partial t} \nabla_\omega f_i(\omega_t) + \sum_{i=1}^{k} \frac{\partial \phi}{\partial \omega_i} \mathcal{D}(\nabla_\omega f_i(\omega_t))$$
$$+ \sum_{i=1}^{k}\sum_{j=1}^{k} \frac{\partial^2 \phi}{\partial \omega_i \partial \omega_j} \nabla_\omega f_i(\omega_t) \mathcal{D} f_i(\omega_t), \tag{2.4.1}$$
$$\nabla_\omega(\mathcal{D} f(t,\omega_t)) = \sum_{i=1}^{k} \frac{\partial^2 \phi}{\partial t \partial \omega_i} \nabla_\omega f_i(\omega_t) + \sum_{i=1}^{k} \frac{\partial \phi}{\partial \omega_i} \nabla_\omega(\mathcal{D} f_i(\omega_t))$$
$$+ \sum_{i=1}^{k}\sum_{j=1}^{k} \frac{\partial^2 \phi}{\partial \omega_i \partial \omega_j} \nabla_\omega f_j(\omega_t) \mathcal{D} f_i(\omega_t). \tag{2.4.2}$$

By the functional Lie bracket, we have

$$\begin{aligned}
\mathfrak{L} f(t,\omega_t) &= \sum_{i=1}^{k} \frac{\partial \phi}{\partial \omega_i} \mathcal{D}(\nabla_\omega f_i(\omega_t)) - \sum_{i=1}^{k} \frac{\partial \phi}{\partial \omega_i} \nabla_\omega(\mathcal{D} f_i(\omega_t)) \\
&= \sum_{i=1}^{k} \frac{\partial \phi}{\partial \omega_i} [\mathcal{D}(\nabla_\omega f_i(\omega_t)) - \nabla_\omega(\mathcal{D} f_i(\omega_t))] \\
&= \sum_{i=1}^{k} \frac{\partial \phi}{\partial \omega_i} \mathfrak{L} f_i(\omega_t).
\end{aligned} \tag{2.4.3}$$

□

Example 2.4.4. The operators ∇_ω and $\mathcal{D}$ only commute when applied to functionals of the form $F(t,Y_t) = \phi(t,y_t)$, where ϕ is smooth. Now set,

$$\mathbb{I}(Y_t) = \int_0^t \int_0^s y_u \mathrm{d}u \mathrm{d}s$$

and notice that

$$\mathcal{D}\mathbb{I}(Y_t) = \int_0^t y_s \mathrm{d}s$$
$$\nabla_\omega \mathbb{I}(Y_t) = 0$$

which implies that

$$\nabla_\omega(\mathcal{D}\mathbb{I})(Y_t) = 0 = \mathcal{D}(\nabla_\omega \mathbb{I})(Y_t).$$

Chapter 3
Pathwise integration and functional change of variable formula

Föllmer (1981) in his *seminal paper Calcul d'Itô sans probabilités* showed that if a cádlág function x has finite quadratic variation along some sequence $\pi_n = (0 = t_0^n < t_1^n < \cdots < t_n^n = T)$ of partitions of $[0,T]$ with step size decreasing to zero, then for $f \in C^2(\mathbb{R}^d)$ one can define the pathwise integral

$$\int_0^T \nabla f(x(t))\mathrm{d}^\pi x = \lim_{n\to\infty} \sum_{i=0}^{n-1} \nabla f(x(t_i^n))(x(t_{i+1}^n) - x(t_i^n)) \tag{3.0.1}$$

as a limit of Riemann sums along the sequence $\pi = (\pi_n)_{n\geq 1}$ of partitions and obtain a change of variable formula for this integral, see Föllmer (1981). It is through this approach and Dupire's derivatives that we obtain a pathwise change of variable formula for functionals in $\mathbb{C}^{1,2}_{\mathrm{loc}}(\Lambda_T)$, see Dupire (2019).

We denote $\langle A, B\rangle = tr({}^tA \cdot B)$ the Hilbert-Schmidt scalar product of two real $d \times d$ matrices.

3.1 Quadratic variation

We now introduce the notion of quadratic variation of a path along a sequence of partitions Cont (2012); Föllmer (1981). In this section we denote by $\pi = (\pi_n)_{n\geq 1}$ a sequence of partitions of $[0,T]$ into intervals:

$$\pi_n = \left(0 = t_0^n < t_1^n < t_2^n < \cdots < t_{k(n)}^n = T\right).$$

We denote $|\pi_n| = \sup\{|t_{i+1}^n - t_i^n|,\ i = 1,2,3,\ldots,k(n)\} \to 0$, as $n \to \infty$, the mesh size of the partition.

Definition 3.1.1. *(Quadratic variation of a path along a sequence of partitions). Let $\pi_n = (0 = t_0^n < t_1^n < t_2^n < \cdots < t_{k(n)}^n = T)$ be a sequence of partitions of $[0,T]$ with step size decreasing to zero, that is, $|\pi_n| \to 0$ as $n \to \infty$. A cádlág path $x \in D([0,T],\mathbb{R})$ is said to have finite quadratic variation along the sequence of partitions π if for any $t \in [0,T]$ the limit*

$$[x](t) := \lim_{n\to\infty} \sum_{t_{i+1}^n \leq t} \left(x\left(t_{i+1}^n\right) - x\left(t_i^n\right)\right)^2 < \infty \tag{3.1.1}$$

exists and the increasing function $[x]$ has Lebesgue decomposition

$$[x]_\pi(t) = [x]_\pi^c(t) + \sum_{0<s\leq t} |\Delta x(s)|^2 \tag{3.1.2}$$

where $[x]_\pi^c$ is a continuous and increasing function.

The increasing function $[x] : [0,T] \to \mathbb{R}_+$ defined by Equation (3.1.1) is called the quadratic variation of x along the sequence of partitions $\pi = (\pi_n)_{n\geq 1}$ and $[x]^c_\pi$ is the continuous quadratic variation of x along π.

In the following example we will show that in general the quadratic variation of a path x along a sequence of partitions π depend on the choice of the sequence π Cont (2012).

Example 3.1.2. Let $\omega \in C^0([0,T],\mathbb{R})$ be an arbitrary continuous function. Let us construct recursively a sequence π_n of partitions of $[0,T]$ such that

$$|\pi_n| \leq \frac{1}{n} \text{ and } \sum_{\pi_n} |\omega(t^n_{k+1}) - \omega(t^n_k)|^2 \leq \frac{1}{n}. \tag{3.1.3}$$

Assume we have constructed π_n with property (3.1.3) above. If we add to π_n the points $\frac{k}{n+1}$, $k = 1,\ldots,n$, we obtain a new partition $\sigma_n = (s^n_i,\ i = 0,\ldots,M_n)$ with mesh size $|\sigma_n| \leq \frac{1}{n+1}$. For $i = 0,\ldots,M_n - 1$, we further refine $[s^n_i, s^n_{i+1}]$ as follows. Let $J(i)$ be an integer with

$$J(i) \geq (n+1)M_n|\omega(s^n_{i+1}) - \omega(s^n_i)|^2 \tag{3.1.4}$$

and define $\tau^n_{i,1} = s^n_i$ and for $k = 1,\ldots,J(i)$,

$$\tau^n_{i,k+1} = \inf\left\{t \geq \tau^n_{i,k},\ \omega(t) = \omega(s^n_i) + \frac{k(\omega(s^n_{i+1}) - \omega(s^n_i))}{J(i)}\right\}.$$

Then the points $(\tau^n_{i,k},\ k = 1,\ldots,J(i))$ define a partition of $[s^n_i, s^n_{i+1}]$ with

$$|\tau^n_{i,k+1} - \tau^n_{i,k}| \leq \frac{1}{n+1} \text{ and } |\omega(\tau^n_{i,k+1}) - \omega(\tau^n_{i,k})| = \frac{|\omega(s^n_{i+1}) - \omega(s^n_i)|}{J(i)}$$

so using (3.1.4) we have

$$\sum_{k=1}^{J(i)} |\omega(\tau^n_{i,k+1}) - \omega(\tau^n_{i,k})|^2 \leq J(i)\frac{|\omega(s^n_{i+1}) - \omega(s^n_i)|^2}{J(i)^2} = \frac{1}{(n+1)M_n}.$$

Sorting $(\tau^n_{i,k},\ i = 0,\ldots,M_n,\ k = 1,\ldots,J(i))$ in increasing order we thus obtain a partition $\pi_{n+1} = (t^{n+1}_j)$ of $[0,T]$ such that

$$|\pi_{n+1}| \leq \frac{1}{n+1} \text{ and } \sum_{\pi_{n+1}} |\omega(t^n_{i+1}) - \omega(t^n_i)|^2 \leq \frac{1}{n+1}. \tag{3.1.5}$$

Taking limits along this sequence $\pi = (\pi_n)_{n\geq 1}$, then yields $[\omega]_\pi = 0$.

Remark 3.1.3. The notion of quadratic variation along a sequence of partitions depends on the chosen partition. We are going to drop the subscript π in $[x]_\pi$ because we fix the sequence $\pi = (\pi_n)_{n\geq 1}$ of partitions with $|\pi_n| \to 0$ and all limits will be considered along the same sequence π.

We denote by $Q^\pi([0,T],\mathbb{R}^d)$ the set of $\mathbb{R}^d$-valued cádlág paths with finite quadratic variation with respect to the partition $\pi = (\pi_n)_{n\geq 1}$. The notion of quadratic variation is extended to vector-valued paths as follows Cont (2012).

Definition 3.1.4. *A d-dimensional path $x = (x^1, x^2, \ldots, x^d) \in D([0,T], \mathbb{R}^d)$ is said to have finite quadratic variation along $\pi = (\pi_n)_{n\geq 1}$ if $x^i \in Q^\pi([0,T], \mathbb{R})$ and $x^i + x^j \in Q^\pi([0,T], \mathbb{R})$ for all $i, j = 1, 2, \ldots, d$. Then for any $i, j = 1, 2, \ldots, d$ and $t \in [0,T]$ we have*

$$\sum_{t_k^n \in \pi_n,\ t_k^n \leq t} \left(x^i(t_{k+1}^n) - x^i(t_k^n)\right) \left(x^j(t_{k+1}^n) - x^j(t_k^n)\right) \xrightarrow{n\to\infty} [x]_{ij}(t) = \frac{[x^i + x^j](t) - [x^i](t) - [x^j](t)}{2}.$$

The matrix-valued function $[x] : [0,T] \to S_d^+$ whose elements are given by

$$[x]_{ij}(t) = \frac{[x^i + x^j](t) - [x^i](t) - [x^j](t)}{2}$$

is called the quadratic covariation of the path x: for any $t \in [0,T]$,

$$\sum_{t_i^n \in \pi_n,\ t_i^n \leq t} \left(x(t_{i+1}^n) - x(t_i^n)\right)^t \left(x(t_{i+1}^n) - x(t_i^n)\right) \xrightarrow{n\to\infty} [x](t) \in S_d^+$$

and $[x]$ is increasing in the sense of the order on the positive symmetric matrices: for $h \geq 0,\ [x](t+h) - [x](t) \in S_d^+$.

Definition 3.1.5. *Let $\omega \in Q^\pi([0,T], \mathbb{R}),\ f \in D([0,T], \mathbb{R})$. Then the integral $\int_0^t f \mathrm{d}[\omega]$ may be defined as a limit of Riemann sums*

$$\int_0^t f \mathrm{d}[\omega] = \lim_{n\to\infty} \sum_{\pi_n} f(t_i^n)([\omega](t_{i+1}^n) - [\omega](t_i^n)).$$

Proposition 3.1.6. (Uniform convergence of quadratic Riemann sums).

$$\forall \omega \in Q^\pi([0,T], \mathbb{R}^d),\ \forall f \in \mathbb{C}_b^0([0,T], \mathbb{M}_d),\ \forall t \in [0,T],$$

$$\sum_{t_i^n \in \pi_n,\ t_i^n \leq t} \mathrm{tr}(f(t_i^n))(\omega(t_{i+1}^n) - \omega(t_i^n))^t(\omega(t_{i+1}^n) - \omega(t_i^n)) \to \int_0^t \langle f, \mathrm{d}[\omega]\rangle,$$

as $n \to \infty$. Furthermore, if $\omega \in \mathbb{C}_b^{0,0}([0,T], \mathbb{R}^d)$, the convergence is uniform for $t \in [0,T]$.

Proof. From Definition 3.1.5, using the definition of $[\omega]$, the result holds for $f : [0,T] \to \mathbb{R}$ of the form $f = \sum_{\pi_n} a_k \mathbb{1}_{[t_k^n, t_{k+1}^n)}$. Now, consider $f = \mathbb{C}_b^{0,0}([0,T], \mathbb{R})$ and define the piecewise constant approximations

$$f_n = \sum_{\pi_n} f(t_k^n) \mathbb{1}_{[t_k^n, t_{k+1}^n)}.$$

Then f_n converges uniformly to f : $\|f - f_n\|_\infty \to 0$ as $n \to \infty$, and for each $n \geq 1$,

$$\sum_{t_i^k \in \pi_n,\ t_i^k \leq t} f_n(t_i^k)(\omega(t_{i+1}^k) - \omega(t_i^k))^2 \to \int_0^t f_n \mathrm{d}[\omega], \quad \text{as } k \to \infty.$$

Since f is bounded on $[0,T]$ this sequence is dominated, we can then conclude that the result holds, by using a diagonal convergence argument. □

Proposition 3.1.6 implies weak convergence on $[0,T]$ of the discrete measures

$$\xi^n = \sum_{i=0}^{k(n)-1} (\omega(t_{i+1}^n) - \omega(t_i^n))^2 \delta_{t_i^n} \Rightarrow \xi = \mathrm{d}[\omega] \text{ as } n \to \infty$$

where δ_t is the Dirac measure at t.

Lemma 3.1.7. *Let $(\mu_n)_{n\geq 1}$ be a sequence of Radon measures on $[0,T]$ converging weakly to a Radon measure μ with no atoms, and $(f_n)_{n\geq 1}$, f be left-continuous functions defined on $[0,T]$ with*

$$\forall t \in [0,T], \ \lim_{n\to\infty} f_n(t) = f(t), \ |f_n(t)| \leq K$$

then

$$\int_s^t f_n(u)\mathrm{d}\mu_n(u) \to \int_s^t f(u)\mathrm{d}\mu(u), \text{ as } n \to \infty. \tag{3.1.6}$$

For the proof of Lemma 3.1.7, see Cont and Fournié (2010), Appendix C, Lemma 12.

We consider a path $\omega \in Q^\pi([0,T],\mathbb{R}^d)$ with finite quadratic variation along π. The path ω has a countable set of jump times, so we suppose that the partition exhausts the jump times in the sense that

$$\sup_{t\in[0,T]\setminus\pi_n} |\omega(t) - \omega(t-)| \to 0, \text{ as } n \to \infty. \tag{3.1.7}$$

Then the piecewise-constant approximation

$$\omega^n(t) = \sum_{i=0}^{k(n)-1} \omega(t_{i+1}-)\mathbb{1}_{[t_i,t_{i+1})}(t) + \omega(T)\mathbb{1}_{\{T\}}(t) \tag{3.1.8}$$

converges uniformly to ω, that is,

$$\sup_{t\in[0,T]} \|\omega^n(t) - \omega(t)\|_\infty = 0 \text{ as } n \to \infty.$$

We notice that according to (3.1.8), $\omega^n(t_i^n-) = \omega(t_i^n-)$ but $\omega^n(t_i^n) = \omega(t_{i+1}^n-)$. If we then define

$$\omega^{n,\Delta\omega(t_i^n)} = \omega^n + \Delta\omega(t_i^n)\mathbb{1}_{[t_i^n,T]}, \text{ then } \omega_{t_i^n-}^{n,\delta\omega(t_i^n)}(t_i^n) = \omega(t_i^n), \tag{3.1.9}$$

where $\Delta\omega(t_i^n) = \omega(t_i^n) - \omega(t_i^n-)$ is the path's discontinuity at t_i^n.

3.2 Change of variable

Now, we give a detailed version of the result in Cont (2012):

Theorem 3.2.1. *(Change of variable formula for functionals of continuous paths). Let* $\omega \in Q^\pi([0,T],\mathbb{R}^d)$ *and verifies* (3.1.7)*. Then for any* $F \in \mathbb{C}^{1,2}_{\text{loc}}(\Lambda_T)$ *the limit*

$$\int_0^T \nabla_\omega F(t,\omega_{t-})\mathrm{d}^\pi\omega := \lim_{n\to\infty}\sum_{i=0}^{k(n)-1} \nabla_\omega F(t_i^n, \omega_{t_i^n-}^{n,\Delta\omega(t_i^n)})(\omega(t_{i+1}^n)-\omega(t_i^n)) \tag{3.2.1}$$

exists and

$$F(T,\omega_T)-F(0,\omega_0) = \int_0^T \mathcal{D}F(t,\omega_t)\mathrm{d}t + \int_0^T \frac{1}{2}\mathrm{tr}({}^t\nabla_\omega^2 F(t,\omega_{t-})\mathrm{d}[\omega]^c(t)) + \int_0^T \nabla_\omega F(t,\omega_{t-})\mathrm{d}^\pi\omega$$
$$+\sum_{t\in(0,T]}[F(t,\omega_t)-F(t,\omega_{t-})-\nabla_\omega F(t,\omega_{t-})\Delta\omega(t)]$$

Proof. We first note that up to localization by a sequence of stopping times, we can assume that $F \in \mathbb{C}^{1,2}_b(\Lambda_T)$. Denoting $\delta\omega_i^n = \omega(t_{i+1}^n)-\omega(t_i^n)$. Since ω is continuous on $[0,T]$, it is uniformly continuous, so

$$\eta_n = \sup\{|\omega(u)-\omega(t_{i+1}^n)|+|t_{i+1}^n-t_i^n|,\ 0\le i\le k(n)-1,\ u\in[t_i^n,t_{i+1}^n)\}\to 0,\ \text{as } n\to\infty.$$

Since $F \in \mathbb{C}^{1,2}_b(\Lambda_T)$, then by Definition 2.3.12, $\nabla_\omega^2 F$, $\mathcal{D}F$ satisfy the boundedness-preserving property, for n large enough there exists $C>0$ such that

$$\forall t\in[0,T),\ \forall\omega'\in\Lambda_T,\ d_\infty((t,\omega),(t',\omega'))<\eta_n \Rightarrow |\mathcal{D}F(t',\omega')|\le C,\ |\nabla_\omega^2 F(t',\omega')|\le C.$$

For $i\le k(n)-1$, consider the decomposition:

$$F(t_{i+1}^n,\omega_{t_{i+1}^n-}^n)-F(t_i^n,\omega_{t_i^n-}^n) = \left(F(t_{i+1}^n,\omega_{t_{i+1}^n-}^n)-F(t_i^n,\omega_{t_i^n}^n)\right)+\left(F(t_i^n,\omega_{t_i^n}^n)-F(t_i^n,\omega_{t_i^n-}^n)\right), \tag{3.2.2}$$

where the first term on the RHS of Equation (3.2.2) is the horizontal increment while the second term is the vertical increment. If we let $\psi(u)=F(t_i^n+u,\omega_{t_i^n}^n)$, then we can write the horizontal increment as $\psi(h_i^n)-\psi(0)$, where $h_i^n=t_{i+1}^n-t_i^n$. Since $F\in\mathbb{C}^{1,2}_b(\Lambda_T)$, ψ is right-differentiable, and ψ is left-continuous (see Cont (2012), Proposition 5.5), so

$$F(t_{i+1}^n,\omega_{t_i^n}^n)-F(t_i^n,\omega_{t_i^n}^n)=\int_0^{h_i^n}\mathcal{D}F(t_i^n+u,\omega_{t_i^n}^n)\mathrm{d}u. \tag{3.2.3}$$

According to Equation (3.1.9) we can write the vertical increment in Equation (3.2.2) as

$$F(t_i^n,\omega_{t_i^n-}^{n,\delta\omega_i^n})-F(t_i^n,\omega_{t_i^n-}^n)=F(t_i^n,\omega_{t_i^n-}^n+\delta\omega_i^n\mathbb{1}_{[t_i^n,T]})-F(t_i^n,\omega_{t_i^n-}^n).$$

If we let $\phi(u)=F(t_i^n,\omega_{t_i^n-}^n+u\mathbb{1}_{[t_i^n,T]})$, then we can write the vertical increment in Equation (3.2.2) as $\phi(\delta\omega_i^n)-\phi(0)$. Since $F\in\mathbb{C}^{1,2}_b(\Lambda_T)$, $\phi\in C^2(\mathbb{R}^d)$ on the convex set $\mathcal{B}(0,\eta_n)$, with

$$\nabla\phi(u)=\nabla_\omega F(t_i^n,\omega_{t_i^n-}^n+u\mathbb{1}_{[t_i^n,T]}),$$
$$\nabla^2\phi(u)=\nabla_\omega^2 F(t_i^n,\omega_{t_i^n-}^n+u\mathbb{1}_{[t_i^n,T]}).$$

A second order Taylor expansion of ϕ at $u = 0$ gives

$$F(t_i^n, \omega_{t_i^n}^n) - F(t_i^n, \omega_{t_i^n-}^n) = \nabla_\omega F(t_i^n, \omega_{t_i^n-}^n)\delta\omega_i^n + \frac{1}{2}\mathrm{tr}(\nabla_\omega^2 F(t_i^n, \omega_{t_i^n-}^n)^t(\delta\omega_i^n)\delta\omega_i^n) + r_i^n$$

where

$$r_i^n \le K|\delta\omega_i^n|^2 \sup_{x\in\mathcal{B}(0,\eta_n)} |\nabla_\omega^2 F(t_i^n, \omega_{t_i^n-}^n + x\mathbb{1}_{[t_i^n,T]}) - \nabla_\omega^2 F(t_i^n, \omega_{t_i^n-}^n)|.$$

Denote by $i^n(t)$ the index such that $t \in [t_{i^n(t)}^n, t_{i^n(t)+1}^n)$. We now sum all the terms above from $i = 0$ to $k(n) - 1$: The LHS of Equation (3.2.2) gives $F(T, \omega_{T-}^n) - F(0, \omega_0^n)$, which converges to $F(T, \omega_{T-}) - F(0, \omega_0)$ since F is left-continuous and this quantity is equal to $F(T, \omega_T) - F(0, \omega_0)$ since ω is continuous. The horizontal increment in Equation (3.2.2) can also be written as,

$$\int_0^T \mathcal{D}F(u, \omega_{t_{i^n(u)}^n}^n)\mathrm{d}u \tag{3.2.4}$$

where the integrand converges to $\mathcal{D}F(u, \omega_u)$ and is bounded by C. Hence the dominated convergence theorem applies and the expression (3.2.4) converges to

$$\int_0^T \mathcal{D}F(u, \omega_u)\mathrm{d}u.$$

The vertical increment in Equation (3.2.2) can also be written as

$$\sum_{i=0}^{k(n)-1} \nabla_\omega F(t_i^n, \omega_{t_i^n-}^n)(\omega(t_{i+1}^n) - \omega(t_i^n)) + \sum_{i=0}^{k(n)-1} \frac{1}{2}\mathrm{tr}\left(\nabla_\omega^2 F(t_i^n, \omega_{t_i^n-}^n)^t(\delta\omega_i^n)\delta\omega_i^n\right) + \sum_{i=0}^{k(n)-1} r_i^n.$$

The term $\nabla_\omega^2 F(t_i^n, \omega_{t_i^n-}^n)\mathbb{1}_{(t_i^n, t_{i+1}^n]}$ is bounded by C, and converges to $\nabla_\omega^2 F(t, \omega_t)$ by left-continuity of $\nabla_\omega^2 F$ and the path is left-continuous. Applying Lemma 3.1.7 to the second term in the sum, we obtain

$$\int_0^T \frac{1}{2}\mathrm{tr}({}^t\nabla_\omega^2 F(t_i^n, \omega_{t_i^n-}^n)\mathrm{d}\xi^n) \to \int_0^T \frac{1}{2}\mathrm{tr}({}^t\nabla_\omega^2 F(u, \omega_u)\mathrm{d}[\omega](u)), \text{ as } n \to \infty.$$

We then use the same lemma, since r_i^n is bounded,

$$\sum_{i=i^n(s)+1}^{i^n(t)-1} r_i^n \to 0, \text{ as } n \to \infty.$$

Since all other terms converge as $n \to \infty$, we conclude that the limit of the Riemann sums

$$\sum_{i=0}^{k(n)-1} \nabla_\omega F(t_i^n, \omega_{t_i^n-}^n)(\omega(t_{i+1}^n) - \omega(t_i^n i))$$

exists, and this is the pathwise integral $\int \nabla_\omega F(t, \omega)\mathrm{d}^\pi\omega$. □

From Theorem 3.2.1, for $\omega \in Q^\pi([0, T], \mathbb{R}^d)$, we can now define the pathwise integral $\int_0^T \phi\mathrm{d}^\pi\omega$ as a limit of non-anticipative Riemann sums:

$$\int_0^T \phi\mathrm{d}^\pi\omega := \lim_{n\to\infty} \sum_{i=0}^{k(n)-1} \phi\left(t_i^n, \omega_{t_i^n-}^{n,\Delta\omega(t_i^n)}\right)\left(\omega(t_{i+1}^n) - \omega(t_i^n)\right) \tag{3.2.5}$$

for any integrand of the form

$$\phi(t,\omega) = \nabla_\omega F(t,\omega)$$

where $F \in \mathbb{C}^{1,2}_{\mathrm{loc}}(\Lambda_T)$, and the path ω is not required to be of finite variation. This extends the pathwise integral in Föllmer (1981) for integrands of the form $\phi(t,\omega)$ where ϕ belongs to the space of vertical $1-$forms integrands given by

$$V(\Lambda_T) = \{\nabla_\omega F(\cdot,\cdot),\ F \in \mathbb{C}^{1,2}_b(\Lambda^d_T)\}$$

where Λ^d_T denotes the space of $\mathbb{R}^d-$valued stopped cádlág paths. Since the horizontal and vertical derivatives do not commute, then $V(\Lambda_T)$ and $\mathbb{C}^{1,1}_b(\Lambda_T)$ do not coincide.

This set of integrands has a natural vector space structure which includes the space of simple predictable cylindrical functionals, as subsets, and we denote by $\mathbb{S}(\Lambda_T)$ the space of simple predictable cylindrical functionals. Then for $\phi = \nabla_\omega F \in V(\Lambda^d_T)$, the pathwise integral $\int_0^t \phi(\cdot,\omega_-)\mathrm{d}^\pi\omega$ is given by

$$\begin{aligned}\int_0^T \phi(t,\omega_{t-})\mathrm{d}^\pi\omega = F(T,\omega_T) - F(0,\omega_0) - \frac{1}{2}\int_0^T \langle \nabla_\omega\phi(t,\omega_{t-})\mathrm{d}[\omega]_\pi\rangle \\ - \int_0^T \mathcal{D}F(t,\omega_{t-})\mathrm{d}t - \sum_{0\le s\le T} F(t,\omega_t) - F(t,\omega_{t-}) - \phi(t,\omega_{t-})\Delta\omega(t). \qquad (3.2.6)\end{aligned}$$

The following result summarizes some key properties of this integral, for prove see Bally et al. (2016).

Proposition 3.2.2. Let $\omega \in Q^\pi([0,T],\mathbb{R}^d)$. The pathwise integral Equation (3.2.6) defines a map

$$\begin{aligned}I_w : & V(\Lambda^d_T) \to Q^\pi([0,T],\mathbb{R}^d) \\ & \phi \mapsto \int_0^\cdot \phi(t,\omega_{t-})\mathrm{d}^\pi\omega(t)\end{aligned}$$

with the following properties:

1. pathwise isometry formula: $\forall\phi \in \mathbb{S},\ \forall t \in [0,T]$,

$$[I_w(\phi)]_\pi(t) = \left[\int_0^\cdot \phi(t,\omega_{t-})\mathrm{d}^\pi\omega\right]_\pi(t) = \int_0^t \langle \phi(u,\omega_{u-})^t\phi(u,\omega_{u-}),\mathrm{d}[\omega]\rangle;$$

2. quadratic covariation formula: for $\phi,\psi \in \mathbb{S}$, the limit

$$[I_w(\phi),I_w(\psi)]_\pi(T) := \lim_{n\to\infty}\sum_{\pi_n}\left(I_w(\phi)(t^n_{k+1}) - I_w(\phi)(t^n_k)\right)\left(I_w(\psi)(t^n_{k+1}) - I_w(\psi)(t^n_k)\right)$$

exists and is given by

$$[I_w(\phi),I_w(\psi)]_\pi(T) = \int_0^T \langle \psi(t,\omega_{t-})^t\phi(t,\omega_{t-}),\mathrm{d}[\omega]\rangle;$$

3. associativity: let $\phi \in V(\Lambda_T^d)$, $\psi \in V(\Lambda_T^1)$ and $x \in D([0,T],\mathbb{R})$ defined by $x \in \int_0^t \phi(u,\omega_{u-})\mathrm{d}^\pi\omega$. Then

$$\int_0^T \psi(t,\omega_{t-})\mathrm{d}^\pi\omega = \int_0^T \psi\left(t, \left(\int_0^t \phi(u,\omega_{u-})\mathrm{d}^\pi\omega\right)_{t-}\right)\phi(t,\omega_{t-})\mathrm{d}^\pi\omega.$$

Remark 3.2.3. There is some interesting applications in mathematical finance Cont and Riga (2014) where the integrals of such vertical $1-$forms appears as hedging strategies.

We define the space

$$\mathbb{C}_b^{1,2}(\omega) := \{F(\cdot,\omega),\ F \in \mathbb{C}_b^{1,2}(\Lambda_T^d) \subset Q^\pi([0,T],\mathbb{R}^d)$$

which is a vector space of paths with finite quadratic variation whose properties are controlled by ω, which is well defined along the sequence of partitions π. This is not the case with the space $Q^\pi([0,T],\mathbb{R}^d)$, hence, the space $\mathbb{C}_b^{1,2}(\omega)$ is the appropriate space to study the pathwise calculus.

Now, we denote by $BV([0,T])$ the space of functions with bounded variations over the interval $[0,T]$. In the case where ω has finite variation then the Föllmer integral reduces to the Riemann-Stieltjes integral and we obtain the following result

Proposition 3.2.4. For any $F \in \mathbb{C}_{\mathrm{loc}}^{1,1}(\Lambda_T)$, $\omega \in BV([0,T]) \cap D([0,T],\mathbb{R}^d)$,

$$\begin{aligned} F(T,\omega_T) - F(0,\omega_0) &= \int_0^T \mathcal{D}F(t,\omega_{t-})\mathrm{d}t + \int_0^T \nabla_\omega F(t,\omega_{t-})\mathrm{d}\omega \\ &\quad + \sum_{t\in(0,T]} \left[F(t,\omega_t) - F(t,\omega_{t-}) - \nabla_\omega F(t,\omega_{t-})\Delta\omega(t)\right]. \end{aligned} \tag{3.2.7}$$

Remark 3.2.5. 1. The integrals in Equation (3.2.7) are defined as limits of Riemann sums along any sequence of partitions $(\pi_n)_{n\geq 1}$ with $|\pi_n| \to 0$.

2. For the case when ω is continuous with finite variation, then $\forall F \in \mathbb{C}_{\mathrm{loc}}^{1,1}(\Lambda_T)$, $\forall \omega \in BV([0,T]) \cap C^0([0,T],\mathbb{R}^d)$ Equation (3.2.7) is given by

$$F(T,\omega_T) - F(0,\omega_0) = \int_0^T \mathcal{D}F(t,\omega_t)\mathrm{d}t + \int_0^T \nabla_\omega F(t,\omega_t)\mathrm{d}\omega. \tag{3.2.8}$$

This shows that the restriction of any functional $F \in \mathbb{C}_{\mathrm{loc}}^{1,1}(\Lambda_T)$ to $BV([0,T]) \cap C^0([0,T],\mathbb{R}^d)$ may be decomposed into horizontal and vertical components.

The following result shows that, for a continuous semimartingale X, any smooth non-anticipative functional Y depends on F and its derivatives only through their values on continuous paths, $\mathcal{W}_T$:

Corollary 3.2.6. *Let X be a continuous semimartingale and $F \in \mathbb{C}_{\mathrm{loc}}^{1,2}(\mathcal{W}_T)$. For any $t \in [0,T)$,*

$$\begin{aligned} F(t,X_t) - F(0,X_0) &= \int_0^t \mathcal{D}F(u,X_u)\mathrm{d}u + \int_0^t \nabla_\omega F(u,X_u)\mathrm{d}X_u \\ &\quad + \frac{1}{2}\int_0^t \nabla_\omega^2 F(u,X_u)\mathrm{d}[X] \quad a.s. \end{aligned} \tag{3.2.9}$$

In particular, $Y(t) = F(t,X_t)$ is a continuous semimartingale.

Example 3.2.7. Let B be a standard Brownian motion and consider the integral

$$x = \int_0^t \sigma(s, B_s)\mathrm{d}B_s.$$

Then

$$[x] = \int_0^t \sigma^2(s, B_s)\mathrm{d}s.$$

3.3 Pathwise derivatives of an adapted process

We consider an $\mathcal{F}_t^X$-adapted process $(Y_t)_{t\in[0,T]}$ is given by a functional representation

$$Y_t = F(t, X_t) \tag{3.3.1}$$

where $\mathbb{C}_l^{0,0}(\Lambda_T)$ has left-continuous and vertical derivatives $\mathcal{D}F \in \mathbb{C}_l^{0,0}(\Lambda_T)$ and $\nabla_\omega F \in \mathbb{C}_l^{0,0}(\Lambda_T)$. It is noted that since X has continuous paths, Y only depends on the restriction of F to

$$\mathcal{W}_T = \{(t, \omega) \in \Lambda_T,\ \omega \in C^0([0,T], \mathbb{R}^d)\}.$$

Therefore, the representation (3.3.1) is not unique. However, the definition of $\nabla_\omega F$ which involves evaluating F on cádlág paths, seems to depend on the choice of representation, in particular, on the values taken by F outside $\mathcal{W}_T$. It is crucial to resolve this non-uniqueness if one is to deal with functionals of continuous processes or, more generally, processes for which the topological support of the law is not the full space $D([0,T], \mathbb{R}^d)$. The following result shows that if Y has a functional representation (3.3.1) where F is differentiable in the sense of horizontal and vertical derivatives and the derivatives define elements of $\mathbb{C}_l^{1,1}(\Lambda_T)$ then $\nabla_\omega F(t, X_t)$ is uniquely defined, independent of the choice of the representation F.

Theorem 3.3.1. *Consider $F^1, F^2 \in \mathbb{C}_l^{1,1}(\Lambda_T)$ with left-continuous horizontal and vertical derivatives. If F^1 and F^2 coincide on continuous paths:*

$$\forall t \in [0,T],\ \forall \omega \in C^0([0,T], \mathbb{R}^d),\ F^1(t, \omega_t) = F^2(t, \omega_t),$$

then

$$\forall t \in [0,T],\ \forall \omega \in C^0([0,T], \mathbb{R}^d),\ \nabla_\omega F^1(t, \omega_t) = \nabla_\omega F^2(t, \omega_t).$$

Proof. Let $F = F^1 - F^2 \in \mathbb{C}_l^{1,1}(\Lambda_T)$ and $\omega \in C^0([0,T], \mathbb{R}^d)$. Then $F(t, \omega_t) = 0$ for all $t \le T$. It is then clear that $\mathcal{D}F(t, \omega_t)$ is also 0 on continuous paths. Assume now that there exists some $\omega \in C^0([0,T], \mathbb{R}^d)$ such that for some $1 \le i \le d$ and $t_0 \in [0,T)$, $\partial_i F(t_0, \omega_{t_0}) > 0$. Let $\alpha = \frac{1}{2}\partial_i F(t_0, \omega_{t_0})$. By the left-continuity of $\partial_i F$ and using the fact that $\mathcal{D}F \in \mathbb{B}(\Lambda_T)$, there exists $\epsilon > 0$ such that for any $(t', \omega') \in \Lambda_T$,

$$(t' < t_0,\ d_\infty((t_0, \omega), (t', \omega')) < \epsilon) \Rightarrow (\partial_i F(t', \omega') > \alpha \text{ and } |\mathcal{D}F(t', \omega')| < 1)\,. \tag{3.3.2}$$

We choose $t < t_0$ such that $d_\infty(\omega_t, \omega_{t_0}) < \frac{\epsilon}{2}$, define $h = t_0 - t$ and define the following extension of ω_t to $[0, T]$:

$$\begin{cases} z(u) = \omega(u), & u \le t \\ z_j(u) = \omega_j(t) + \mathbb{1}_{i=j}(u-t), & t \le u \le T,\ 1 \le j \le d. \end{cases} \tag{3.3.3}$$

We define the following sequence of piecewise constant approximation of z_{t+h}:

$$\begin{cases} z^n(u) = \tilde{z}^n = z(u) & \text{if } u \le t \\ z_j^n(u) = \omega_j(t) + \mathbb{1}_{i=j} \frac{h}{n} \sum_{k=0}^{n} \mathbb{1}_{\frac{kh}{n} \le u-t} & \text{if } t \le u \le t+h,\ 1 \le j \le d \end{cases} \tag{3.3.4}$$

Since$\left\| z_{t+h} - z_{t+h}^n \right\|_\infty = \frac{h}{n} \to 0$, as $n \to \infty$, then

$$\left| F(t+h, z_{t+h}) - F(t+h, z_{t+h}^n) \right| \to 0, \text{ as } n \to \infty.$$

We can now decompose $F(t+h, z_{t+h}^n) - F(t, \omega)$ as

$$\begin{aligned} F(t+h, z_{t+h}^n) - F(t, \omega) = & \sum_{k=1}^{n} \left(F\left(t + \frac{kh}{n}, z^n_{t+\frac{kh}{n}}\right) - F\left(t + \frac{kh}{n}, z^n_{t+\frac{kh}{n}-}\right)\right) \\ & + \sum_{k=1}^{n} \left(F\left(t + \frac{kh}{n}, z^n_{t+\frac{kh}{n}-}\right) - F\left(t + \frac{(k-1)h}{n}, z^n_{t+\frac{(k-1)h}{n}}\right)\right) \end{aligned} \tag{3.3.5}$$

where the first sum corresponds to jumps of z^n at times $t + \frac{kh}{n}$ and the second sum to the horizontal variations of z^n on $[t + \frac{(k-1)h}{n}, t + \frac{kh}{n}]$.

$$F\left(t + \frac{kh}{n}, z^n_{t+\frac{kh}{n}}\right) - F\left(t + \frac{kh}{n}, z^n_{t+\frac{kh}{n}-}\right) = \phi\left(\frac{h}{n}\right) - \phi(0) \tag{3.3.6}$$

where

$$\phi(u) = F\left(t + \frac{kh}{n}, z^n_{t+\frac{kh}{n}-} + u e_i \mathbb{1}_{[t+\frac{kh}{n}, T]}\right).$$

Since F is vertically differentiable, ϕ is differentiable and

$$\phi'(u) = \partial_i F\left(t + \frac{kh}{n}, z^n_{t+\frac{kh}{n}-} + u e_i \mathbb{1}_{[t+\frac{kh}{n}, T]}\right)$$

is continuous. For $u \le \frac{h}{n}$ we have

$$d_\infty\left((t, \omega_t), \left(t + \frac{kh}{n}, z^n_{t+\frac{kh}{n}-} + u e_i \mathbb{1}_{[t+\frac{kh}{n}, T]}\right)\right) \le h$$

so $\phi'(u) > \alpha$ hence

$$\sum_{k=1}^{n} F\left(t + \frac{kh}{n}, z^n_{t+\frac{kh}{n}}\right) - F\left(t + \frac{kh}{n}, z^n_{t+\frac{kh}{n}-}\right) > \alpha h.$$

On the other hand,

$$F\left(t+\frac{kh}{n}, z^n_{t+\frac{kh}{n}-}\right) - F\left(t+\frac{(k-1)h}{n}, z^n_{t+\frac{(k-1)h}{n}}\right) = \psi\left(\frac{h}{n}\right) - \psi(0)$$

where

$$\psi(u) = F\left(t+\frac{(k-1)h}{n} + u, z^n_{t+\frac{(k-1)h}{n}}\right)$$

so that ψ is right-differentiable on $(0, \frac{h}{n})$ with right derivative

$$\psi'_r(u) = \mathcal{D}F\left(t+\frac{(k-1)h}{n} + u, z^n_{t+\frac{(k-1)h}{n}}\right).$$

Since $F \in \mathbb{C}^{1,1}_l(\Lambda_T)$, ψ is left-continuous, so

$$\sum_{k=1}^{n}\left[F\left(t+\frac{kh}{n}, z^n_{t+\frac{kh}{n}-}\right) - F\left(t+\frac{(k-1)h}{n}, z^n_{t+\frac{(k-1)h}{n}}\right)\right] = \int_0^h \mathcal{D}F(t+u, z^n_t)\mathrm{d}u.$$

We note that

$$d_\infty\left((t+h, z^n_{t+h}), (t+h, z_{t+h})\right) \leq \frac{h}{n},$$

then we obtain

$$\mathcal{D}F(t+h, z^n_{t+h}) \to \mathcal{D}F(t+h, z_{t+h}) = 0, \text{ as } n \to \infty,$$

since the path of z_{t+u} is continuous. Moreover, $|\mathcal{D}F(t+h, z^n_{t+h})| \leq 1$ since $d_\infty((t+h, z^n_{t+h})) \leq \epsilon$, so by dominated convergence the integral converges to 0 as $n \to \infty$. Writing

$$F(t+h, z_{t+h}) - F(t,\omega) = \left(F(t+h, z_{t+h}) - F(t+h, z^n_{t+h})\right) + \left(F(t+h, z^n_{t+h}) - F(t,\omega)\right)$$

and taking the limit as $n \to \infty$ leads to $F(t+h, z_{t+h}) - F(t,\omega) \geq \alpha h$, a contradiction. □

Remark 3.3.2. This result implies that, if $\nabla_\omega F^i \in \mathbb{C}^{1,1}(\Lambda_T)$, $\mathcal{D}(\nabla_\omega F) \in \mathbb{B}(\Lambda_T)$ and $F^1 = F^2$ for any continuous path ω, then $\nabla^2_\omega F^1$ and $\nabla^2_\omega F^2$ must also coincide on continuous paths. Under the weaker assumption that $F^i \in \mathbb{C}^{1,2}(\Lambda_T)$, we obtain the result in Theorem 3.3.1 using probabilistic arguments.

We let W be a real Brownian motion on an auxiliary probability space $(\tilde{\Omega}, \mathcal{B}, \mathbb{P})$ whose generic element we will denote ω, $(\mathcal{B}_s)_{s\geq 0}$ its natural filtration, and let $\tau = \inf\{s > 0,\ |w(s)| = \frac{\epsilon}{2}\}$.

Theorem 3.3.3. *If $F^1, F^2 \in \mathbb{C}^{1,2}_b(\Lambda_T)$ coincide on continuous paths $\forall\omega \in C^0([0,T], \mathbb{R}^d),\ \forall t \in [0,T)$,*

$$F^1(t, \omega_t) = F^2(t, \omega_t)$$

then their second vertical derivatives also coincide on continuous paths; $\forall\omega \in C^0([0,T], \mathbb{R}^d),\ \forall t \in [0,T)$,

$$\nabla^2_\omega F^1(t, \omega_t) = \nabla^2_\omega F^2(t, \omega_t).$$

Proof. Let $F = F^1 - F^2$. Assume that there exists $\omega \in C^0([0,T], \mathbb{R}^d)$ such that for some $1 \leq i \leq d$ and $t_0 \in [0,T)$ and some direction $h \in \mathbb{R}^d$, $\|h\| = 1$, ${}^t h \nabla_\omega^2 F(t_0, \omega_{t_0}) \cdot h > 0$, and denote $\alpha = \frac{1}{2} {}^t h \nabla_\omega^2 F(t_0, \omega_{t_0}) \cdot h$. We want to show that this leads to a contradiction. According to Theorem 3.3.1 $\nabla_\omega F(t, \omega_t) = 0$. There exists $\eta > 0$ such that

$$\forall (t', \omega') \in \Lambda_T, \ \{t' \leq t_0, \quad d_\infty((t_0, \omega), (t', \omega')) < \eta$$

which implies that

$$\max\left(|F(t', \omega') - F(t_0, \omega_{t_0})|, |\nabla_\omega F(t', \omega')|, |\mathcal{D}F(t', \omega')|\right) < 1, \quad {}^t h \nabla_\omega^2 F(t', \omega') \cdot h > \alpha. \quad (3.3.7)$$

We define for $t' \in [0,T]$, the Brownian extrapolation

$$U_{t'}(\omega) = \omega(t')\mathbb{1}_{t' \leq t} + \left(\omega(t) + W((t'-t) \wedge \tau)h\right)\mathbb{1}_{t' > t}.$$

For all $s < \frac{\epsilon}{2}$, we have

$$d_\infty((t+s, U_{t+s}(\omega)), (t, \omega_t)) < \epsilon \ \ \mathbb{P} - a.s.$$

We define the following piecewise-constant approximation of the stopped process w_τ for $0 \leq s \leq \frac{\epsilon}{2}$:

$$W^n(s) = \sum_{i=0}^{n-1} W\left(i\frac{\epsilon}{2n} \wedge \tau\right)\mathbb{1}_{s \in \left[i\frac{\epsilon}{2n}, (i+1)\frac{\epsilon}{2n}\right)} + W\left(\frac{\epsilon}{2} \wedge \tau\right)\mathbb{1}_{s = \frac{\epsilon}{2}}. \quad (3.3.8)$$

We denote

$$\begin{aligned} &Z(s) = F(t+s, U_{t+s}), \quad s \in [0, T-t], \quad Z^n(s) = F(t+s, U^n_{t+s}), \\ &U^n_{t'}(\omega) = \omega(t')\mathbb{1}_{t' \leq t} + \left(\omega(t) + W^n((t'-t) \wedge \tau)h\right)\mathbb{1}_{t' > t}, \end{aligned}$$

which gives the following decomposition:

$$\begin{aligned} Z\left(\frac{\epsilon}{2}\right) - Z(0) = Z\left(\frac{\epsilon}{2}\right) - Z^n\left(\frac{\epsilon}{2}\right) + \sum_{i=1}^{n}\left(Z^n\left(i\frac{\epsilon}{2n}\right) - Z^n\left(i\frac{\epsilon}{2n}-\right)\right) \\ + \sum_{i=0}^{n-1}\left(Z^n\left((i+1)\frac{\epsilon}{2n}-\right) - Z^n\left(i\frac{\epsilon}{2n}\right)\right). \quad (3.3.9) \end{aligned}$$

As $n \to \infty$, $\left\|U_{t+\frac{\epsilon}{2}} - U^n_{t+\frac{\epsilon}{2}}\right\|_\infty \to 0$, as a result the first term in the RHS of Equation (3.3.9) vanishes almost surely. The second term may now be expressed as

$$Z^n\left(i\frac{\epsilon}{2n}\right) - Z^n\left(i\frac{\epsilon}{2n}-\right) = \phi_i\left(W\left(i\frac{\epsilon}{2n}\right) - W\left((i-1)\frac{\epsilon}{2n}\right)\right) - \phi_i(0) \quad (3.3.10)$$

where

$$\phi_i(u, \omega) = F\left(t + i\frac{\epsilon}{2n}, U^n_{t+i\frac{\epsilon}{2n}-}(\omega) + uh\mathbb{1}_{[t+i\frac{\epsilon}{2n}, T]}\right).$$

We note that $\phi_i(u,\omega)$ is measurable with respect to $\mathcal{B}_{(i-1)\frac{\epsilon}{2n}}$ whereas Equation (3.3.10) is independent with respect to $\mathcal{B}_{(i-1)\frac{\epsilon}{2n}}$. Let $\Omega_1 \subset \tilde{\Omega}$, $\mathbb{P}(\Omega_1) = 1$ such that ω has continuous sample paths on Ω_1. Then, on Ω_1, $\phi_i(\cdot,\omega) \in C^2(\mathbb{R})$ and the following relations hold $\mathbb{P}-$almost surely:

$$\phi_i'(u,\omega) = \nabla_\omega F\left(t + i\frac{\epsilon}{2n}, U^n_{t+i\frac{\epsilon}{2n}-}(\omega_t) + uh\mathbb{1}_{[t+i\frac{\epsilon}{2n},T]}\right)\cdot h$$
$$\phi_i''(u,\omega) = ^t h\nabla^2_\omega F\left(t + i\frac{\epsilon}{2n}, U^n_{t+i\frac{\epsilon}{2n}}(\omega_t) + uh\mathbb{1}_{[t+i\frac{\epsilon}{2n},T]}\right)\cdot h.$$

So, using the above arguments we can apply the Itô formula to Equation (3.3.10) on Ω_1. We denote by $i(s)$ the index such that $s \in [(i(s)-1)\frac{\epsilon}{2n}, i(s)\frac{\epsilon}{2n})$ and summing on i we obtain that

$$\sum_{i=1}^{n} Z^n\left(i\frac{\epsilon}{2n}\right) - Z^n\left(i\frac{\epsilon}{2n}-\right)$$
$$= \int_0^{\frac{\epsilon}{2}} \nabla_\omega F\left(t + i(s)\frac{\epsilon}{2n}, U^n_{t+i(s)\frac{\epsilon}{2n}-} + \left(W(s) - W((i(s)-1)\frac{\epsilon}{2n})\right)h\mathbb{1}_{[t+i(s)\frac{\epsilon}{2n},T]}\right)\mathrm{d}W_s$$
$$+ \frac{1}{2}\int_0^{\frac{\epsilon}{2}} \left\langle h, \nabla^2_\omega F\left(t + i(s)\frac{\epsilon}{2n}, U^n_{t+i(s)\frac{\epsilon}{2n}-} + \left(W(s) - W((i(s)-1)\frac{\epsilon}{2n})\right)h\mathbb{1}_{[t+i(s)\frac{\epsilon}{2n},T]}\right)\cdot h\right\rangle \mathrm{d}s.$$

Since the first derivative is bounded, according to Equation (3.3.7), the stochastic integral is a martingale, so taking expectation leads to

$$\mathbb{E}\left[\sum_{i=1}^{n} Z^n\left(i\frac{\epsilon}{2n}\right) - Z^n\left(i\frac{\epsilon}{2n}-\right)\right] \geq \alpha\frac{\epsilon}{2}.$$
$$Z^n\left((i+1)\frac{\epsilon}{2n}-\right) - Z^n\left(i\frac{\epsilon}{2n}\right) = \psi\left(\frac{\epsilon}{2n}\right) - \psi(0)$$

where

$$\psi(u) = F\left(t + i\frac{\epsilon}{2n} + u, U^n_{t+i\frac{\epsilon}{2n}}\right)$$

is right-differentiable with right derivative

$$\psi'(u) = \mathcal{D}F\left(t + i\frac{\epsilon}{2n} + u, U^n_{t+i\frac{\epsilon}{2n}}\right).$$

Since $F \in \mathbb{C}^{0,0}_l([0,T])$, ψ is left-continuous and the fundamental theorem of calculus yields

$$\sum_{i=0}^{n-1} Z^n\left((i+1)\frac{\epsilon}{2n}-\right) - Z^n\left(i\frac{\epsilon}{2n}\right) = \int_0^{\frac{\epsilon}{2}} \mathcal{D}F\left(t+s, U^n_{t+(i(s)-1)\frac{\epsilon}{2n}}\right)\mathrm{d}s.$$

The integrand converges to $\mathcal{D}F(t+s, U_{t+s}) = 0$ as $n \to \infty$, since $\mathcal{D}F(t+s,\omega) = 0$ whenever ω is continuous. Since this term is also bounded, by the dominated convergence theorem

$$\int_0^{\frac{\epsilon}{2}} \mathcal{D}F\left(t+s, U^n_{t+(i(s)-1)\frac{\epsilon}{2n}}\right)\mathrm{d}s \to 0 \text{ as } n \to \infty.$$

Since $F(t,\omega) = 0$ whenever ω is a continuous path, then $Z(\frac{\epsilon}{2n}) = 0$. Conversely, since all derivatives of F in Equation (3.3.9) are bounded, by the dominated convergence theorem we can take expectations throughout Equation (3.3.9) with respect to the Wiener measure and obtain that $\alpha\frac{\epsilon}{2n} = 0$, which is a contradiction. □

Remark 3.3.4. This result enables us to define the class $\mathbb{C}_b^{1,2}(\mathcal{W}_T)$ of non-anticipative functionals such that their restriction to $\mathcal{W}_T$ fulfills the conditions of Theorem 3.3.3 without having to extend to the entire space Λ_T.

As a result, putting together this with the proof of the change of variable formula for functionals of continuous paths, Theorem 3.2.1, we have the following result:

Theorem 3.3.5 (Pathwise change of variable formula for $\mathbb{C}_b^{1,2}(\mathcal{W}_T)$ functionals)**.** *For any* $F \in \mathbb{C}_b^{1,2}(\mathcal{W}_T),\ \omega \in C^0([0,T],\mathbb{R}^d) \cap Q^\pi([0,T],\mathbb{R}^d)$ *the limit*

$$\int_0^T \nabla_\omega F(t,\omega_t)\mathrm{d}^\pi\omega := \lim_{n\to\infty} \sum_{i=0}^{k(n)-1} \nabla_\omega F\left(t_i^n, \omega_{t_i^n-}^n\right)\left(\omega\left(t_{i+1}^n\right) - \omega\left(t_i^n\right)\right) \tag{3.3.11}$$

exists and

$$F(T,\omega_T) - F(0,\omega_0) = \int_0^T \mathcal{D}F(t,\omega_t)\mathrm{d}t + \int_0^T \nabla_\omega F(t,\omega_t)\mathrm{d}^\pi\omega + \frac{1}{2}\int_0^T tr\left({}^t\nabla_\omega^2 F(t,\omega_t)\mathrm{d}[\omega]\right).$$

Remark 3.3.6. We note that for the change of variable formula to hold we only need to have the finite quadratic variation property, this does not require the process to be semimartingale.

Chapter 4 Extension to square-integrable functionals and pathwise hedging of options

In this chapter, we let $\mathcal{M}^2(X)$ be the space of square-integrable $\mathcal{F}_t^X$-martingales with initial value zero. We now extend the operator (vertical derivative) ∇_X to a continuous functional on $\mathcal{M}^2(X)$. Given a square-integrable Itô process X, we extend the pathwise functional Itô calculus to weak calculus for non-anticipative functionals whose domain of applicability includes all square-integrable semimartingales adapted to the filtration $\mathcal{F}_t^X$ generated by X.

4.1 Weak derivatives

We denote by $X : [0,T] \times D([0,T],\mathbb{R}^d) \to \mathbb{R}^d$ a continuous, $\mathbb{R}^d$-valued semimartingale defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider the canonical space $D([0,T],\mathbb{R}^d)$, and $X(t,\omega) = \omega(t)$ to be the coordinate process. We denote by $(\mathcal{F}_t^X)_{t\geq 0}$ the natural filtration of X and $\mathbb{C}_b^{1,2}(X)$ the set of $(\mathcal{F}_t^X)$-adapted process Y which admit a functional representation in $\mathbb{C}_b^{1,2}(\Lambda_T)$:

$$\mathbb{C}_b^{1,2}(X) = \{Y : \exists F \in \mathbb{C}_b^{1,2}(\Lambda_T),\ Y(t) = F(t, X_t)\ \mathrm{d}t \times \mathrm{d}\mathbb{P} - a.e.\} \tag{4.1.1}$$

We assume that

$$[X](t) = \int_0^t A(u)\mathrm{d}u \tag{4.1.2}$$

for some cádlág process A with values in S_d^+.

Any non-anticipative functional $F : \Lambda_T \to \mathbb{R}$ applied to X generates an $\mathcal{F}_t^X$-adapted process

$$Y(t) = F(t, X_t) = F(t, \{X(u \wedge t),\ u \in [0,T]\}). \tag{4.1.3}$$

The functional representation of the process Y is not unique. Y is not affected by the values taken by F outside $\mathcal{W}_T$, meanwhile the definition of the vertical derivative $\nabla_\omega F$ depends on the value of F for paths such as $\omega + e\mathbb{1}_{[t,T]}$ which is not continuous. Now we recall the definition of vertical derivative of a process:

Definition 4.1.1 (Vertical derivative of a process). *Define $\mathbb{C}_{\mathrm{loc}}^{1,2}(X)$ the set of $\mathcal{F}_t$-adapted process Y which admit a functional representation in $\mathbb{C}_{\mathrm{loc}}^{1,2}(\Lambda_T)$:*

$$\mathbb{C}_{\mathrm{loc}}^{1,2}(X) = \{Y,\ \exists F \in \mathbb{C}_{\mathrm{loc}}^{1,2}(\Lambda_T),\ Y(t) = F(t, X_t)\ \mathrm{d}t \times \mathrm{d}\mathbb{P} - a.e.\}. \tag{4.1.4}$$

In the case where X is a standard Brownian motion this construction is still applicable, and gives the existence of a vertical derivative of processes for $\mathbb{C}^{1,2}_{\text{loc}}(\Lambda_T)$ Brownian functionals:

Definition 4.1.2 (Vertical derivative of non-anticipative Brownian functionals)**.** *Let W be a standard d-dimensional Brownian motion. For any $Y \in \mathbb{C}^{1,2}_{\text{loc}}(W)$ with representation $Y(t) = F(t, W_t)$, the predictable process*

$$\nabla_W Y(t) = \nabla_\omega F(t, W_t)$$

is uniquely defined up to an evanescent set, independently of the choice of $F \in \mathbb{C}^{1,2}_{\text{loc}}(\Lambda_T)$.

The vertical derivative ∇_X defines a linear operator

$$\nabla_X : \mathbb{C}^{1,2}_b(X) \to \mathbb{C}^{1,2}_l(X)$$

which maps any process $Y \in \mathbb{C}^{1,2}_b(X)$ to an $\mathcal{F}^X_t$-adapted process $\nabla_X Y$. This linear operator is defined on the algebra of smooth functionals $\mathbb{C}^{1,2}_b(X)$, with the following properties: for any $Y, Z \in \mathbb{C}^{1,2}_b(X)$ and any $\mathcal{F}^X_t$-predictable process λ, we have

1. $\mathcal{F}^X_t$-linearity: $\nabla_X(Y + \lambda Z)(t) = \nabla_X Y(t) + \lambda(t)\nabla_X Z(t)$.
2. Differentiation of products: $\nabla_X(YZ)(t) = Z(t)\nabla_X Y(t) + Y(t)\nabla_X Z(t)$.
3. Composition rule: If $U \in \mathbb{C}^{1,2}_b(Y)$, then $U \in \mathbb{C}^{1,2}_b(X)$ and $\forall \in [0, T]$,

$$\nabla_X U(t) = \nabla_Y U(t) \cdot \nabla_X Y(t).$$

The operator ∇_X has crucial links with the Itô stochastic integral, given by the following proposition:

Proposition 4.1.3. For any local martingale $Y \in \mathbb{C}^{1,2}_{\text{loc}}(X)$ we have the representation

$$Y(T) = Y(0) + \int_0^T \nabla_X Y \mathrm{d}M \tag{4.1.5}$$

where M is the local martingale component of X.

Proof. Since $Y \in \mathbb{C}^{1,2}_{\text{loc}}(X)$, there exists $F \in \mathbb{C}^{1,2}_{\text{loc}}(\mathcal{W}_T)$ such that $Y(t) = F(t, X_t)$. Theorem 3.2.6 then implies that for any $t \in [0, T)$, we have

$$\begin{aligned} Y(t) - Y(0) = \int_0^t \mathcal{D}F(u, X_u)\mathrm{d}u + \frac{1}{2}\int_0^t \operatorname{tr}\left({}^t\nabla^2_\omega F(u, X_u)\mathrm{d}[X](u)\right) \\ + \int_0^t \nabla_\omega F(u, X_u)\mathrm{d}Z(u) + \int_0^t \nabla_\omega F(u, X_u)\mathrm{d}M(u). \end{aligned} \tag{4.1.6}$$

Since F is a locally regular non-anticipative functional, all the terms of Equation (4.1.6) are continuous process with finite variation while the last is a continuous local martingale. By uniqueness of the semimartingale decomposition of Y, we have $Y(t) = \int_0^t \nabla_\omega F(u, X_u)\mathrm{d}M(u)$. Since $F \in \mathbb{C}^{1,2}_l([0, T])$, $Y(t) \to F(T, X_T)$, as $t \to T$ so the stochastic integral is well defined for $t \in [0, T]$. □

Now we explore how it links with the martingale representation theorem.

4.2 Martingale representation formula

From Tikanmäki (2013) and Bender et al. (2008), introduce the concept of full support which means that any path of a stochastic process is possible. We give the following definitions:

Definition 4.2.1. *The support of $X \in (\mathcal{F}_t^X)_{t\in[0,T]}$, denoted by supp$(\mathbb{P}(X))$, is a closed subset of C_{x_0} with $\mathbb{P}-$null complement, such that $\mathbb{P}(G) > 0$ for all open subsets G of C_{x_0} having non-empty intersection with it.*

The process X has full support if

$$\text{supp}(\mathbb{P}(X_T)) = C_{x_0}. \tag{4.2.1}$$

We assume that the following small ball condition is verified:

Given $\eta \in C_{\sigma,x_0}$ and $\epsilon \geq 0$

$$\mathbb{P}(\|X - \eta\|_\infty < \epsilon) > 0 \tag{4.2.2}$$

Definition 4.2.2 (Discounted market model)**.** *A five-tuple $(\Omega, \mathcal{F}, X, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ is called a discounted market model if $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions and $X = (X_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t)_{t\in[0,T]}$ progressively measurable quadratic variation process with continuous paths starting from x_0.*

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with linear growth.

Definition 4.2.3. *Let f_σ be the unique solution to the ODE*

$$f'(x) = \sigma(f(x)), \quad f(0) = x_0.$$

Since σ is continuously differentiable, f_σ belongs to $C^2(\mathbb{R})$ and

$$f''_\sigma(x) = f'_\sigma(x)\sigma'(f_\sigma(x)).$$

Define the space

$$C_{\sigma,x_0} = \{f_\sigma(\theta(\cdot)) \mid \theta \in C([0,T]),\ \theta(0) = 0\}.$$

Definition 4.2.4 (Model class $\mathcal{M}_\sigma$)**.** *The model class $\mathcal{M}_\sigma$ corresponding to σ is defined to be the class of discounted market models satisfying the quadratic variation property*

$$\mathrm{d}[X](t) = \sigma^2(X_t)\mathrm{d}t \ \ \textit{a.s.}$$

and the non-degeneracy property: $\mathbb{P}(X \in C_{\sigma,x_0}) = 1$ and X has full support in C_{x_0}.

The functional Itô formula given by Equation (3.2.9) leads to an explicit martingale representation formula for $\mathcal{F}_t^X-$martingales. This may be looked at as a non-anticipative version of the Clark-Haussmann-Ocone formula.

We now consider the case where X is a Brownian martingale

$$X(t) = x_0 + \int_0^t \sigma(u)\mathrm{d}W_u,$$

where σ is an $\mathcal{F}_t^W$-adapted process satisfying

$$\mathbb{E}\left(\int_0^T \|\sigma(t)\|^2 \mathrm{d}t\right) < \infty, \quad \det(\sigma(t)) \neq 0 \ \ \mathrm{d}t \times \mathrm{d}\mathbb{P} - a.e. \tag{4.2.3}$$

Let X be a square-integrable martingale with the predictable representation property. Let $(\mathcal{F}_t^X)_{t\in[0,T]}$ be the filtration generated by X, for any square-integrable $\mathcal{F}_t^X$-measurable random variable H such that $\mathbb{E}|H| < \infty$, alternatively, any square-integrable $\mathcal{F}_t^X$-martingale Y defined by $Y(t) = \mathbb{E}[H|\mathcal{F}_t]$.

Theorem 4.2.5. *If $Y \in \mathbb{C}_b^{1,2}(X)$, then*

$$Y(T) = \mathbb{E}[Y(T)] + \int_0^T \nabla_\omega F(t,\omega_t)\mathrm{d}X_t,$$

where $Y(t) = F(t,\omega_t)$ and the stochastic integral is the Itô integral.

We now define the following spaces:

Definition 4.2.6. *Tikanmäki (2013) We call $\mathcal{L}^2(X)$ the Hilbert space of $\mathcal{F}_t^X-$predictable processes ϕ such that*

$$\|\phi\|^2_{\mathcal{L}^2(X)} = \mathbb{E}\left[\int_0^T \phi^2(t)\mathrm{d}[X](t)\right] < \infty.$$

Definition 4.2.7. *We call $\mathcal{M}^2(X)$ the space of square-integrable $\mathcal{F}_t^X-$martingales and is defined by*

$$\mathcal{M}^2(X) = \{Y = \int_0^\cdot \phi(t)\mathrm{d}X_t \mid \phi \in \mathcal{L}^2(X)\},$$

equipped with the norm $\|Y\|_2^2 = \mathbb{E}\left[Y(T)^2\right]$.

We define the space $D(X)$ as

$$D(X) := \mathbb{C}_b^{1,2}(X) \cap \mathcal{M}^2(X).$$

This space consist of $\mathcal{F}_t^X-$adapted processes which admits a $\mathbb{C}^{1,2}-$representation and are also taken from the space of square-integrable stochastic integrals with respect to X.

This definitions plays a major role in the computation of the Greeks, which will be covered in the next chapter.

Theorem 4.2.8. *The vertical derivative $\nabla_X : D(X) \to \mathcal{L}^2(X)$ admits a closure in $\mathcal{M}^2(X)$. Its closure is a bijective isometry*

$$\nabla_X : \mathcal{M}^2(X) \to \mathcal{L}^2(X)$$
$$\int_0^\cdot \phi \mathrm{d}X \to \phi$$

characterised by the property that, for any $Y \in \mathcal{M}^2(X)$, $\nabla_X Y$ is the unique element of $\mathcal{L}^2(X)$ such that

$$\forall Z \in D(X) \ \ \mathbb{E}[Y(T)Z(T)] = \mathbb{E}\left[\int_0^T \nabla_X Y(t)\nabla_X Z(t)\mathrm{d}[X](t)\right]. \tag{4.2.4}$$

In particular, ∇_X is the adjoint of the Itô stochastic integral

$$I_X : \mathcal{L}^2(X) \to \mathcal{M}^2(X)$$
$$\phi \to \int_0^{\cdot} \phi \mathrm{d}X$$

in the following sense: $\forall \phi \in \mathcal{L}^2(X),\ \forall Y \in \mathcal{M}^2(X)$

$$\mathbb{E}\left[Y(T)\int_0^T \phi \mathrm{d}X(t)\right] = \mathbb{E}\left[\int_0^T \nabla_X Y(t)\phi(t)\mathrm{d}[X](t)\right]. \tag{4.2.5}$$

Remark 4.2.9. This can be understood as an integration by by parts formula on $[0,T] \times D([0,T],\mathbb{R}^d)$ with respect to the measure $\mathrm{d}[X] \times \mathrm{d}\mathbb{P}$.

Proof. Any $Y \in \mathcal{M}^2(X)$ may be written as $Y(t) = \displaystyle\int_0^t \phi(s)\mathrm{d}X_s$ with the integrand $\phi \in \mathcal{L}^2(X)$, which is defined uniquely on $\mathrm{d}[X] \times \mathrm{d}\mathbb{P}$ $a.e.$ The Itô isometry formula then guarantees that Equation (4.2.4) holds for ϕ. To show that Equation (4.2.4) uniquely characterizes ϕ, we consider $\psi \in \mathcal{L}^2(X)$ which satisfies Equation (4.2.4) also, denoting $I_X(\psi) = \int_0^{\cdot} \psi \mathrm{d}X$ its stochastic integral with respect to X, then Equation (4.2.4) implies that

$$\forall Z \in D(X),\ \ \langle I_X(\psi) - Y, Z\rangle_{\mathcal{M}^2(X)} = \mathbb{E}\left[\left(Y(T) - \int_0^T \psi \mathrm{d}X\right)Z(T)\right] = 0$$

which implies $I_X(\psi) = Y\mathrm{d}[X] \times \mathrm{d}\mathbb{P}$ $a.e.$, since by construction $D(X)$ is dense in $\mathcal{M}^2(X)$. Hence, $\nabla_\omega : D(X) \to \mathcal{L}^2(X)$ is closable on $\mathcal{M}^2(X)$. □

Remark 4.2.10. ∇_X extends the pathwise vertical derivative ∇_ω to a weak derivative defined for all square-integrable martingales, which is the inverse of the Itô integral I_X with respect to X:

$$\forall \phi \in \mathcal{L}^2(X),\ \ \nabla_X\left(\int_0^{\cdot} \phi \mathrm{d}X\right) = \phi$$

holds in the sense of equality in $\mathcal{L}^2(X)$.

This result helps us in computing the sensitivity of the price of an underlying asset, which is done in the next chapter.

Suppose that X is a continuous martingale with respect to its filtration $(\mathcal{F}_t^X)_{t\in[0,T]}$. Let $H \in L^1$ be $\mathcal{F}_T^X-$measurable random variable and $Y(t) = \mathbb{E}\left[H|\mathcal{F}_t^X\right]$. If $Y \in \mathbb{C}_b^{1,2}(X)$ such that $Y(t) = F(t,X_t)$ then by Theorem 4.2.8 for all $t \in [0,T]$ we have the representation

$$\begin{aligned} Y(t) &= \mathbb{E}[Y(t)] + \int_0^t \nabla_\omega F(s,X_s)\mathrm{d}X_s \\ &= F(0,X_0) + \int_0^t \nabla_\omega F(s,X_s)\mathrm{d}X_s, \end{aligned}$$

where the stochastic integral is the Itô integral, and the first term can be understood as the initial capital. We now extend the hedging result to the model class $\mathcal{M}_\sigma$ for the functional F. This differs to the martingale case in the sense that the first term (initial capital) can not be understood as an expectation and the stochastic integral is understood as the Föllmer integral not the Itô integral.

Theorem 4.2.11. *Tikanmäki (2013) Let* $(\Omega, \mathcal{F}, X, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}) \in \mathcal{M}_\sigma$ *such that* $Y \in \mathbb{C}_b^{1,2}(X)$ *with* $Y(t) = F(t, X_t)$*. Then for all* $(\tilde{\Omega}, \tilde{\mathcal{F}}, Z, (\tilde{\mathcal{F}})_{t\in[0,T]}, \tilde{\mathbb{P}}) \in \mathcal{M}_\sigma$ *and for all* $t \in [0, T]$

$$F(t, Z_t) = F(0, Z_0) + \int_0^t \nabla_\omega F(s, Z_s)\mathrm{d}Z_s, \quad \tilde{\mathbb{P}} - \text{a.s.}$$

with the stochastic integral is in this case the Föllmer integral.

Proof. For the continuous semimartingale X, according to Equation (3.2.9) we have

$$F(t, Z_t) - F(0, Z_0) = \int_0^t \mathcal{D}F(u, Z_u)\mathrm{d}u + \frac{1}{2}\int_0^t \nabla_\omega^2 F(u, Z_u)\mathrm{d}[Z](u) + \int_0^t \nabla_\omega F(u, Z_u)\mathrm{d}Z_u$$

where the last term is now understood as the Föllmer integral. Now, we prove this result by contradiction. Assume that there exists $\tilde{\Omega}_1 \subset \Omega$ with $\tilde{\mathbb{P}}(\tilde{\Omega}_1) > 0$ such that

$$\left|\int_0^t \mathcal{D}F(u, Z_u)\mathrm{d}u + \frac{1}{2}\int_0^t \nabla_\omega^2 F(u, Z_u)\mathrm{d}[Z](u)\right| > \epsilon > 0$$

in $\tilde{\Omega}_1$. On the other hand, for $Z \in C_{\sigma,x_0}$, $\tilde{\mathbb{P}}(Z) = 1$. Thus, there exists $X \in C_{\sigma,x_0}$ with quadratic variation given by $\mathrm{d}[X](u) = \sigma^2(X_u)\mathrm{d}u$ such that

$$\left|\int_0^t \mathcal{D}F(u, X_u)\mathrm{d}u + \frac{1}{2}\int_0^t \nabla_\omega^2 F(u, X_u)\mathrm{d}[X](u)\right| > \epsilon > 0.$$

Let $(X^n)_{n=1}^\infty \subset C_{\sigma,x_0}$ with $\mathrm{d}[X^n](u) = \sigma^2(X_u^n)\mathrm{d}u$ such that $X^n \to X$ in the supremum norm. We recall that $\mathcal{D}F$ and $\nabla_\omega^2 F$ are continuous at fixed times and locally bounded. Thus, by the dominated convergence theorem

$$\int_0^t \mathcal{D}F(u, X_u^n)\mathrm{d}u + \frac{1}{2}\int_0^t \nabla_\omega^2 F(u, X_u)\sigma^2(X_u^n)\mathrm{d}u \to \int_0^t \mathcal{D}F(u, X_u)\mathrm{d}u + \frac{1}{2}\int_0^t \nabla_\omega^2 F(u, X_u)\mathrm{d}[X](u),$$

as $n \to \infty$. Thus, each path $\tilde{\omega} \in \tilde{\Omega}_1$ has a surrounding small ball $B_{\tilde{\omega}} \subset C_{\sigma,x_0}$ and $B'_{\tilde{\omega}} = \{X \in B_{\tilde{\omega}} \mid \mathrm{d}[X](u) = \sigma^2(X_u)\mathrm{d}u\}$ such that for $\xi \in B'_{\tilde{\omega}}$, we have

$$\left|\int_0^t \mathcal{D}F(u, \xi_u)\mathrm{d}u + \frac{1}{2}\int_0^t \nabla_\omega^2 F(u, \xi_u)\mathrm{d}[\xi](u)\right| > \frac{\epsilon}{2}. \tag{4.2.6}$$

By the full support property we have that

$$\mathbb{P}(X_u \in B'_{\tilde{\omega}}) = \mathbb{P}(X_u \in B_{\tilde{\omega}}) > 0.$$

On the other hand, by comparing Equation (3.2.9) and Theorem 4.2.5, we have

$$\int_0^t \mathcal{D}F(u, X_u)\mathrm{d}u + \frac{1}{2}\int_0^t \nabla_\omega^2 F(u, X_u)\mathrm{d}[X](u) = 0 \;\; \mathbb{P} - \text{a.s.} \tag{4.2.7}$$

Hence, Equation (4.2.6) and (4.2.7) shows that $\tilde{\mathbb{P}}(\tilde{\Omega}_1) = 0$, which is a contradiction. Thus, the claim holds $\tilde{\mathbb{P}}-$a.s. □

Example 4.2.12. Let $H = \int_0^T X_t \mathrm{d}t$. Then

$$Y(t) = \mathbb{E}[H|\mathcal{F}_t^X] = \int_0^t X_s \mathrm{d}s + (T-t)X_t \in \mathbb{C}_b^{1,2}(X).$$

Thus, the functional $F(t, X_t) = \int_0^t X_s \mathrm{d}s + (T-t)X_t$. Then we have

$$\mathcal{D}F(t, X_t) = 0$$

and

$$\nabla_\omega F(t, X_t) = T - t.$$

Hence,

$$\int_0^T Z_s \mathrm{d}s = TZ(0) + \int_0^T (T-s)\mathrm{d}Z_s,$$

the stochastic integral is understood in the Föllmer sense.

Example 4.2.13. Let X be a continuous path and F be a cylindrical functional such that $\nabla_\omega F$ is a cylindrical integrand. Then $\nabla_\omega^2 F = 0$ and $\mathcal{D}F = 0$ and Equation (3.2.9) implies that

$$F(t, X_t) = F(0, X_0) + \int_0^t \nabla_\omega F(s, X_s)\mathrm{d}X_s.$$

Remark 4.2.14. For cylindrical functionals of the type in Example 4.2.13, we realize that we do not need the support property to obtain the hedging strategy. Thus, we realise that the hedging strategy is not only robust in model class $\mathcal{M}_\sigma$ but for all quadratic variation models.

We require the following lemma for the proof of the next theorem:

Lemma 4.2.15. *Tikanmäki (2013) Let $F \in \mathbb{C}_b^{1,2}$ and $X \in C_{\sigma,x_0}$. Then the mapping*

$$\begin{aligned} C_{\sigma,x_0} &\to C([0,T]) \\ X &\mapsto \int_0^\cdot \nabla_\omega F(s, X_s)\mathrm{d}X_s \end{aligned}$$

is continuous.

Proof. Let $(X^n)_{n=0}^\infty \subset C_{\sigma,x_0}$ such that $\mathrm{d}[X^n](u) = \sigma^2(X_u^n)\mathrm{d}u$ be a sequence that converges to X in the supremum norm. By the functional change of variables formula, Equation (3.2.9), we obtain that

$$\begin{aligned} \int_0^t \nabla_\omega F(s, X_s)\mathrm{d}X_s - \int_0^t \nabla_\omega F(s, X_s^n)\mathrm{d}X_s^n &= F(t, X_t) - F(t, X_t^n) - \frac{1}{2}\int_0^t \nabla_\omega^2 F(s, X_s)\mathrm{d}[X](s) \\ &+ \frac{1}{2}\int_0^t \nabla_\omega^2 F(s, X_s^n)\mathrm{d}[X^n](s) - \int_0^t \mathcal{D}F(s, X_s)\mathrm{d}s + \int_0^t \mathcal{D}F(s, X_s^n)\mathrm{d}s. \end{aligned}$$

We recall that F is continuous at fixed times. Thus, $F(t, X_t^n) \to F(t, X_t)$. Then second-order vertical derivative, $\nabla_\omega^2 F$ is also a functional and is locally bounded and continuous at fixed times, and σ is a function with linear growth. Thus, by the dominated convergence theorem, we get

$$\begin{aligned} &\int_0^t \nabla_\omega^2 F(s, X_s^n)\mathrm{d}[X^n](s) = \int_0^t \nabla_\omega^2 F(s, X_s^n)\sigma^2(X_s^n)\mathrm{d}s \\ \to &\int_0^t \nabla_\omega^2 F(s, X_s)\sigma^2(X_s)\mathrm{d}s = \int_0^t \nabla_\omega^2 F(s, X_s)\mathrm{d}[X](s). \end{aligned}$$

We also recall that $\mathcal{D}F$ is locally bounded and continuous at fixed times. Thus, by the dominated convergence theorem, we get

$$\int_0^t \mathcal{D}F(s, X_s^n)\mathrm{d}s \to \int_0^t \mathcal{D}F(s, X_s)\mathrm{d}s.$$

Thus, the mapping is continuous and hence, completes the proof. □

We commence by giving the definition of a hindsight factor:

Definition 4.2.16. *A mapping $g : [0,T] \times C_{\sigma,x_0} \to \mathbb{R}$ is a hindsight factor if:*

1. $g(t,\eta) = g(t,\tilde{\eta})$ *whenever* $\eta(s) = \tilde{\eta}(s)$ *for all* $0 \leq s \leq t$*,*
2. *for every* $\eta \in C_{\sigma,x_0}$*,* $g(\cdot,\eta)$ *is of bounded variation and continuous,*
3. *for every continuous function* f*, there exists a constant* K *such that*

$$\left|\int_0^t f(s)\mathrm{d}g(s,\eta) - \int_0^t f(s)\mathrm{d}g(s,\tilde{\eta})\right| \leq K \max_{0\leq r\leq t} |f(r)| \, \|\eta - \tilde{\eta}\|_\infty .$$

Remark 4.2.17. Property 1 is the assumption that the factors contains no information about the future stock prices. Property 2 and 3 insures continuity as the integrands are not continuous, naturally.

Suppose that the hindsight factors $g_i,\ i = 1,\ldots,n$ and a function $\varphi : [0,T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are given. We consider strategies of the form

$$\Phi_t = \varphi(t, X_t, g_1(t,X_t),\ldots,g_n(t,X_t)). \tag{4.2.8}$$

Here, Φ_t denotes the number of stocks held at time t by an investor, and we call Φ_t the allowed strategy. Thus, the wealth process corresponding to the allowed strategy Φ is

$$V_t(\Phi, v_0, X) = v_0 + \int_0^t \Phi_u \mathrm{d}X_u,$$

where $v_0 \in \mathbb{R}$ denotes the investor's initial capital. We note that the stochastic integral is the Föllmer's integral, which is defined as the limit of forward sums. Hence, this definition has a similar meaning to the classical case of self-financing portfolio.

Definition 4.2.18. *A strategy Φ is called smooth if it is of the form (4.2.8) with $\varphi \in C^1([0,T] \times \mathbb{R} \times \mathbb{R}^n)$ and it is no-doubling-strategies-admissible in the classical sense, that is, there exists a constant $a > 0$ such that for all $t \in [0,T]$,*

$$\int_0^t \Phi_u \mathrm{d}X_u \geq -a \quad \mathbb{P} - \text{a.s.}$$

Now, we recall that a strategy Φ is an arbitrage in the market model $(\Omega, \mathcal{F}, X, (\mathcal{F}_t^X)_{t\in[0,T]}, \mathbb{P})$ if

$$V_T(\Phi, 0, X) \geq 0 \quad \mathbb{P} - \text{a.s.} \quad \text{and } \mathbb{P}(V_T(\Phi,0,X) > 0) > 0.$$

Lemma 4.2.19. *Let $0 \leq t \leq T$ and let Φ be a smooth strategy induced by φ in the market $(\Omega, \mathcal{F}, X, (\mathcal{F}_t^X)_{t\in[0,T]}, \mathbb{P}) \in \mathcal{M}_\sigma$. Then for $0 \leq t \leq T$, we have*

$$V_t(\Phi, v_0, X) = v_0 + V(t, X, \varphi) \quad \mathbb{P} - \text{a.s.}$$

Moreover, the map

$$\begin{aligned} C_{\sigma,x_0} &\to C([0,T]) \\ \eta &\mapsto V(\cdot, \eta, \varphi) \end{aligned}$$

is continuous.

Proof. See Bender et al. (2008) for the proof. □

The next result is important such that it is in some sense a non-probabilistic version of the uniqueness part of the martingale representation theorem.

Theorem 4.2.20. *Tikanmäki (2013) Let $(\Omega, \mathcal{F}, X, (\mathcal{F}_t^X)_{t\in[0,T]}, \mathbb{P}) \in \mathcal{M}_\sigma$ be a discounted market model such that X is a continuous and square-integrable martingale. Let a claim $H \in \mathcal{F}_t^X$ such that $\mathbb{E}[H^2] < \infty$. Assume that H can be hedged using an allowed strategy (4.2.8), such that $\Phi_t \in \mathcal{L}^2(X)$. Assume also that $Y(t) = \mathbb{E}[H|\mathcal{F}_t^X] \in D(X)$ such that for $F \in \mathbb{C}_b^{1,2}$ we have $Y(t) = F(t, X_t)$. Then for all $X \in C_{\sigma,x_0}$ satisfying $\mathrm{d}[X](s) = \sigma^2(X_s)\mathrm{d}s$ and $t \in [0,T]$*

$$\int_0^t \nabla_\omega F(s, X_s)\mathrm{d}X_s = \int_0^t \varphi(s, X_s, g_1(s, X_s), \ldots, g_n(s, X_s))\mathrm{d}X_s.$$

Proof. We write $\psi(t, X_t) = \varphi(t, X_t, g_1(t, X_t), \ldots, g_n(t, X_t))$. Then we prove this result by contradiction. Assume that for some $X \in C_{\sigma,x_0}$ satisfying $\mathrm{d}[X](s) = \sigma^2(X_s)\mathrm{d}s$ and some $t \in [0,T]$

$$\left| \int_0^t \psi(s, X_s)\mathrm{d}X_s - \int_0^t \nabla_\omega F(s, X_s)\mathrm{d}X_s \right| = \epsilon > 0.$$

Without any loss of generality, we assume that $t < T$. We now use the property (4.2.2) and Lemma 4.2.19 to obtain that

$$\mathbb{P}\left(\int_0^t \psi(s, X_s)\mathrm{d}X_s \neq \int_0^t \nabla_\omega F(s, X_s)\mathrm{d}X_s \right) > 0. \tag{4.2.9}$$

On the other hand, by the assumptions

$$\int_0^T \psi(s, X_s)\mathrm{d}X_s = H - \mathbb{E}[H] = \int_0^T \nabla_\omega F(s, X_s)\mathrm{d}X_s.$$

Now, by the Itô isometry

$$\mathbb{E}\left[\int_0^T (\psi(s, X_s) - \nabla_\omega F(s, X_s))^2 \sigma^2(X_s)\mathrm{d}s \right] = 0.$$

Since $\sigma^2(X_s) \neq 0$, this implies that

$$\psi(s, X_s) = \nabla_\omega F(s, X_s).$$

Thus, for all $t \in [0, T]$

$$\mathbb{P}\left(\int_0^t \psi(s, X_s)\mathrm{d}X_s = \int_0^t \nabla_\omega F(s, X_s)\mathrm{d}X_s\right) = 1,$$

which contradicts Inequality (4.2.9). Hence, the proof is complete. □

Example 4.2.21. The continuous average can be represented using the following non-anticipative functional

$$F(t, X_t) = \frac{1}{T}\int_0^t X_s \mathrm{d}s + \frac{T-t}{T} X_t.$$

If X is a martingale, then $(F(t, X_t))_{t\in[0,T]}$ is also a martingale. Thus

$$\nabla_\omega F(t, X_t) = \frac{T-t}{T}$$

is a pathwise hedging strategy. We note that

$$\nabla^2_\omega F(t, X_t) = 0$$

and

$$\mathcal{D}F(t, X_t) = 0.$$

Hence, the following integral representation holds for X without referencing the property (4.2.2)

$$\frac{1}{T}\int_0^T X_s \mathrm{d}s = X_0 + \int_0^T \frac{T-s}{T}\mathrm{d}X_s.$$

Example 4.2.22. The discrete average can be hedged using the cylindrical functional

$$F(t, X_t) = \sum_{i=1}^N f_i(t, X_t, X(t_0), \ldots, X(t_{i-1}))\mathbb{1}_{(t_{i-1}, t_i]},$$

where

$$f_i(t, X_t, X(t_0), \ldots, X(t_i)) = \frac{1}{N+1}\left(\sum_{j=0}^{i-1} X(t_j) + \frac{t - t_{i-1}}{t_i - t_{i-1}} X_t\right),$$

where $t_j = \frac{j}{N}$, $j = 0, \ldots, N$. Now, the derivatives are

$$\nabla_\omega F(t, X_t) = \frac{N}{N+1}\sum_{i=1}^N (t - t_{i-1})\mathbb{1}_{(t_{i-1}, t_i]}(t),$$
$$\nabla^2_\omega F(t, X_t) = 0,$$

and

$$\mathcal{D}F(t, X_t) = \frac{N}{N+1} X_t.$$

Hence, the following change of variable holds

$$\frac{1}{N+1}\sum_{j=0}^N X(t_j) = x_0 + \frac{N}{N+1}\int_0^T \sum_{j=1}^N (s - t_{j-1})\mathbb{1}_{(t_{j-1}, t_j]}(s)\mathrm{d}X_s + \frac{N}{N+1}\int_0^T X_s \mathrm{d}s.$$

This gives the hedging strategy for the difference of discrete and continuous averages.

4.3 Euler approximations for path-dependent SDEs

In this section we consider the following distance structure, d, defined on Λ_T:

$$d((t,\omega),(t',\omega')) := \sup_{u\in[0,T]} |\omega(u\wedge t) - \omega'(u\wedge t')| + \sqrt{|t-t'|} = \|\omega_t - \omega'_{t'}\|_\infty + \sqrt{|t-t'|}.$$

We consider the general SDE with path-dependent coefficients

$$\mathrm{d}X(t) = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \tag{4.3.1}$$

where $X(0) = x_0 \in \mathbb{R}^d$,

$$b : \Lambda_T \to \mathbb{R}^d$$
$$\sigma : \Lambda_T \to \mathbb{M}_d(\mathbb{R})$$

are non-anticipative functionals, assumed to be Lipschitz continuous with respect to the distance d.

Assumption 4-$\mathcal{A}$. (Lipschitz continuity of coefficients) The non-anticipative functionals

$$b : (\Lambda_T, d) \to \mathbb{R}^d$$
$$\sigma : (\Lambda_T, d) \to \mathbb{M}_d(\mathbb{R})$$

are Lipschitz continuous if

$$\forall t, t' \in [0,T],\ \forall \omega, \omega' \in D([0,T], \mathbb{R}^d),\ \exists \kappa > 0,$$

such that

$$|b(t,\omega) - b(t',\omega')| + \|\sigma(t,\omega) - \sigma(t',\omega')\| \le \kappa d((t,\omega),(t',\omega')).$$

Remark 4.3.1. This Lipschitz condition allows for a Hölder smoothness of degree $^1/_2$ in t. The conditions are weaker than the case where the distance d_∞ is used in place of the distance d.

Remark 4.3.2. Assumption 4-$\mathcal{A}$ is still strong. The next proposition holds under weaker conditions. For instance, the Hölder condition with respect to the differences in t can be by the weaker condition:

$$\sup_{t\in[0,T]} |b(t,\bar{0})| + \left\|\sigma(t,\bar{0})\right\| < \infty$$

where $\bar{0}$ denotes the path which takes constant value 0.

The pathwise version of the famous Burkholder-Davis-Gundy (BDG) inequalities is given by the following theorem

Theorem 4.3.3 (Burkholder-Davis-Gundy (BDG) inequalities)**.** *For $1 \le p < \infty$, there exist constants $a_p, b_p < \infty$ such that the following holds: for every $N \in \mathbb{N}$ and every martingale $(x_k)_{k=0}^N$*

$$\mathbb{E}[X]_N^{\frac{p}{2}} \le a_p \mathbb{E}\left[(X_N^*)^p\right],\ \mathbb{E}\left[(X_N^*)^p\right] \le b_p \mathbb{E}[X]_N^{\frac{p}{2}}. \tag{4.3.2}$$

Now, the following result is useful for the proof of the next proposition: Given real numbers $x_n,\ h_n,\ n \in \mathbb{N}$, we write

$$x_n^* := \max_{k \leq n} |x_k|,\ [x]_n := x_0^2 + \sum_{k=0}^{n-1} (x_{k+1} - x_k)^2,\ (h \cdot x)_n := \sum_{k=0}^{n-1} h_k (x_{k+1} - x_k).$$

Proposition 4.3.4. Under Assumption 4-$\mathcal{A}$, there exists a unique $\mathcal{F}_t^X$-adapted process X satisfying (4.3.1). Moreover for $p \geq 1$, we have

$$\mathbb{E}\left[\|X_T\|_\infty^{2p}\right] \leq C(1 + |x_0|^{2p}) e^{CT} \tag{4.3.3}$$

for some constant C depending on $p,\ T$ and κ.

Proof. (Protter, 2004, Chapter 5, Theorem 7) shows the existence and uniqueness of this result. We want to show that the Inequality (4.3.3) holds. Using the Burkholder-Davis-Gundy and Hölder's inequalities, we have

$$\mathbb{E}\left[\|X_T\|_\infty^{2p}\right] \leq C(p) \left(|x_0|^{2p} + \mathbb{E}\left[\left(\int_0^T |b(t, X_t)|^2 \mathrm{d}t\right)^p\right] + \mathbb{E}\left[\left(\int_0^T \|\sigma(t, X_t)\|^2 \, \mathrm{d}t\right)^p\right]\right).$$

By Hölder's inequality, $\|f\|_p = (\int_s |f|^p \mathrm{d}\mu)^{\frac{1}{p}}$, we have

$$\begin{aligned} \mathbb{E}\left[\|X_T\|_\infty^{2p}\right] &\leq C(p, T) \left(|x_0|^{2p} + \mathbb{E}\left[\int_0^T |b(t, X_t)|^{2p} \mathrm{d}t\right] + \mathbb{E}\left[\int_0^T \|\sigma(t, X_t)\|^{2p} \, \mathrm{d}t\right]\right) \\ &\leq C(p, T) \left(|x_0|^{2p} + \mathbb{E}\left[\int_0^T (|b(t, X_t)| + \|\sigma(t, X_t)\|)^{2p} \, \mathrm{d}t\right]\right). \end{aligned}$$

According to the weaker condition in the remark above, we have

$$\mathbb{E}\left[\|X_T\|_\infty^{2p}\right] \leq C(p, T) \left(|x_0|^{2p} + \mathbb{E}\left[\int_0^T \left(|b(0, \bar{0})| + \left\|\sigma(0, \bar{0})\right\| + \kappa(\sqrt{t} + \|X_t\|_\infty)\right)^{2p} \mathrm{d}t\right]\right),$$

by Hölder's inequality we have

$$\mathbb{E}\left[\|X_T\|_\infty^{2p}\right] \leq C(p, T, \kappa) \left(|x_0|^{2p} + 1 + \int_0^T \mathbb{E}\, \|X_t\|_\infty^{2p} \, \mathrm{d}t\right).$$

Gronwall's inequality yields the result. □

4.4 Euler approximations as non-anticipative functionals

We consider an Euler approximation for the path-dependent SDE (4.3.1) and study its properties as a non-anticipative functional. Suppose that $n \in \mathbb{N},\ \delta = \frac{T}{n}$. Then the Euler approximation $\bar{X}$ of X on $(t_j = j\delta,\ j = 0, \ldots, n)$ as its grid is defined as follows

Definition 4.4.1 (Euler scheme). *For $\omega \in D([0,T],\mathbb{R}^d)$, we denote by $\bar{X}(\omega) \in D([0,T],\mathbb{R}^d)$ the piecewise-constant Euler approximation for (4.3.1) computed along the path ω, defined as follows:*
$\bar{X}(\omega)$ is constant in each interval $[t_j,t_{j+1})$ $\forall 0 \leq j \leq n-1$ with $\bar{X}(0,\omega) = x_0$ and

$$\bar{X}(t_{j+1},\omega) = \bar{X}(t_j,\omega) + b(t_j,\bar{X}_{t_j}(\omega))\delta + \sigma(t_j,\bar{X}_{t_j}(\omega))(\omega(t_{j+1}-) - \omega(t_j-)), \tag{4.4.1}$$

where $\bar{X}_t(\omega) = \bar{X}(t \wedge \cdot, \omega)$ and by convention $\omega(0-) = \omega(0)$.

Remark 4.4.2. When the approximation is computed along the path of the Brownian motion W, $\bar{X}(W)$ is the piecewise-constant Euler-Maruyama scheme, Pardoux and Talay (1985) for the SDE (4.3.1).

By definition the path $\bar{X}(\omega)$ depends only on a finite number of increments of ω:

$$\omega(t_1-) - \omega(0), \ldots, \omega(t_n-) - \omega(t_{n-1}-).$$

We define a map

$$p_n : \mathbb{M}_{d,n}(\mathbb{R}) \to D([0,T],\mathbb{R}^d)$$

such that for $\omega \in D([0,T],\mathbb{R}^d)$ we have

$$p_n\left(\omega(t_1-) - \omega(0), \omega(t_2-) - \omega(t_1-), \ldots, \omega(t_n-) - \omega(t_{n-1}-)\right) =_n X(\omega). \tag{4.4.2}$$

We now, denote the path $p_n(y)$ stopped at t by $p_n(t,y)$. Then the following lemma shows that the map $p_n : \mathbb{M}_{d,n}(\mathbb{R}) \to (D([0,T],\mathbb{R}^d), \|\cdot\|_\infty)$ is locally Lipschitz continuous:

Lemma 4.4.3. *For every $\eta > 0$, there exists a constant C depending on η, T and κ such that for any $y = (y_1,\ldots,y_n), y' = (y'_1,\ldots,y'_n) \in \mathbb{M}_{d,n}(\mathbb{R})$,*

$$\max_{1\leq\kappa\leq n} |y_\kappa| \vee |y'_\kappa| \leq \eta \;\Rightarrow\; \|p_n(y) - p_n(y')\|_\infty \leq C(\eta,T,\kappa) \max_{1\leq\kappa\leq n} |y_\kappa - y'_\kappa|.$$

Proof. The paths $p_n(y)$ and $p_n(y')$ are constructed such that are stepwise constant, it suffices to show the inequality at times $(t_j)_{0\leq j\leq n}$. By mathematical induction we show that

$$\|p_n(t_j,y) - p_n(t_j,y')\|_\infty \leq C(\eta,T,\kappa) \max_{1\leq\kappa\leq n} |y_\kappa - y'_\kappa| \tag{4.4.3}$$

with some constant C depending on η, T and κ. For $j = 0$, we clearly have $p(y)(0) = p(y')(0) = x_0$. Assume that the Inequality (4.4.3) is verified for some $0 \leq j \leq n-1$, we consider

$$\|p_n(t_{j+1},y) - p_n(t_{j+1},y')\|_\infty,$$

we have

$$p_n(t_{j+1},y) = p_n(t_j,y) + b(t_j,p_n(t_j,y))\delta + \sigma(t_j,p_n(t_j,y))y_{j+1}$$

and

$$p_n(t_{j+1},y') = p_n(t_j,y') + b(t_j,p_n(t_j,y'))\delta + \sigma(t_j,p_n(t_j,y'))y'_{j+1}.$$

Thus

$$\begin{aligned}|p_n(t_{j+1},y)-p_n(t_{j+1},y')| &\le |p_n(t_j,y)-p_n(t_j,y')|+|b(t_j,p_n(t_j,y))-b(t_j,p_n(t_j,y'))|\delta\\ &+\|\sigma(t_j,p_n(t_j,y))\|\,|y_{j+1}-y'_{j+1}|+\|\sigma(t_j,p_n(t_j,y))-\sigma(t_j,p_n(t_j,y'))\|\,|y'_{j+1}|\\ &\le C(\eta,T,\kappa)\max_{1\le\kappa\le j}|y_\kappa-y'_\kappa|+\kappa C(\eta,T,\kappa)\max_{1\le\kappa\le j}|y_\kappa-y'_\kappa|\delta\\ &\quad+\left(\left\|\sigma(0,\bar{0})\right\|+\kappa(\sqrt{t_j}+\|p_n(t_j,y)\|_\infty)\right)|y_{j+1}-y'_{j+1}|\\ &\quad\;+\kappa\eta C(\eta,T,\kappa)\max_{1\le\kappa\le j}|y_\kappa-y'_\kappa|\\ &\le C(\eta,T,\kappa)\max_{1\le\kappa\le j+1}|y_\kappa-y'_\kappa|.\end{aligned}$$

Then we have

$$\|p_n(t_{j+1},y)-p_n(t_{j+1},y')\|_\infty\le C(\eta,T,\kappa)\max_{1\le\kappa\le j+1}|y_\kappa-y'_\kappa|,$$

And we conclude by mathematical induction. □

4.5 Strong convergence

Lemma 4.5.1. *Let $Y_1,\dots,Y_n$ be non-negative random variables with the same distribution satisfying $\mathbb{E}\left[e^{\lambda Y_i}\right]<\infty$ for some $\lambda>0$. Then we have: For all $p>0$,*

$$\|\max(Y_1,\dots,Y_n)\|_p\le\frac{1}{\lambda}(\log n+C(p,Y_1,\lambda)).$$

The following result gives a uniform estimate of the discretization error, $X_T-\bar{X}_T$ to the path-dependent SDE (4.3.1):

Proposition 4.5.2. For a path-dependent SDE (4.3.1) with coefficients satisfying Assumption (4-$\mathcal{A}$) we have the following estimate in L^{2p} for the strong error of the piecewise-constant Euler-Maruyama scheme:

$$\mathbb{E}\left[\sup_{s\in[0,T]}\left\|X(s)-\bar{X}(s)\right\|^{2p}\right]\le C(x_0,p,T,\kappa)\left(\frac{1+\log n}{n}\right)^p,\quad\forall p\ge 1$$

with constant C depending on $x_0,\ p,\ T$ and κ only.

Proof. We want to construct a Brownian interpolation $\hat{X}_T$ of $\bar{X}_T$:

$$\hat{X}(s)=x_0+\int_0^s b(\underline{u},\bar{X}_{\underline{u}})\mathrm{d}u+\int_0^s\sigma(\underline{u},\bar{X}_{\underline{u}})\mathrm{d}W_u,$$

where $\underline{u}=\left\lfloor\frac{u}{\delta}\right\rfloor$.

$\hat{X}$ is a continuous semimartingale and

$$\left\| \sup_{s\in[0,T]} \left|X(s)-\bar{X}(s)\right| \right\|_{2p} \leq \left\| \sup_{s\in[0,T]} \left|X(s)-\hat{X}(s)\right| \right\|_{2p} + \left\| \sup_{s\in[0,T]} \left|\hat{X}(s)-\bar{X}(s)\right| \right\|_{2p}. \tag{4.5.1}$$

We consider the term $\left\| \sup_{s\in[0,T]} \left|X(s)-\hat{X}(s)\right| \right\|_{2p}$ first. Using the BDG and Hölder's inequality, we have

$$\begin{aligned}
\mathbb{E}\left\|X_T-\hat{X}_T\right\|_\infty^{2p} &\leq C(p)\left(\mathbb{E}\left[\int_0^T \left|b(s,X_s)-b(\underline{s},\hat{X}_{\underline{s}})\right|\mathrm{d}s\right]^{2p}+\mathbb{E}\left[\int_0^T \left\|\sigma(s,X_s)-\sigma(\underline{s},\bar{X}_{\underline{s}})\right\|^2\mathrm{d}s\right]^{p}\right)\\
&\leq C(p,T)\left(\mathbb{E}\left[\int_0^T \left|b(s,X_s)-b(\underline{s},\bar{X}_{\underline{s}})\right|^{2p}\mathrm{d}s\right]+\mathbb{E}\left[\int_0^T \left\|\sigma(s,X_s)-\sigma(\underline{s},\bar{X}_{\underline{s}})\right\|^{2p}\mathrm{d}s\right]\right)\\
&\leq C(p,T,\kappa)\mathbb{E}\left[\int_0^T\left((s-\underline{s})^p+\left\|X_s-\bar{X}_s\right\|_\infty^{2p}\right)\mathrm{d}s\right]\\
&\leq C(p,T,\kappa)\left(\frac{1}{n^p}+\int_0^T\mathbb{E}\left\|X_s-\bar{X}_s\right\|_\infty^{2p}\mathrm{d}s\right).
\end{aligned} \tag{4.5.2}$$

Since $\bar{X}$ is a piecewise-constant, then we used $\bar{X}_{\underline{s}}=\bar{X}_s$. Now, we consider the second term $\left\| \sup_{s\in[0,T]} \left|\hat{X}(s)-\bar{X}(s)\right| \right\|_{2p}$. We note that

$$\begin{aligned}
\hat{X}(s)-\bar{X}(s) &= \hat{X}(s)-\hat{X}(\underline{s})\\
&= b(\underline{s},\bar{X}_{\underline{s}})(s-\underline{s})+\sigma(\underline{s},\bar{X}_{\underline{s}})(W(s)-W(\underline{s})),
\end{aligned}$$

then we have

$$\left\|\hat{X}_T-\bar{X}_T\right\|_\infty \leq C(T,\kappa)(1+\left\|\bar{X}_T\right\|_\infty)\left(\frac{1}{n}+\sup_{s\in[0,T]}|W(s)-W(\underline{s})|\right)$$

and

$$\begin{aligned}
\mathbb{E}\left\|\hat{X}_T-\bar{X}_T\right\|_\infty^{2p} &\leq C(p,T,\kappa)\frac{1}{n^{2p}}\mathbb{E}\left[(1+\left\|\bar{X}_T\right\|_\infty)^{2p}\right]\\
&\quad + C(p,T,\kappa)\mathbb{E}\left[(1+\left\|\bar{X}_T\right\|_\infty)\sup_{s\in[0,T]}|W(s)-W(\underline{s})|\right]^{2p}.
\end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\mathbb{E}\left\|\hat{X}_T-\bar{X}_T\right\|_\infty^{2p} \leq C(p,T,\kappa)\left(1+\sqrt{\mathbb{E}\left\|\bar{X}_T\right\|_\infty^{4p}}\right)\left(\frac{1}{n^{2p}}+\sqrt{\mathbb{E}\left[\sup_{s\in[0,T]}|W(s)-W(\underline{s})|\right]^{4p}}\right). \tag{4.5.3}$$

Now, using Lemma 4.5.1, we have

$$\sqrt{\mathbb{E}\left[\sup_{s\in[0,T]}|W(s)-W(\underline{s})|\right]^{4p}} \leq C(p,T)\left(\frac{1+\log n}{n}\right)^p. \tag{4.5.4}$$

We consider $\mathbb{E}\left\|\bar{X}_T\right\|_\infty^{4p}$ and using the BDG inequality, we have

$$\begin{aligned}\mathbb{E}\left\|\bar{X}_T\right\|_\infty^{4p} &\le \mathbb{E}\left\|\hat{X}_T\right\|_\infty^{4p}\\ &\le C(p)\left(x_0^{4p}+\mathbb{E}\left[\int_0^T\left|b(\underline{s},\bar{X}_{\underline{s}})\right|\mathrm{d}s\right]^{4p}+\mathbb{E}\left[\int_0^T\left\|\sigma(\underline{s},\bar{X}_{\underline{s}})\right\|^2\mathrm{d}s\right]^{2p}\right)\\ &\le C(x_0,p,T)\left(1+\int_0^T\left(\mathbb{E}\left|b(\underline{s},\bar{X}_{\underline{s}})\right|^{4p}+\mathbb{E}\left\|\sigma(\underline{s},\bar{X}_{\underline{s}})\right\|^{4p}\right)\mathrm{d}s\right)\\ &\le C(x_0,p,T,\kappa)\left(1+\int_0^T\mathbb{E}\left\|\bar{X}_s\right\|_\infty^{4p}\mathrm{d}s\right).\end{aligned}$$

From Gronwall's inequality $\mathbb{E}\left\|\bar{X}_T\right\|_\infty^{4p}$ is bounded by some constant depending only on $x_0,\ p,\ T$ and κ. Hence, Equation (4.5.3) and (4.5.4) gives,

$$\mathbb{E}\left\|\hat{X}_T-\bar{X}_T\right\|_\infty^{2p}\le C(x_0,p,T,\kappa)\left(\frac{1+\log n}{n}\right)^p. \tag{4.5.5}$$

Now, we recall Equation (4.5.1) in expectation form

$$\begin{aligned}\mathbb{E}\left\|X_T-\bar{X}_T\right\|_\infty^{2p} &\le \mathbb{E}\left\|X_T-\hat{X}_T\right\|_\infty^{2p}+\mathbb{E}\left\|\hat{X}_T-\bar{X}_T\right\|_\infty^{2p}\\ &\le C(p)\left(\mathbb{E}\left\|X_T-\hat{X}_T\right\|_\infty^{2p}+\mathbb{E}\left\|\hat{X}_T-\bar{X}_T\right\|_\infty^{2p}\right).\end{aligned} \tag{4.5.6}$$

Using Equation (4.5.2) and (4.5.5) in Equation (4.5.6) we get

$$\mathbb{E}\left\|X_T-\bar{X}_T\right\|_\infty^{2p}\le C(x_0,p,T,\kappa)\left(\left(\frac{1+\log n}{n}\right)^p+\int_0^T\mathbb{E}\left\|X_s-\bar{X}_s\right\|_\infty^{2p}\mathrm{d}s\right). \tag{4.5.7}$$

We conclude by Gronwall's inequality. □

Corollary 4.5.3. *Under Assumption (4-$\mathcal{A}$), for all $\alpha\in[0,\frac{1}{2})$,*

$$n^\alpha\left\|X_T-\bar{X}_T\right\|_\infty\to 0,\ \text{as } n\to\infty-\text{a.s.}$$

Proof. Suppose that $\alpha\in[0,\frac{1}{2})$. For p large enough, by Proposition 4.5.2, we have

$$\mathbb{E}\left[\sum_{n\ge1}n^{2p\alpha}\left\|X_T-\bar{X}_T\right\|_\infty^{2p}\right]<\infty.$$

Thus

$$\sum_{n\ge1}n^{2p\alpha}\left\|X_T-\bar{X}_T\right\|_\infty^{2p}<\infty-\text{a.s.},$$

and $n^\alpha\left\|X_T-\bar{X}_T\right\|_\infty\to 0$ as $n\to\infty$, almost surely. □

Chapter 5
Greeks for path-dependent functionals

In this chapter, we compute the sensitivity of the price, also known as Greeks, because the most common of these sensitivities are denoted by Greek letters (Delta, Gamma, Theta, etc.). The Greeks are the quantities representing the sensitivity of the price of derivatives such as options to a change in underlying parameters on which the value of an instrument or portfolio of financial instruments is dependent. However, some are not denoted by Greek letters, for instance, the Vega.

5.1 Path-dependent functionals

We consider a semimartingale X which can be represented as a solution of a stochastic differential equation whose coefficients are allowed to be path-dependent, left-continuous functionals:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t \tag{5.1.1}$$

where b and σ are non-anticipative functionals with values in $\mathbb{R}^d$ and $\mathbb{R}^{d\times n}$, respectively, whose coordinates are in $\mathbb{C}_l^{0,0}(\Lambda_T)$ such that Equation (5.1.1) has a unique weak solution $\mathbb{P}$ on $(\Omega, \mathcal{F}^0)$.

This kind of processes are natural path-dependent extension of the class of diffusion processes. Various conditions are required to ensure existence and uniqueness of solutions, such as functional Lipschitz property and boundedness.

Proposition 5.1.1 (Strong solutions for path-dependent SDEs)**.** Assume that the non-anticipative functionals b and σ satisfy the following Lipschitz and linear growth conditions:

$$|b(t,\omega) - b(t,\omega')| + |\sigma(t,\omega) - \sigma(t,\omega')| \le K \sup_{s\le t} |\omega(s) - \omega'(s)|, \tag{5.1.2}$$

and

$$|b(t,\omega)| + |\sigma(t,\omega)| \le K \left(1 + \sup_{s\le t} |\omega(s)| + |t|\right) \tag{5.1.3}$$

for all $t \ge t_0$, $\omega, \omega' \in C^0([0,t], \mathbb{R}^d)$.

Then for any $\xi \in C^0([0,T], \mathbb{R}^d)$, the SDE (5.1.1) has a unique strong solution X with initial condition $X_{t_0} = \xi_{t_0}$. The paths of X lie in $C^0([0,T], \mathbb{R}^d)$ and

1. There exists a constant $C(T,K)$ such that, for any $t \in [t_0, T]$:

$$\mathbb{E}\left[\sup_{s\in[0,t]} |X(s)|^2\right] \le C\left(1 + \sup_{s\in[0,t_0]} |\xi(s)|^2\right) e^{C(t-t_0)}$$

2.
$$\int_0^{t-t_0} \left[|b(t_0+s, X_{t_0+s})| + |\sigma(t_0+s, X_{t_0+s})|^2 \right] \mathrm{d}s < +\infty \text{ a.s.}$$

3.
$$X_t - X_{t_0} = \int_0^{t-t_0} b(t_0+s, X_{t_0+s})\mathrm{d}s + \int_0^{t-t_0} \sigma(t_0+s, X_{t_0+s})\mathrm{d}w_s,$$

see Protter (2004) for the existence and uniqueness proofs.

Now, for the computation of the Greeks, we first let $b(t, X_t) = rX_t$ and $\sigma(t, X_t) = \sigma(X_t)$ in Equation (5.1.1), then we deal with the following path-dependent volatility model

$$\mathrm{d}X_t = rX_t\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t. \tag{5.1.4}$$

The no-arbitrage price

$$F(t, Y_t) = e^{-r(T-t)}\mathbb{E}[g(X_T)|Y_t], \tag{5.1.5}$$

where $g : \Lambda_T \to \mathbb{R}$ is a functional.

The following result is very important in pricing and hedging path-dependent options with possible path-dependent dynamics:

Theorem 5.1.2 (Pricing PDE)**.** *Let X follow Equation (5.1.4), with σ such that the SDE is well defined, with instantaneous interest rate r, assume that the price of the option for the current path X_t is a $\mathbb{C}_b^{1,2}$ functional $F(t, X_t)$, then*

$$\mathcal{D}F(t, Y_t) + \frac{1}{2}\sigma^2(Y_t)\nabla_\omega^2 F(t, Y_t) + rY_t\nabla_\omega F(t, Y_t) - rF(t, Y_t) = 0, \tag{5.1.6}$$

with final condition $F(T, Y_T) = g(Y_T)$.

The Equation (5.1.6) holds for any continuous path.

Proof. We apply the functional Itô formula to $F(t, X_t)$:

$$\mathrm{d}F = \nabla_\omega F(t, X_t)\mathrm{d}X_t + \mathcal{D}F(t, X_t)\mathrm{d}t + \frac{1}{2}\sigma^2\nabla_\omega^2 F(t, X_t)\mathrm{d}t.$$

The portfolio Π of option F with a short position of $\nabla_\omega F$ stocks gives

$$\mathrm{d}\Pi = \mathcal{D}F(t, X_t)]\mathrm{d}t + \frac{1}{2}\sigma^2\nabla_\omega^2 F(t, X_t)\mathrm{d}t. \tag{5.1.7}$$

This risk-less portfolio has to earn the instantaneous interest rate (in the no-arbitrage setup):

$$\mathrm{d}\Pi = r(F(t, X_t) - \nabla_\omega F(t, X_t)X_t)\mathrm{d}t. \tag{5.1.8}$$

Equations (5.1.7) and (5.1.8) gives

$$\mathcal{D}F(t, X_t) + \frac{1}{2}\sigma^2\nabla_\omega^2 F(t, X_t) - r(F(t, X_t) - \nabla_\omega F(t, X_t)X_t) = 0$$

□

Remark 5.1.3. The Black-Scholes PDE also holds for path-dependent options. However, in this case, the Greeks, namely, Delta, Gamma and Vega are path-dependent themselves.

5.2 Delta

The Delta of a derivative contract is the sensitivity of its price with respect to the initial value of the underlying asset. Hence, if $F(t, X_t)$ denotes the price of the derivative at time t, its Delta is given by $\nabla_\omega F(t, X_t)$.

Now, if we assume that $r = 0$, then Equation (5.1.5) reduces to

$$F(t, Y_t) = \mathbb{E}[g(X_T)|Y_t],$$

for any $(t, Y_t) \in \Lambda_T$. Now, we assume F is as smooth as necessary for us to perform some formal computations. By the Pricing PDE, Theorem 5.1.2, we know that

$$\mathcal{D}F(t, Y_t) + \frac{1}{2}\sigma^2(Y_t)\nabla_\omega^2 F(t, Y_t) = 0, \tag{5.2.1}$$

for any continuous Y_t. We consider the tangent process given by

$$\mathrm{d}Z_t = \nabla_\omega \sigma(X_t) Z_t \mathrm{d}W_t, \tag{5.2.2}$$

where $Z_0 = 1$. For $r = 0$, the volatility model reduces to,

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t. \tag{5.2.3}$$

Now, we apply the Functional Itô Formula, Corollary 3.2.6 to $\nabla_X F(t, X_t) Z_t$. We first note that applying ∇_ω to the Pricing PDE (5.2.1) yields

$$\nabla_\omega(\mathcal{D}F(t, Y_t)) + \sigma(Y_t)\nabla_\omega^2 F(t, Y_t) + \frac{1}{2}\sigma^2(Y_t)\nabla_\omega^3 F(t, Y_t) = 0. \tag{5.2.4}$$

In order for Equation (5.2.4) to hold we require that the following result holds:

Theorem 5.2.1. *If $F(t, Y_t) = 0$, for all Y continuous path, and $F \in \mathbb{C}^{1,1}$, then $\nabla_\omega F(t, Y_t) = 0$, for all Y continuous.*

Hence,

$$\mathrm{d}(\nabla_\omega F(t, X_t) Z_t) = Z_t \mathrm{d}(\nabla_\omega F(t, X_t)) + \nabla_\omega F(t, X_t)\mathrm{d}Z_t + \mathrm{d}(\nabla_\omega F(t, X_t))\mathrm{d}Z_t. \tag{5.2.5}$$

Applying Theorem 3.2.6 to $\nabla_\omega F(t, X_t)$ yields,

$$\begin{aligned}\mathrm{d}(\nabla_\omega F(t, X_t)) &= \mathcal{D}(\nabla_\omega F(t, X_t))\mathrm{d}t + \nabla_\omega(\nabla_\omega F(t, X_t))\mathrm{d}X_t + \frac{1}{2}\nabla_\omega^2(\nabla_\omega F(t, X_t))\mathrm{d}[X]. \\ &= \mathcal{D}(\nabla_\omega F(t, X_t))\mathrm{d}t + \nabla_\omega^2 F(t, X_t)\mathrm{d}X_t + \frac{1}{2}\nabla_\omega^3 F(t, X_t)\mathrm{d}[X]. \end{aligned} \tag{5.2.6}$$

Now, we substitute Equation (5.2.6) into Equation (5.2.5)

$$\begin{aligned}\mathrm{d}(\nabla_\omega F(t, X_t) Z_t) = Z_t &\left[\mathcal{D}(\nabla_\omega F(t, X_t))\mathrm{d}t + \nabla_\omega^2 F(t, X_t)\mathrm{d}X_t + \frac{1}{2}\nabla_\omega^3 F(t, X_t)\mathrm{d}[X]\right] + \nabla_\omega F(t, X_t)\mathrm{d}Z_t \\ &+ \left[\mathcal{D}(\nabla_\omega F(t, X_t))\mathrm{d}t + \nabla_\omega^2 F(t, X_t)\mathrm{d}X_t + \frac{1}{2}\nabla_\omega^3 F(t, X_t)\mathrm{d}[X]\right]\mathrm{d}Z_t.\end{aligned}$$

Next we substitute $\mathrm{d}Z_t$ into the above and noting that $\mathrm{d}W_t\mathrm{d}t = \mathrm{d}t\mathrm{d}W_t = 0$ and $\mathrm{d}W_t\mathrm{d}W_t = \mathrm{d}t$, we get

$$\begin{aligned}\mathrm{d}(\nabla_\omega F(t,X_t)Z_t) = \mathcal{D}(\nabla_\omega F(t,X_t))Z_t\mathrm{d}t + \nabla_\omega^2 F(t,X_t)Z_t\mathrm{d}X_t + \frac{1}{2}\nabla_\omega^3 F(t,X_t)Z_t\mathrm{d}[X] \\ + \frac{\nabla_\omega\sigma(X_t)}{\sigma(X_t)}\nabla_\omega F(t,X_t)Z_t\mathrm{d}X_t + \sigma(X_t)\nabla_\omega\sigma(X_t)\nabla_\omega^2 F(t,X_t)Z_t\mathrm{d}t \\ + \frac{1}{2}\nabla_\omega\sigma(X_t)\nabla_\omega^3 F(t,X_t)Z_t\mathrm{d}[X]\mathrm{d}Z_t.\end{aligned}$$

Using the quadratic variation, $\mathrm{d}[X] = (\sigma(X_t)\mathrm{d}W_t)^2 = \sigma^2(X_t)\mathrm{d}t$, leads to

$$\begin{aligned}\mathrm{d}(\nabla_\omega F(t,X_t)Z_t) = \mathcal{D}(\nabla_\omega F(t,X_t))Z_t\mathrm{d}t + \nabla_\omega^2 F(t,X_t)Z_t\mathrm{d}X_t + \frac{1}{2}\sigma^2(X_t)\nabla_\omega^3 F(t,X_t)Z_t\mathrm{d}t \\ + \frac{\nabla_\omega\sigma(X_t)}{\sigma(X_t)}\nabla_\omega F(t,X_t)Z_t\mathrm{d}X_t + \sigma(X_t)\nabla_\omega\sigma(X_t)\nabla_\omega^2 F(t,X_t)Z_t\mathrm{d}t.\end{aligned}$$

Now, we group terms involving $\mathrm{d}t$ and $\mathrm{d}X_t$, respectively, together

$$\begin{aligned}\mathrm{d}(\nabla_\omega F(t,X_t)Z_t) = \left[\mathcal{D}(\nabla_\omega F(t,X_t)) + \sigma(X_t)\nabla_\omega\sigma(X_t)\nabla_\omega^2 F(t,X_t) + \frac{1}{2}\sigma^2(X_t)\nabla_\omega^3 F(t,X_t)\right] Z_t\mathrm{d}t \\ + \left[\frac{\nabla_\omega\sigma(X_t)}{\sigma(X_t)}\nabla_\omega F(t,X_t) + \nabla_\omega^2 F(t,X_t)\right] Z_t\mathrm{d}X_t. \end{aligned} \tag{5.2.7}$$

Using the Pricing PDE (5.2.4), the Equation (5.2.7) becomes

$$\begin{aligned}\mathrm{d}(\nabla_\omega F(t,X_t)Z_t) &= \left[\mathcal{D}(\nabla_\omega F(t,X_t)) - \nabla_\omega(\mathcal{D}F(t,X_t))\right] Z_t\mathrm{d}t \\ &\quad + \left[\frac{\nabla_\omega\sigma(X_t)}{\sigma(X_t)}\nabla_\omega F(t,X_t) + \nabla_\omega^2 F(t,X_t)\right] Z_t\mathrm{d}X_t \\ &= -\mathfrak{L}F(t,X_t)Z_t\mathrm{d}t + \left[\frac{\nabla_\omega\sigma(X_t)}{\sigma(X_t)}\nabla_\omega F(t,X_t) + \nabla_\omega^2 F(t,X_t)\right] Z_t\mathrm{d}X_t\end{aligned}$$

We define the local martingale

$$\mathrm{d}m_t = \left[\frac{\nabla_\omega\sigma(X_t)}{\sigma(X_t)}\nabla_\omega F(t,X_t) + \nabla_\omega^2 F(t,X_t)\right] Z_t\mathrm{d}X_t, \tag{5.2.8}$$

with $m_0 = 0$, and we assume that the integrand is integrable. Hence,

$$\mathrm{d}(\nabla_\omega F(t,X_t)Z_t) = -\mathfrak{L}F(t,X_t)Z_t\mathrm{d}t + \mathrm{d}m_t. \tag{5.2.9}$$

Now, we state the following assumptions on the functional F:

Assumption 5-$\mathcal{A}$.

1. The Lie Bracket of F, $\mathfrak{L}F$, exists;
2. $F \in \mathbb{C}^{2,3}$;
3. $g(X_T) \in L^2$.

Assumption 5-$\mathcal{B}$. $\mathfrak{L}F(t, Y_t) = 0$, for continuous paths Y_t.

In particular, if F is locally weakly path-dependent, then F satisfies Assumption (5-$\mathcal{B}$). Then the following result is true:

Theorem 5.2.2. *Consider a path-dependent derivative with maturity T and payoff function $g : \Lambda_T \to \mathbb{R}$. If the price of this derivative, denoted by F, satisfies Assumptions (5-$\mathcal{A}$) and (5-$\mathcal{B}$), then $(\nabla_\omega F(t, X_t)Z_t)_{t\in[0,T]}$ is a local martingale and the following formula for the Delta is valid*

$$\nabla_\omega F(0, Y_0) = \mathbb{E}\left[g(X_T)\frac{1}{T}\int_0^T \frac{Z_t}{\sigma(X_t)}\mathrm{d}W_t|Y_0\right]. \tag{5.2.10}$$

Proof. We assume that $F \in D(X)$ and that X and m are martingales. From

$$\begin{aligned}\mathrm{d}(\nabla_\omega F(t, X_t)Z_t) &= (\mathcal{D}(\nabla_\omega F(t, X_t)) - \nabla_\omega(\mathcal{D}F(t, X_t)))\, Z_t\mathrm{d}t + \mathrm{d}m_t \\ &= -\mathfrak{L}F(t, X_t)Z_t\mathrm{d}t + \mathrm{d}m_t\end{aligned}$$

and Assumption 5-$\mathcal{B}$, $\mathcal{L}F(t, X_t) = 0$ and since X_t is a continuous path $\mathbb{P} - a.s$, we conclude that

$$\nabla_\omega F(t, X_t)Z_t = \nabla_\omega F(0, X_0) + m_t, \tag{5.2.11}$$

then $(\nabla_\omega F(t, X_t)Z_t)_{t\in[0,T]}$ is clearly a martingale. Now, we integrate Equation (5.2.11) with respect to t and we get

$$\int_0^T \nabla_\omega F(t, X_t)Z_t\mathrm{d}t = \nabla_\omega F(0, X_0)T + \int_0^T m_t\mathrm{d}t.$$

Now, we take the expectations and note that $\mathbb{E}[m_t] = m_0 = 0$

$$\mathbb{E}\left[\int_0^T \nabla_\omega F(t, X_t)Z_t\mathrm{d}t\right] = \mathbb{E}\left[\nabla_\omega F(0, X_0)T\right] + \mathbb{E}\left[\int_0^T m_t\mathrm{d}t\right].$$

By the dominated convergence theorem we can interchange the order of integration with the expectation

$$\begin{aligned}\mathbb{E}\left[\int_0^T \nabla_\omega F(t, X_t)Z_t\mathrm{d}t\right] &= \nabla_\omega F(0, X_0)T + \int_0^T \mathbb{E}[m_t]\mathrm{d}t \\ &= \nabla_\omega F(0, X_0)T.\end{aligned}$$

We make $\nabla_\omega F(0, X_0)$ the subject

$$\nabla_\omega F(0, X_0) = \mathbb{E}\left[\frac{1}{T}\int_0^T \nabla_\omega F(t, X_t)Z_t\mathrm{d}t\right].$$

By $\mathrm{d}t = \frac{\mathrm{d}[X]}{\sigma^2(X_t)}$ we have

$$\begin{aligned}\nabla_\omega F(0, X_0) &= \mathbb{E}\left[\frac{1}{T}\int_0^T \nabla_\omega F(t, X_t)Z_t\frac{\mathrm{d}[X]}{\sigma^2(X_t)}\right] \\ &= \mathbb{E}\left[\frac{1}{T}\int_0^T \nabla_\omega F(t, X_t)\frac{Z_t}{\sigma^2(X_t)}\mathrm{d}[X]\right].\end{aligned}$$

By Equation (4.2.5)

$$\begin{aligned}
\nabla_\omega F(t, X_0) &= \mathbb{E}\left[F(T, X_T)\frac{1}{T}\int_0^T \frac{Z_t}{\sigma^2(X_t)}\mathrm{d}X_t\right] \\
&= \mathbb{E}\left[g(X_T)\frac{1}{T}\int_0^T \frac{Z_t}{\sigma^2(X_t)}\sigma(X_t)\mathrm{d}W_t\right] \\
&= \mathbb{E}\left[g(X_T)\frac{1}{T}\int_0^T \frac{Z_t}{\sigma(X_t)}\mathrm{d}W_t\right].
\end{aligned}$$

□

Remark 5.2.3. For locally weakly path-dependent functionals, Theorem 5.2.2 suggests that the weight takes the following form

$$\pi = \frac{1}{T}\int_0^T \frac{Z_t}{\sigma(X_t)}\mathrm{d}W_t,$$

that is, Equation (5.2.10) takes the form

$$\nabla_\omega F(0, X_0) = \mathbb{E}[g(X_T)\pi | X_0].$$

Example 5.2.4. Suppose now, that we consider the Black-Scholes model, that is, $\sigma(S_t) = \sigma S_t$:

$$\mathrm{d}S_t = \sigma S_t \mathrm{d}W_t,$$

where σ is constant. Then

$$\nabla_\omega F(0, S_0) = \mathbb{E}\left[g(S_T)\frac{1}{T}\int_0^T \frac{Z_t}{\sigma(S_t)}\mathrm{d}W_t\right].$$

In the Black-Scholes setup, $\sigma(S_t) = \sigma S_t$ and $Z_t = \frac{S_t}{S_0}$, then

$$\begin{aligned}
\nabla_\omega F(0, S_0) &= \mathbb{E}\left[g(S_T)\frac{1}{T}\int_0^T \frac{S_t}{S_0\sigma S_t}\mathrm{d}W_t\right] \\
&= \mathbb{E}\left[g(S_T)\frac{1}{S_0\sigma T}\int_0^T \mathrm{d}W_t\right] \\
&= \mathbb{E}\left[g(S_T)\frac{W_T}{S_0\sigma T}\right].
\end{aligned}$$

5.3 Gamma

Assumption 5-$\mathcal{C}$. $\mathcal{D}\sigma = \mathcal{D}(\nabla_\omega\sigma) = 0$ in Λ_T.

This assumption is satisfied by time-homogeneous local volatility models, see Equation (5.1.4).

Then we assume that the tangent process given by Equation (5.2.2) satisfies the following derivative assumptions:

Assumption 5-$\mathcal{D}$. We consider the functional Z such that $Z(X_t) = Z_t$:

1. $\mathcal{D}Z(X_t) = 0$;
2. $\nabla_\omega Z(X_t) = \dfrac{\nabla_\omega \sigma(X_{t-})}{\sigma(X_{t-})} Z(X_{t-})$;
3. $\nabla^2_\omega Z(X_t) = 0$.

The Gamma of a derivative is the sensitivity of its Delta with respect to the initial value of the underlying asset, i.e. $\nabla^2_\omega F(t, X_t)$, if $F(t, X_t)$ denotes the price of the derivative at time t and its Delta exists. Now, we want to compute the Gamma of the price.

Theorem 5.3.1. *Under Assumptions (5-$\mathcal{A}$) and (5-$\mathcal{B}$) for F and $\nabla_\omega F$ and additionally assuming that σ satisfies Assumption (5-$\mathcal{C}$) and Assumption (5-$\mathcal{D}$) on the tangent process $Z(X_t)$, we find*

$$\nabla^2_\omega F(0, X_0) = \mathbb{E}\left[g(X_T) \left(\pi^2 - \frac{\nabla_\omega \sigma(X_0)}{\sigma(X_0)} \pi - \frac{1}{T\sigma^2(X_0)} \right) \right],$$

where the weight π is given by

$$\pi = \frac{1}{T} \int_0^T \frac{Z_t}{\sigma(X_t)} \mathrm{d}W_t$$

.

Proof. We set $\eta_t = \int_0^t \frac{Z_s}{\sigma(X_s)} \mathrm{d}W_s$. Then it follows that $\nabla_\omega \eta_t = \frac{Z(X_t)}{\sigma^2(X_t)}$, $\nabla^2_\omega \eta_t = 0$, and $\mathcal{D}\eta_t = 0$. Now, with $s \in [0, T]$, from Equation (5.2.10) we have

$$(T - s)Z(X_s)\nabla_\omega F(s, X_s) + F(s, X_s)\eta(X_s) = \mathbb{E}[g(X_T)\eta(X_T)|X_s]. \tag{5.3.1}$$

We define

$$\tilde{g}(X_T) = g(X_T)\eta(X_T) \text{ and } \tilde{F}(s, X_s) = \mathbb{E}[\tilde{g}(X_T)|X_s].$$

Hence, Equation (5.3.1) becomes

$$\tilde{F}(s, X_s) = (T - s)Z(X_s)\nabla_\omega F(s, X_s) + F(s, X_s)\eta(X_s). \tag{5.3.2}$$

Now, we take the vertical and horizontal derivative of Equation (5.3.2):

$$\begin{aligned} \nabla_\omega \tilde{F}(s, X_s) &= (T - s)Z(X_s)\nabla^2_\omega F(s, X_s) + (T - s)\frac{\nabla_\omega \sigma(X_s)}{\sigma(X_s)} Z(X_s)\nabla_\omega F(s, X_s) \\ &\quad + \nabla_\omega F(s, X_s)\eta(X_s) + F(s, X_s)\frac{Z(X_s)}{\sigma^2(X_s)}, \qquad (5.3.3) \\ \mathcal{D}\tilde{F}(s, X_s) &= -Z(X_s)\nabla_\omega F(s, X_s) + (T - s)Z(X_s)\mathcal{D}(\nabla_\omega F)(s, X_s) + \mathcal{D}F(s, X_s)\eta(X_s). \end{aligned}$$

Now, we compute the mixed second order derivatives:

$$\begin{aligned} \mathcal{D}(\nabla_\omega \tilde{F})(s, X_s) = &-\frac{\nabla_\omega \sigma(X_s)}{\sigma(X_s)} Z(X_s)\nabla_\omega F(s, X_s) + (T - s)\frac{\nabla_\omega \sigma(X_s)}{\sigma(X_s)} Z(X_s)\mathcal{D}(\nabla_\omega F)(s, X_s) \\ &- Z(X_s)\nabla^2_\omega F(s, X_s) + (T - s)Z(X_s)\mathcal{D}(\nabla^2_\omega F)(s, X_s) + \mathcal{D}(\nabla_\omega F)(s, X_s)\eta(X_s) \\ &+ \mathcal{D}F(s, X_s)\frac{Z(X_s)}{\sigma^2(X_s)}, \end{aligned} \tag{5.3.4}$$

$$\nabla_\omega(\mathcal{D}\tilde{F})(s,X_s) = -\frac{\nabla_\omega\sigma(X_s)}{\sigma(X_s)}Z(X_s)\nabla_\omega F(s,X_s) - Z(X_s)\nabla_\omega^2 F(s,X_s)$$
$$+ (T-s)\frac{\nabla_\omega\sigma(X_s)}{\sigma(X_s)}Z(X_s)\mathcal{D}(\nabla_\omega F)(s,X_s) + (T-s)Z(X_s)\nabla_\omega(\mathcal{D}(\nabla_\omega F))(s,X_s)$$
$$+ \nabla_\omega(\mathcal{D}F)(s,X_s)\eta(X_s) + \mathcal{D}F(s,X_s)\frac{Z(X_t)}{\sigma^2(X_t)}. \tag{5.3.5}$$

Now, to evaluate the Lie bracket of $\tilde{F}$, we subtract Equation (5.3.4) from Equation (5.3.5):

$$\begin{aligned}\mathfrak{L}\tilde{F}(s,X_s) &= \nabla_\omega(\mathcal{D}\tilde{F}) - \mathcal{D}(\nabla_\omega\tilde{F}) \\ &= (T-s)Z(X_s)\left[\nabla_\omega(\mathcal{D}(\nabla_\omega F))(s,X_s) - \mathcal{D}(\nabla_\omega^2 F)(s,X_s)\right] \\ &\quad + \left[\nabla_\omega(\mathcal{D}F)(s,X_s) - \mathcal{D}(\nabla_\omega F)(s,X_s)\right]\eta(X_s).\end{aligned}$$

Since the Lie bracket of F and $\nabla_\omega F$ are zero, then

$$\mathfrak{L}\tilde{F}(s,X_s) = (T-s)Z(X_s)[0] + [0]\eta(X_s) = 0.$$

From Equation (5.3.1) we have

$$(T-s)Z(X_s)\nabla_\omega\tilde{F}(s,X_s) + \tilde{F}(s,X_s)\eta(X_s) = \mathbb{E}\left[\tilde{g}(X_T)\eta(X_T)|X_s\right].$$

Substituting Equation (5.3.3) yield

$$(T-s)Z(X_s)\Bigg[(T-s)Z(X_s)\nabla_\omega^2 F(s,X_s) + (T-s)\frac{\nabla_\omega\sigma(X_s)}{\sigma(X_s)}Z(X_s)\nabla_\omega F(s,X_s) + \nabla_\omega F(s,X_s)\eta(X_s)$$
$$+F(s,X_s)\frac{Z(X_s)}{\sigma^2(X_s)}\Bigg] + \tilde{F}(s,X_s)\eta(X_s) = \mathbb{E}\left[\tilde{g}(X_T)\eta(X_T)|X_s\right].$$

Now, we set $s=0$, then note that $Z(X_0)=1$ and $\eta(X_0) = \int_0^0 \frac{Z_t}{\sigma(X_t)}\mathrm{d}W_t = 0$. Hence,

$$T\left[T\nabla_\omega^2 F(0,X_0) + T\frac{\nabla_\omega\sigma(X_0)}{\sigma(X_0)}\nabla_\omega F(0,X_0) + \frac{1}{\sigma^2(X_0)}F(0,X_0)\right] = \mathbb{E}\left[\tilde{g}(X_T)\eta(X_T)|X_0\right],$$

which implies that

$$T^2\nabla_\omega^2 F(0,X_0) + T^2\frac{\nabla_\omega\sigma(X_0)}{\sigma(X_0)}\nabla_\omega F(0,X_0) + \frac{T}{\sigma^2(X_0)}F(0,X_0) = \mathbb{E}\left[g(X_T)\left(\int_0^T \frac{Z_t}{\sigma(X_t)}\mathrm{d}W_t\right)^2\right].$$

Making $\nabla_\omega^2 F(0,X_0)$ the subject yields

$$\nabla_\omega^2 F(0,X_0) = \mathbb{E}\left[g(X_T)\left(\frac{1}{T}\int_0^T \frac{Z_t}{\sigma(X_t)}\mathrm{d}W_t\right)^2\right] - \frac{\nabla_\omega\sigma(X_0)}{\sigma(X_0)}\nabla_\omega F(0,X_0) - \frac{1}{T\sigma^2(X_0)}F(0,X_0).$$

Substituting $\nabla_\omega F(0, X_0)$ and $F(0, X_0)$ yields the desired result:

$$\begin{aligned}\nabla_\omega^2 F(0, X_0) &= \mathbb{E}\left[g(X_T)\left(\frac{1}{T}\int_0^T \frac{Z_t}{\sigma(X_t)}\mathrm{d}W_t\right)^2\right] - \frac{\nabla_\omega \sigma(X_0)}{\sigma(X_0)}\mathbb{E}\left[g(X_T)\frac{1}{T}\int_0^T \frac{Z_t}{\sigma(X_t)}\mathrm{d}W_t\right] \\ &\quad - \frac{1}{T\sigma^2(X_0)}\mathbb{E}\left[g(X_T)\right] \\ &= \mathbb{E}\left[g(X_T)\left(\pi^2 - \frac{\nabla_\omega \sigma(X_0)}{\sigma(X_0)}\pi - \frac{1}{T\sigma^2(X_0)}\right)\right],\end{aligned}$$

where $\pi = \dfrac{1}{T}\displaystyle\int_0^T \frac{Z_t}{\sigma(X_t)}\mathrm{d}W_t$. □

Example 5.3.2. Now, we consider the Black-Scholes model, that is, $\sigma(S_t) = \sigma S_t$, governed by:

$$\mathrm{d}S_t = \sigma S_t \mathrm{d}W_t, \tag{5.3.6}$$

where σ is constant. Then the weight

$$\pi = \frac{1}{T}\int_0^T \frac{Z_t}{\sigma(S_t)}\mathrm{d}W_t,$$

with $\sigma(S_t) = \sigma S_t$ and $Z_t = \frac{S_t}{S_0}$ can be rewritten to take the form:

$$\begin{aligned}\pi &= \frac{1}{T}\int_0^T \frac{S_t}{S_0\sigma S_t}\mathrm{d}W_t \\ &= \frac{1}{S_0\sigma T}\int_0^T \mathrm{d}W_t \\ &= \frac{W_T}{S_0\sigma T}.\end{aligned} \tag{5.3.7}$$

Then the Gamma is given by

$$\begin{aligned}\nabla_\omega^2 F(0, S_0) &= \mathbb{E}\left[g(S_T)\left(\left(\frac{W_T}{S_0\sigma T}\right)^2 - \frac{\nabla_\omega \sigma S_t}{\sigma S_0}\frac{W_t}{S_0\sigma T} - \frac{1}{T\sigma^2 S_0^2}\right)\right] \\ &= \mathbb{E}\left[g(S_T)\left(\frac{W_T^2}{S_0^2\sigma^2 T^2} - \frac{\sigma W_T}{S_0^2\sigma^2 T} - \frac{1}{S_0^2\sigma^2 T}\right)\right] \\ &= \mathbb{E}\left[g(S_T)\frac{1}{S_0^2\sigma T}\left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right)\right].\end{aligned}$$

5.4 Vega

We consider the time-homogeneous local volatility model, with the diffusion σ taken in the direction of $\tilde{\sigma}$:

$$\mathrm{d}X_t^\epsilon = rX_t^\epsilon \mathrm{d}t + \left[\sigma(X_t^\epsilon) - \epsilon\tilde{\sigma}(X_t^\epsilon)\right]\mathrm{d}W_t, \tag{5.4.1}$$

where ϵ is very small and can be either positive or negative but not zero. Then we assume that $r = 0$ to obtain:

$$\mathrm{d}X_t^\epsilon = \left[\sigma(X_t^\epsilon) - \epsilon\tilde{\sigma}(X_t^\epsilon)\right]\mathrm{d}W_t. \tag{5.4.2}$$

We also introduce the $\mathbb{R}^n$-valued tnagent process of the process with respect to ϵ

$$\mathrm{d}Z_t = \tilde{\sigma}(X_t^\epsilon)\mathrm{d}W_t + \sum_{i=1}^{n}\left[\sigma_i'(X_t^\epsilon) - \epsilon\tilde{\sigma}_i'(X_t^\epsilon)\right] Z_t \mathrm{d}W_{t_i}.$$

The pricing PDE given by Equation (5.1.6) takes the following form under the SDE given by Equation (5.4.2):

$$\mathcal{D}F(t, X_t^\epsilon) = -\frac{1}{2}\sigma^2(X_t^\epsilon)\nabla_\omega^2 F(t, X_t^\epsilon) + \frac{\epsilon}{2}\tilde{\sigma}^2(X_t^\epsilon)\nabla_\omega^2 F(t, X_t^\epsilon). \tag{5.4.3}$$

For $g \in \Lambda_T$, define $F \in \Lambda_T$ by

$$\Pi_g(\sigma(X_t^\epsilon)) = F(t, X_t^\epsilon) = \mathbb{E}\left[g(X_T^\epsilon)|X_t^\epsilon\right]. \tag{5.4.4}$$

Now, under the local volatility model Equation (5.4.2) we have the following path-dependent version of the Itô formula:

$$\begin{aligned} F(T, X_T^\epsilon) = F(0, X_0^\epsilon) &+ \int_0^T \mathcal{D}F(t, X_t^\epsilon)\mathrm{d}t + \int_0^T \nabla_\omega F(t, X_t^\epsilon)\mathrm{d}X_t^\epsilon \\ &+ \frac{1}{2}\int_0^T \nabla_\omega^2 F(t, X_t^\epsilon)\mathrm{d}[X^\epsilon]. \end{aligned} \tag{5.4.5}$$

We substitute Equation (5.4.2), (5.4.3) and $\mathrm{d}[X^\epsilon] = \sigma^2(X_t^\epsilon)\mathrm{d}t$ into Equation (5.4.5), we have

$$\begin{aligned} F(T, X_T^\epsilon) &= F(0, X_0^\epsilon) - \frac{1}{2}\int_0^T \sigma^2(X_t^\epsilon)\nabla_\omega^2 F(t, X_t^\epsilon)\mathrm{d}t + \frac{\epsilon}{2}\int_0^T \tilde{\sigma}^2(X_t^\epsilon)\nabla_\omega^2 F(t, X_t^\epsilon)\mathrm{d}t \\ &\quad + \int_0^T \sigma(X_t^\epsilon)\nabla_\omega F(t, X_t^\epsilon)\mathrm{d}W_t - \epsilon\int_0^T \tilde{\sigma}(X_t^\epsilon)\nabla_\omega F(t, X_t)\mathrm{d}W_t + \frac{1}{2}\int_0^T \sigma^2(X_t^\epsilon)\nabla_\omega^2 F(t, X_t^\epsilon)\mathrm{d}t \\ &= F(0, X_0^\epsilon) + \frac{\epsilon}{2}\int_0^T \tilde{\sigma}^2(X_t^\epsilon)\nabla_\omega^2 F(t, X_t^\epsilon)\mathrm{d}t + \int_0^T \sigma(X_t^\epsilon)\nabla_\omega F(t, X_t^\epsilon)\mathrm{d}W_t - \epsilon\int_0^T \tilde{\sigma}(X_t^\epsilon)\nabla_\omega F(t, X_t^\epsilon)\mathrm{d} \end{aligned}$$

Taking the expectation throughout, yields

$$F(T, X_T^\epsilon) = F(0, X_0^\epsilon) + \frac{\epsilon}{2}\mathbb{E}\left[\int_0^T \tilde{\sigma}^2(X_t^\epsilon)\nabla_\omega^2 F(t, X_t^\epsilon)\mathrm{d}t\right]. \tag{5.4.6}$$

Equation (5.4.6) can be written in particular, as

$$\Pi_g(\sigma(X_T^\epsilon) - \epsilon\tilde{\sigma}(X_T^\epsilon)) = \Pi_g(\sigma(X_0^\epsilon) - 0\tilde{\sigma}(X_0^\epsilon)) + \frac{\epsilon}{2}\mathbb{E}\left[\int_0^T \tilde{\sigma}^2(X_t^\epsilon)\nabla_\omega^2 F(t, X_t^\epsilon)\mathrm{d}t\right],$$

according to Equation (5.4.4). We define the Vega of $F(t, X_t)$ as the Fréchet derivative of $F(t, X_t^\epsilon)$ with respect to the volatility, σ. This definition is consistent with the work by Dupire (2019).

Then we can now evaluate the Fréchet derivative as

$$\begin{aligned}
\left.\frac{\partial F}{\partial \epsilon}\right|_{\epsilon=0} &= \lim_{\epsilon\to 0} \frac{\Pi_g(\sigma - \epsilon\tilde{\sigma}) - \Pi_g(\sigma)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{\Pi_g(\sigma) + \frac{\epsilon}{2}\mathbb{E}\left[\int_0^T \tilde{\sigma}^2(X_t^\epsilon)\nabla_\omega^2 F(t, X_t^\epsilon)\mathrm{d}t\right] - \Pi_g(\sigma)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \mathbb{E}\left[\frac{1}{2}\int_0^T \tilde{\sigma}^2(X_t^\epsilon)\nabla_\omega^2 F(t, X_t^\epsilon)\mathrm{d}t\right].
\end{aligned}$$

As $\epsilon \to 0$, $X_t^\epsilon \to X_t$. Hence, $\tilde{\sigma}X_t^\epsilon \to \sigma(X_t)$. Then we have

$$\left.\frac{\partial F}{\partial \epsilon}\right|_{\epsilon=0} = \mathbb{E}\left[\frac{1}{2}\int_0^T \sigma^2(X_t)\nabla_\omega^2 F(t, X_t)\mathrm{d}t\right].$$

Under the similar assumptions as Theorem 5.3.1, this gives the following important result:

Theorem 5.4.1. *Under the hypotheses of Theorem 5.3.1, the Vega satisfies*

$$\mathcal{V} = \left.\frac{\partial f}{\partial \epsilon}\right|_{\epsilon=0} = \mathbb{E}\left[g(X_T)\frac{1}{2}\int_0^T \sigma^2(X_0)\left(\pi^2 - \frac{\nabla_\omega \sigma(X_0)}{\sigma(X_0)}\pi - \frac{1}{T\sigma^2(X_0)}\right)\mathrm{d}t\right]. \tag{5.4.7}$$

Remark 5.4.2. Theorem 5.4.1 avoids the computation of the second-order derivative, $\nabla_\omega^2 F(t, X_t)$.

Example 5.4.3. Again, we consider the Black-Scholes model, given by Equation (5.3.6) and retain the weight (5.3.7). We observe that $\nabla_\omega \sigma(X_0) = \nabla_\omega(\sigma S_t) = \sigma$ so that from (5.4.7) the Vega is given by

$$\begin{aligned}
\mathcal{V} = \left.\frac{\partial F}{\partial \epsilon}\right|_{\epsilon=0} &= \mathbb{E}\left[g(S_T)\frac{1}{2}\int_0^T \sigma^2 S_0^2\left(\left(\frac{W_T}{S_0\sigma T}\right)^2 - \frac{\sigma}{\sigma S_0}\frac{W_t}{S_0\sigma T} - \frac{1}{T\sigma^2 S_0^2}\right)\mathrm{d}t\right] \\
&= \mathbb{E}\left[g(S_T)\frac{1}{2}\int_0^T \frac{\sigma^2 S_0^2}{S_0^2\sigma T}\left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right)\mathrm{d}t\right] \\
&= \mathbb{E}\left[g(S_T)\frac{\sigma}{2T}\left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right)\int_0^T \mathrm{d}t\right] \\
&= \mathbb{E}\left[g(S_T)\frac{\sigma}{2T}\left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right)T\right] \\
&= \mathbb{E}\left[g(S_T)\frac{1}{2}\sigma\left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right)\right].
\end{aligned}$$

Chapter 6
Conclusion

In this thesis, we study the pathwise functional Itô calculus for non-anticipative functionals, inspired by the work done by Föllmer (1981), Dupire (2019) and Cont and Fournié (2010); Cont (2012); Cont and Fournié (2013). We introduced the quadratic variation for non-anticipative functionals on a sequence of partitions, this allowed us to establish a pathwise change of variable formula for functionals of continuous paths. Upon weakening some conditions, the vertical derivative was extended to accommodate adapted processes and ultimately establish a probability-free Itô formula.

We have shown, in Chapter 4, that the hedging strategy is not only robust in the model $\mathcal{M}_\sigma$ for the functional F but for all quadratic variational models. This is due to the special structure of functionals. In addition, we obtained the pathwise hedging strategy for Asian options, see Example 4.2.21. The weak Euler approximation was used to approximate square-integrable martingale adapted to the filtration X by using a sequence of smooth functionals of X. This was done in the sense of functional Itô calculus. Explicit expressions for the functional derivatives were derived.

The price of a path-dependent claim written as a function of the price path satisfies a functional partial differential equation, see Theorem 5.1.2. For the computation of Greeks we make use of the functional Itô formula together with the property of Lie bracket. In particular, we used the weak derivative operator and the Itô stochastic integral. We were able to explicitly compute analytically the formulae for Delta, Gamma and Vega for path-dependent volatility models. We observed that under the Black-Scholes model the delta is not a martingale. The loss of martingale is due to the fact that the stock price models through its tangent process Z and the path-dependence of the derivative contract in question. We provided examples of Greeks (delta, gamma and vega) in the Black-Scholes model case. As in several other studies, the Greeks obtained are expressed as an expectation of a product of the payoff functional and a weight function.

Future work will entail relaxing the regularity property of the functional F(i.e. the assumption that $F \in \mathbb{C}^{2,3}$) and see if we can develop a density argument for the functional F. In our computation of Greeks we relied on the assumption that $F \in \mathbb{C}^{2,3}$. It will also be interesting to extend our results to include discontinuous case as most interesting contingent claims are discontinuous, for example the digital option. Another future work, would be to study or establish the relationship between our approach, the pathwise functional Itô calculus, and other approaches like the Malliavin calculus, and approaches involving the Markovian property. One would like to indicate clearly the relationship between these approaches.

The pathwise functional Itô calculus is useful in studying non-anticipative events and promises a lot of applications in mathematical finances and other fields.

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