Malliavin calculus and its applications to Mathematical Finance

by

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Abstract

In this study, we consider two problems. The first one is the problem of computing hedging portfolios for options that may have discontinuous payoff functions. For this problem we use the Malliavin property called the Clark-Ocone formula and give some examples for different types of payoff functions of the options of European type. The second problem is based on the computation of price sensitivities (derivatives of the probabilistic representation of the payoff functions with respect to the underlying parameters of the model) also known as 'Greeks' of discontinuous payoff functions and also give some examples. We restrict ourselves to the computation of Delta, Gamma and Vega. For this problem we make use of the properties of the Malliavin calculus like the integration by parts formula and the chain rule. We find the representations of the price sensitivities in terms of the expected value of the random variables that do not involve the actual direct differentiation of the payoff function, that is, $\mathbb{E}[g(X_T)\pi]$ where g is a payoff function which depend on the stochastic differential equation X_T at maturity time T and π is the Malliavin weight function. In general, we study the regularity of the solutions of the stochastic differential equations in the sense of Malliavin calculus and explore its applications to Mathematical finance.

Declaration

I declare that the dissertation hereby submitted to the University of Limpopo, for the degree of Master of science in Applied Mathematics has not previously been submitted by me for a degree at this or any other university; that it is my work in design and in execution, and that all materials contained herein has been duly acknowledged.

Mr S.M Kgomo November 2020

Dedication

To my family and my late father.

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Chapter 1

Introduction

In mathematical finance, the method most widely used in practice for evaluating hedging portfolios of options in standard diffusion models is based on the fact that the optimal number of shares to be held is typically obtained by differentiating the option price with respect to the underlying factors. For example, one would use the resampling method [12] which involves computing different values of a factor $X(\alpha)$ for some close values of the parameter α and then forming some appropriate differences to approximate the derivatives. However, this approach is costly when the dimension of the parameter α is large and it also provides biased estimators [16].

Other approaches that have been used include the pathwise method and the likelihood ratio method. The pathwise method computes the derivative of the option price with respect to the parameter of interest. This method only works for specific option prices, hence its implementation cannot be generalized. The method gives unbiased results when applicable (see [12]). This approach cannot be applied to non-differentiable option prices as in the case of barrier and digital options. The likelihood ratio method avoids computing the derivative of the option price [14]. Instead, the derivative of the probability density of the underlying variable is computed rather than the derivative of the option price. It has been proven that when applicable the likelihood ratio method gives results with minimal variance [3]. However, in general, the density function of the underlying factor is not explicitly known as in the case of Asian options [9].

In many practical cases, the option price is either discontinuous or the density of the underlying factors is not explicitly known, hence the above mentioned approaches cannot be applied.

To circumvent these difficulties, Fournie *et al.* [9] used a Malliavin calculus approach. Using this approach, the authors showed that the calculation can can be transformed to avoid the need for computing the derivative of the option price. This is quite useful as typical option prices are not everywhere differentiable and the density of the underlying factors is not explicitly known.

In this study we are particularly interested in computing hedging portfolios and price sensitivies where the option prices are discontinuous following the Malliavin calculus approach. A typical example is the digital option. The main difficulty is that the discontinuity of the option price will cause many technical problems using the above-mentioned standard approaches.

The aim of this study is to derive hedging portfolios and compute price sensitivities using the Malliavin calculus approach which does not involve differentiating the payoff function. The objectives of the study are to:

1. Review the main features of Malliavin calculus on the Wiener space.

2. Investigate the regularity of solutions of stochastic differential equations.

3. Establish the Malliavin derivatives of stochastic integrals and solutions of stochastic differential equations.

4. Apply Malliavin calculus to mathematical finance, in particular, to compute hedging portfolios and price sensitivities.

1.1 Construction of Hedging portfolio

In order to make our goal precise, we introduce mathematical notations. We assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space on which a d-dimensional Brownian motion $W = \{W_t\}_{t\geq 0}$ is defined and $\{\mathcal{F}_t\}_{t\geq 0}$ denotes the natural filtration generated by W, argument by the p-null sets of $\mathcal{F}$ and let $\mathcal{F} = \mathcal{F}_\infty$. We consider the market model which consists of one risky asset and one riskless asset, whose prices at time t are denoted by S_t^0 and S_t^1, respectively. We assume that the prices follow the following stochastic differential equations in the integral form:

$$S_t^0 = S_0^0 + \int_0^T r_t S_t^0 dt \tag{1.1}$$
$$S_t^1 = S_0^1 + \int_0^T r_t S_t^1 dt + \int_0^T \sigma_t S_t^1 dW_t. \tag{1.2}$$

We suppose that the volatility matrix σ_t is invertible and the discounted stock prices are martingales. We recall the standard option pricing framework. Suppose the seller (hereafter called the investor) of the option is trying to replicate the option payoff by investing in the market. We denote by π the amount of money is invested in the stock at time t. Typically, the payoff of the option is given by $\tilde{g}(S_T)$ for some function $\tilde{g}$, and by definition, the (option) price process is equal to the wealth process which replicates the option at the maturity time T. The discounted price process satisfies

$$Z_t = Z_0 + \int_0^t R_s \pi_s \sigma_s dW_s,$$

where $R_t = \exp\{-\int_0^t r_s ds\}$. Since σ and R are both invertible, we can simply set $\phi_t = R_t \pi_t \sigma_t$ so that the discounted wealth process Y is now described by a simple form

$$Z_t = Z_0 + \int_0^t \phi_s dW_s. \tag{1.3}$$

To be more precise, we consider a state process X governed by the following stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t \tag{1.4}$$

where the process $\{W_t : 0 \leq t \leq T\}$ is a standard Brownian motion. We can then set the discounted payoff function to be $g(X) = R\tilde{g}(S)$, and thus the discounted price process Y of the option at each time is given by

$$\begin{aligned} Z_t &= \mathbb{E}[g(X_T) \mid \mathcal{F}_t] \\ &= Z_0 + \int_t^T \phi_s dW_s. \\ &= g(X_T) - \int_t^T \phi_s dW_s. \end{aligned} \tag{1.5}$$

In order to perform hedging in our model, we have to find an efficient method for computing the portfolio π, or equivalently, to compute the process ϕ that makes $Z_T = R_T\tilde{g}(S_T)$. ϕ_t can be computed by differentiating the payoff function with respect to the underlying asset. We are particularly interested in computing hedging portfolio ϕ where the payoff function is discontinuous using the Malliavin calculus approach. A typical example is the digital option, that is

$$\tilde{g}(s) = \mathbf{1}_{\{s \geq K\}}$$

for some $K > 0$. The main difficulty is that the discontinuity of the option price will cause many technical problems using the above mentioned standard approaches.

We will also focus on the application of Malliavin calculus in mathematical finance which includes the derivatives of the probabilistic representation (option price) of the payoff function called price sensitivities (also known as Greeks). These are important sensitivities of an option prices with respect to the involved underlying parameters. The Greeks cannot be expressed in closed form in many cases, as a results, they require numerical methods for computation. These derivatives are basically the derivatives of the expectations. In our case, we focus on the case of the digital and barrier options where we have the discontinuous payoff functions or where the payoff function is non-differentiable. Mathematically, a Greek is defined as follows: We first consider a general Itô diffusion process $\{X_t : 0 \leq t \leq T\}$ given by the stochastic differential equation of the form (1.4) where $\{W_t : 0 \leq t \leq T\}$ is a standard Brownian motion. The coefficients $b(\cdot)$ and $\sigma(\cdot)$ are deterministic functions which are assumed to satisfy some usual conditions which ensures the existence and uniqueness of the solution to stochastic differential equation given by equation (1.4). The diffusion coefficient $\sigma(\cdot)$ is also assumed to satisfy a uniform ellipticity conditions of which will be stated later. We let Φ to be a payoff function which depend on the whole sample path of the stochastic process $\{X_t : 0 \leq t \leq T\}$, $\Phi : [0, T] \longrightarrow \mathbb{R}$ and satisfying

$$\mathbb{E}[\Phi(X(\cdot))^2] < \infty$$

and being given by

$$\Phi = \Phi(X_T)$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}$ is infinitely differentiable function of which all its partial derivatives have polynomial growth. We define an option price $u(\alpha)$ as the probabilistic representation of the payoff function Φ which depends on the stochastic process X_T given by

$$u(\alpha) = \mathbb{E}[\Phi(X_T)] \tag{1.6}$$

where α represent the underlying parameter of the model, $\mathbb{E}$ denotes the expectation and T denote the maturity time. We are much interested in the options of the European type. The parameter α could be the initial stock price, the price volatility, the interest rate or the maturity time. A Greek is generally computed as follows:

$$Greek =: \frac{\partial}{\partial \alpha}\mathbb{E}[\Phi(X_T)]$$

where the underlying parameter α can be the initial price x, the drift coefficient b if it is constant, the volatility coefficient σ if it is also a constant or any other underlying constant parameter of the model.
Different Greeks are defined as follows:

i. If α is the initial price x, then we have the Delta which is defined as the derivative of the option price with respect to the initial price, and is denoted by Δ.
The value of delta measures how sensitive the option price is to the underlying security and it plays an important role in portfolio hedging.

ii. Gamma is defined as the second partial derivative of an option price with respect to the initial price x and is denoted by Γ. It is used to measures the sensitivity of Delta.

iii. If α is the volatility, then we have Vega of a which is defined as the derivative of an option price with respect to the volatility σ and is denoted by $\mathcal{V}$.

iv. If α is the interest rate then we have Rho which is defined as the derivative of an option price with respect to the interest rate r and is denoted by ρ.

v. If α is the time then we have Theta which is defined as the derivative of an option price with respect to the maturity time and is denoted by Θ.

We will focus on Delta, Gamma and Vega. This study is motivated by the increasingly complicated problems in mathematical finance which cannot be solved using the existing traditional methods. We are further motivated by the fact that computations of solutions of stochastic differential equations amount to pricing an option in the financial mathematics framework where one is required to represent the solution as an expectation. Our motivation to use Malliavin calculus is that we want a method which is applicable to a wide class of option prices. This approach allows us to derive explicit hedging strategies without much restrictions. The Malliavin calculus approach is applicable to both complicated and discontinuous option prices. The approach also allows for more general option prices than the Markovian ones, in particular, we want to allow path-dependent option prices. The calculus enables us to obtain tractable formulae for hedging portfolios. Such formulae can be simulated using Monte Carlo methods.

1.2 Literature review

The Malliavin calculus, sometimes referred to as the calculus of variation of stochastic processes, is an infinite-dimensional differential calculus defined on the Wiener space. Much of the theory builds on Itô's stochastic calculus. It was first introduced by Paul Malliavin in the 1970's [19]. The purpose of this calculus was to prove results about the smoothness of densities of solutions of stochastic differential equations driven by Brownian motion. For several years this was the only known application [24]. The Malliavin calculus was considered quite complicated by many, as a result it remained a relatively unknown theory among mathematicians for some time. Many mathematicians simply considered the theory as too difficult compared to the results it produced. Moreover, to a large extent these results could also be obtained by using Hörmander's earlier theory on hypo-elliptic operators ([31] and the references therein).

In 1984, Ocone [29] obtained an explicit interpretation of the Clark representation formula in terms of the Malliavin derivative. This remarkable result later became known as the Clark-Ocone formula, sometimes also called Clark-Haussmann-Ocone formula in view of the contribution of Haussmann, [15]. In 1991, Ocone and Karatzas [30] applied this result to mathematical finance. The authors proved that the Malliavin derivative can be used to obtain explicit formulae for the replicating portfolios of contingent claims in markets driven by Brownian motion. This hugely increased in the interest in the Malliavin calculus both among mathematicians and finance researchers ([1, 7, 8, 24, 22], and the introductory lecture notes [23]).
The next breakthrough came in 1999, when Fournié *et al.* [9] obtained numerically tractable formulae for the computation of the so-called *Greeks* in mathematical finance also known as parameters of sensitivity. In recent years many new applications of the Malliavin calculus have been found including in partial information optimal control, insider trading, and more generally, anticipative stochastic calculus [24].

Malliavin calculus has also been extended from the original setting of Brownian motion to more general Lévy processes [8]. These extensions were at first motivated by and tailored to the original application within the study of smoothness of densities and regularity of of solutions of stochastic differential equations in the sense of Malliavin calculus [25]. Today the range of applications has extended even further to include numerical methods, stochastic control and insider trading, not just for systems driven by Brownian motion, but for systems driven by general Lévy processes [10]. The main computational tool of the Malliavin calculus is the integration by parts formula which can be applied to transform a derivative into a weighted integral of random variables.

In this study, we are particularly interested in computing hedging portfolios for option prices that are discontinuous following the Malliavin calculus approach. We will consider models with the following features:

1. the number of factors may be larger than the number of Brownian motions, and

2. the option price functions are discontinuous.

A typical model with feature (1) is one in which the underlying stock is driven by Brownian motion, but the interest rate and volatility are also diffusion processes driven by the same Brownian motion. The prototypical example of feature (2) is the digital option, which will be the focus of our study.
We are also interested in the computation of Greeks. We consider the option price of the form (1.6) for a given payoff function Φ and for a fixed time T.

1.3 Structure of the dissertation

In chapter 2, we give a brief introduction of the Malliavin calculus. We first concentrate on the Wiener's construction of Brownian motion and look at some of the definitions and examples that will be useful in other chapters. We introduce a powerful tool of Malliavin calculus called the integration by parts formula also known as the duality formula which will be very useful in the application to mathematical finance, most importantly for the computation of price sensitivities. We also look at the Skorohod integral which is actually the extension of the Malliavin derivative for non-adapted stochastic processes. We conclude this chapter by stating the Clark-Ocone formula which is important in the computation of hedging portfolios.
In chapter 3, we consider the stochastic differential equation where the drift and the diffusion coefficients are assumed to be functions. We then take the Malliavin derivative of the stochastic differential equation. We also take the partial derivative of the considered stochastic differential equation with respect to the initial value where we obtain what we call the first variational process. We again considering the stochastic differential equation

for a geometric Brownian motion as an example where the drift and the diffusion coefficients are constants.

In chapter 4 and 5, we apply Malliavin calculus to mathematical finance. In chapter 4 we apply the Clark-Haussman Ocone formula to compute the general representation formula for replicating portfolio for options hedging and give some examples based on different pay-off functions. In chapter 5, we apply some important properties of Malliavin calculus to compute the general representation formulas of the price sensitivities 'Greeks' like Delta, Gamma and Vega where we consider the standard Brownian motion. We consider the geometric Brownian motion as an example.

Lastly we consider the hybrid stochastic volatility model for 3-dimensional standard Brownian motion and use the general formula to compute price sensitivities where we include the computation of Rho. We consider two cases, the first one consist of correlated Brownian motions and the second one consist of uncorrelated (independent) Brownian motions. In chapter 6 we give the general conclusion of the study.

Chapter 2

Introduction to Malliavin calculus

We look at the important definitions which we will make use of in the next chapters. We introduce the Gaussian Hilbert spaces [18] which are real or complex inner product spaces that are also a complete metric space with respect to the distance function induced by the inner product. We also look at some properties of Malliavin calculus like the Malliavin derivative and the Skorohod integral in the Brownian motion sense. For a detailed account of Malliavin calculus we refer to [24].

2.1 Gaussian Hilbert spaces

We work on a measure space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathbb{P}$ is a probability measure (non negative) and $\mathbb{P}(\Omega) = 1$.

Definition 2.1.1 *A random variable is a real-valued measurable function on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*.*

Let $\mathbb{B}$ denote the σ-algebra of Borel sets.

Definition 2.1.2 *The distribution of a random variable* X *is the measure* μ_X *defined on* $(\mathbb{R}, \mathbb{B}, \mu_X)$ *by*

$$\mu_X((-\infty, t]) = \mathbb{P}(\{X(\omega) \leq t\}), \qquad t \in \mathbb{R}. \tag{2.1}$$

Definition 2.1.3 *The function* $F_X(t) = \mu_X((-\infty, t])$ *is called the cumulative distribution function of* X *and if* $F_X(t)$ *is differentiable, then* $f_X(t) = F'_X(t)$ *is called the probability* *density of X. Thus we can write*

$$\mathbb{P}(\{X(\omega) \leq t\}) = \int_{-\infty}^{t} d\mu_X = \int_{-\infty}^{t} dF_X(s) = \int_{-\infty}^{t} f_X(s)ds. \tag{2.2}$$

Definition 2.1.4 *The expected value of a function $g(x)$, $g : \mathbb{R} \longrightarrow \mathbb{R}$ is defined as follows*

$$\begin{aligned}\mathbb{E}[g(X)] := \int_{\Omega} g(X(\omega))d\mathbb{P}(\omega) &= \int_{\mathbb{R}} g d\mu_X = \int_{\mathbb{R}} g(s)dF_X(s) \\ &= \int_{\mathbb{R}} g(s)f_X(s)ds\end{aligned}$$

where $\omega \in \Omega$.

For a random variable X, we define

$$\text{mean}(X) = \mathbb{E}[X],$$

$$\text{var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

We define the space of square integrable functions as

$$L^2(\Omega) = \left\{ g : \int_{\Omega} g^2(\omega)d\mathbb{P}(\omega) < \infty \right\}.$$

Then $L^2(\Omega)$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int_{\Omega} f(\omega)g(\omega)d\mathbb{P}(\omega)$$

and norm

$$\|f\|^2 = \langle f, f \rangle.$$

The Hilbert space is complete with respect to convergence in the norm topology:

$$f_n \longrightarrow f \iff \|f_n - f\| \longrightarrow 0,$$

that is, every Cauchy sequence has a limit.

Every separable Hilbert space H has a denumerable orthonormal basis. Let $\{e_i\}$ be such a basis. Then

1. $\|e_i\| = 1$, $\langle e_i, e_j \rangle = 0$ when $i \neq j$.

2. For every $x \in H$ we have

$$x = \sum_{i=0}^{\infty} \langle x, e_i \rangle e_i. \tag{2.3}$$

3. $\langle x, y\rangle = \sum_{i=0}^{\infty}\langle x, e_i\rangle\langle y, e_i\rangle$.

4. $\|x\|^2 = \sum_{i=0}^{\infty}\langle x, e_i\rangle$.

5. If $\{f_i\}$ is an orthogonal sequence, then

$$\sum_{i=0}^{\infty} f_i \quad \text{converges if and only if} \quad \sum_{i=0}^{\infty} \|f_i\|^2 < \infty.$$

The series in (2.3) converges in the norm of H. $L^2(\Omega)$ is a separable Hilbert space.

Definition 2.1.5 *A random variable X is called centred Gaussian if there is a $\sigma > 0$ such that*

$$f_X(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}.$$

Then $X \sim N(\mu, \sigma^2)$.

The term centred refers to the fact that the mean is zero. (i.e $\mu = 0$) A Gaussian linear space can always be completed to a Gaussian Hilbert space which is well stated by the following theorem.

Theorem 2.1.6 *If $X_n \longrightarrow X$ in $L^2(\Omega)$ and for each n, X_n is a centred Gaussian then X is centred Gaussian*

Proof:
Convergence in $L^2(\Omega)$ implies that $\sigma_n^2 = \text{var}(X_n) = \|X_n\|^2 \longrightarrow \|X\| = \text{var}(X) = \sigma^2$ and that

$$\mathbb{P}(X_n \leq t) \longrightarrow \mathbb{P}(X \leq t)$$

so that $X_n \longrightarrow X$ in distribution. Clearly $N(0, \sigma_n^2) \longrightarrow N(0, \sigma^2)$ in distribution such that $X \sim N(0, \sigma^2)$. □

Definition 2.1.7 *Two normally distributed random variables X and Y are independent if and only if they are orthogonal, that is, if and only if*

$$\mathbb{E}[XY] = 0.$$

Definition 2.1.8 *A Gaussian Hilbert space is a Hilbert space whose elements are centred Gaussian random variables.*

Example 2.1.9 *Consider the probability space* $(\mathbb{R}, \mathbb{B}, \gamma)$ *where*

$$d\gamma = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx. \tag{2.4}$$

We show that $g(x) = x$ *is a centred Gaussian random variable. We have*

$$\mathbb{P}(\{g(x) \leq t\}) = \gamma(\{x \leq t\}) = \gamma((-\infty, t]) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{t} e^{-\frac{x^2}{2}} dx.$$

This shows that g *is centred Gaussian.*

Example 2.1.10 *Let* Y *be a set of random variables such that every finite subset is centred multivariate normal. Then the closure in* $L^2(\Omega)$ *of* Y *is a Gaussian Hilbert space.*

Example 2.1.11 *Set* $Y = \{W_t\}_{t \in \mathbb{R}^+}$ *where* $W_t(\omega)$ *is a Brownian motion on* Ω.
It is well known that finite subsets of Y *are centred multivariate normal. The Gaussian Hilbert space* $W(H)$ *generated by* Y *has a well known characterisation given in the following theorem.*

Theorem 2.1.12 *Let* f *be a deterministic function. Then*

$$H(W) = \left\{ \int_0^T f(t) dW(t, \omega) : f \in L^2([0, T], dt) \right\}.$$

where the process $W(t)$ *at time* t *is the standard Brownian motion.*

Proof
Consider a linear map I defined by

$$I(\sum a_i 1_{[0,t_i]}) = \sum a_i W_{t_i}$$

which we write as $I(f) = F$. The functions of type $\sum a_i 1_{[0,t_i]}$ are called simple functions and belongs to $L^2([0, T])$. We can write such functions in the form

$$f = \sum \hat{a}_i 1_{[t_{i-1}, t_i]}$$

for some new constants $\hat{a}_i$. The functions of type $\Sigma a_i W(t_i)(\omega)$ are also called simple functions and belong to $L^2(\Omega)$. They can be written in the form

$$F = \sum a_i \triangle W(t_i)$$

with some constants a_i. Riemann sums suggests that we define

$$F = \int_0^T f(t)dW(t).$$

We extend this definition to the closure of these simple functions. Taking expected values and using

$$\mathbb{E}[W_s W_t] = \int_0^T 1_{[0,s]}(u)1_{[0,t]}(u)du,$$

it follows that

$$\begin{aligned} \|I(f)\|^2_{L^2(\Omega)} = \|F\|^2_{L^2(\Omega)} &= \mathbb{E}\left[\sum_{i,j} a_i W_{t_i} a_j W_{t_j}\right] \\ &= \int_0^T \left(\sum_{i,j} a_i 1_{[0,t_i]} a_j 1_{[0,t_j]}\right) du \\ &= \int_0^T f^2(u)du \\ &= \|f\|^2_{L^2([0,T])}. \end{aligned}$$

Any function $f \in L^2([0,T])$ is a limit of simple functions of the above type. Given such a function f, we construct the approximate sequence f_n which is then a Cauchy sequence. Then

$$\|F_i - F_j\|_{L^2(\Omega)} = \|f_i - f_j\|_{L^2([0,T])}.$$

This shows that F_n is a Cauchy sequence. Thus F_n converges to a limit point F in $L^2(\Omega)$. Also

$$\|F_n\| = \|I(f_n)\| = \|f_n\|$$

so that

$$\|F\|_{L^2(\Omega)} = \|I(f)\|_{L^2(\Omega)} = \|f\|_{L^2([0,T])},$$

where $f \in L^2([0,T])$. Thus, I is an isometry of $L^2([0,T])$ into H. Conversely, a similar argument shows that for any $F \in H$ there exists an $f \in L^2([0,T])$ such that

$$F = I(f),$$

and so I is a surjective map. □

Theorem 2.1.13 *There is an isometry $I : L^2([0,T]) \longrightarrow H$ which is onto. If $I(f) = F$ then we define*

$$\int_0^T f(t)dW(t) := F.$$

By the isometry, we have

$$\begin{aligned} Var(F) &= \mathbb{E}\left[\left(\int_0^T f(t)dW(t)\right)^2\right] \\ &= \|\int_0^T f(t)dW(t)\|^2_{L^2([0,T])} \\ &= \|f\|^2_{L^2([0,T])}. \end{aligned} \tag{2.5}$$

If f_n is a sequence of simple functions converging to f in $L^2([0,T])$, then

$$\lim_{n\to\infty}\int_0^T f_n(t)dW(t) = \int_0^T f(t)dW(t).$$

We have shown that $H(W)$ is the set of all integrals of the form

$$\int_0^T f(t)dW(t), \qquad f \in L^2([0,T]).$$

We have also defined a stochastic integral for square integrable deterministic functions. We note that $\int_0^T f(t)dW(t)$ does not exist for an arbitrary limit of sums for arbitrary continuous functions but the limit does exist in $L^2(\Omega)$.

2.2 1-dimensional Gaussian Hilbert space

Let

$$\begin{aligned} \langle x, y\rangle = \mathbb{E}[x(t)y(t)] &= \int_{\mathbb{R}} x(t)y(t)d\gamma(t) \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x(t)y(t)e^{-\frac{t^2}{2}}dt \end{aligned}$$

denote the inner product in $L^2(\gamma)$. This space includes all functions bounded above in absolute value by some exponential e^{ct}.

Define

$$(\partial x)(t) = x'(t).$$

Then integration by parts shows that

$$\begin{aligned} \langle \partial x, y\rangle &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(\partial x)(ye^{-\frac{t^2}{2}})dt \\ &= -\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x\partial(ye^{-\frac{t^2}{2}})dt \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x(-\partial y + ty)e^{-\frac{t^2}{2}}dt \\ &= \langle x, \partial^* y\rangle \end{aligned}$$

where ∂^* is the adjoint of ∂, and hence we have

$$\partial^* x = -\partial x + tx$$

$$\langle \partial x, y \rangle = \langle x, \partial^* y \rangle$$

Then

$$\partial\partial^* - \partial^*\partial = 1. \tag{2.6}$$

By induction, we have

$$\partial(\partial^*)^n - (\partial^*)^n\partial = n(\partial^*)^{n-1}, \qquad n \geq 2 \tag{2.7}$$

Define the Hermite polynomials by

$$H_n(t) =: (\partial^*)^n(1) \qquad n = 0, 1, 2... \tag{2.8}$$

Then

$$H_0(t) = 1, \qquad H_1(t) = t, \qquad H_2(t) = t^2 - 1$$

and clearly all H_n are monic of degree n (coefficient of t^n is 1). Applying (2.6) with value n to the function 1, and using $\partial(1) = 0$, we obtain

$$H_n' = nH_{n-1}. \tag{2.9}$$

Further, we have

$$\langle H_m, H_n \rangle = \langle (\partial^*)^m 1, H_n \rangle = \langle 1, \partial^m H_n \rangle = \mathbb{E}[\partial^m H_n].$$

Here if $m < n$, then $H_m \perp H_n$. This means that H_m is orthogonal to H_n. By symmetry of the relation, this holds also for $m > n$ and hence, for $m \neq n$. Since the polynomials are monic of degree n, it follows that

$$\|H_n\|^n = \langle H_n, H_n \rangle = \mathbb{E}[\partial^n H_n] = n!$$

Therefore

$$\left\{ \frac{1}{\sqrt{n!}} H_n(t) \right\}$$

is an orthonormal sequence. It can be shown that it forms an orthogonal basis of $L^2(\gamma)$. For $x(t) \in L^2(\gamma)$, we have

$$x = \sum \frac{1}{n!} \langle x, H_n \rangle H_n$$

and if all derivatives of x are in $L^2(\gamma)$, we have

$$\langle x, H_n\rangle = \langle x, (\partial^*)^n 1\rangle = \langle \partial^n x, 1\rangle = \mathbb{E}[\partial^n x].$$

Hence, for such x,

$$x = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbb{E}[\partial^n x] H_n.$$

Applying this to $x(t) = e^{ct}$, we obtain

$$e^{ct} = \mathbb{E}[e^{cu}] \sum_{n=0}^{\infty} \frac{c^n}{n!} H_n.$$

By calculating the expectation integral

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{cu} e^{-\frac{u^2}{2}} du &= e^{\frac{c^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(u-c)^2}{2}} du \\ &= e^{\frac{c^2}{2}} \end{aligned}$$

this becomes

$$e^{ct-\frac{c^2}{2}} = \sum_{n=0}^{\infty} \frac{c^n}{n!} H_n.$$

The operator ∂ is called the annihilation operator, the operator ∂^* is called the creation operator.

2.3 Wiener's construction of Brownian motion and the stochastic integral.

In the construction of Brownian motion on an arbitrary state Ω, the space has no algebraic structure. We will see that if we want to differentiate with respect to the states, we need some sort of structure to make it a vector space. We thus construct Ω explicitly as a vector space, the space of all continuous functions on the non-negative real axis.

Firstly, we notice that $W(t, \omega)$ is a function of two variables. If we fix ω then this becomes a continuous function of one variable t. So to each state ω, the Brownian motion W associates a continuous function. Different states will give rise to different functions with probability 1 so we can identify the states with continuous functions.

Here we need a measure over the continuous functions which tells us what is the probability of getting paths lying in a cylinder about the given path. We can construct this measure on the continuous functions by working at the probability of getting paths lying within a finite number of intervals at times $t_1, t_2, ..., t_k$, one interval at each time, finding probability using a Gaussian probability measure so that the probabilities at all times come out to be normal.

This can be extended to a measure on the continuous functions. The measure of all the differentiable functions turns out to be zero. Sets of functions which have non-zero measure are made up of Holder continuous functions of a certain functional index. Once we have defined the measure, $\mu(\omega)$ say, we have

$$\int F(\omega)d\mu(\omega)$$

which is well defined by measure theory. The integral has as domain some set of continuous functions. The Brownian motion is defined by

$$W(t,\omega) = W(t).$$

We can then define the stochastic integral in the same way as before using some state space and measure to define the probability structure. We thus obtain an integral

$$\int f(t)dW(t,\omega) = \int f(t)dW(t),$$

for deterministic functions in the same way we indicated above using the isometry and thus this may be extended to stochastic functions $f(t,\omega)$ in one of the usual ways.

2.4 The Mallaivin derivative

In this section we denote the Gaussian Hilbert space of Example 2.1.10 by $W(H)$ where $H = L^2([0,T])$. and we use the notation

$$W(h) = \int_0^T h(t)dW(t), \qquad h \in H. \tag{2.10}$$

Having defined our state space as a vector space, we can try to define the derivative, $DF(\omega, h)$ of random variable $F(\omega)$ in the direction $h \in H$ in a standard way, where H is a Gaussian Hilbert space.

$$DF(\omega,h) = \frac{d}{d\epsilon}F(\omega+\epsilon h)|_{\epsilon=0} = \lim_{\epsilon\to 0}\frac{F(\omega+\epsilon h)-F(\omega)}{\epsilon}.$$

However, there is a problem with this definition. We recall that random variables are elements of L^2 and hence only defined $a.s$ with respect to the Wiener measure. Functions different in value only on a set of measure zero are regarded as the same. Given a random variable $F(\omega)$, define

$$G(\omega) = F(\omega + k)$$

where k is a continuous function. If k lies in the Cameron-Martin space then a set A has a measure zero if and only if $A + K$ has measure zero. Thus G is well-defined function in L^2. However, it can be shown that if k lies in the complement, any ω-set has translated measure zero. This cannot be the case for a function in L^2. Thus G is not a well-defined function in L^2 and cannot be used to define the directional derivative. Hence the directional derivative can be defined only for directions in the Cameron-Martin space. To be able to extend the derivative to more general functions, we extend the space W(H).

2.4.1 The space of polynomials in elements of W(H)

Denote by $\mathcal{P}_n(H)$ the space of all polynomials of degree n in elements of $W(H)$ and denote by $\mathcal{P}(\mathcal{H})$ of all such polynomials. We note that $W(H)$ is not an algebra, that is, product of Gaussian random variables generally have non-Gaussian distribution, so most of the elements of $\mathcal{P}(\mathcal{H})$ are not in $W(H)$. However, the polynomials are all in $L^2(\Omega)$ since this space is an algebra. If $f \in \mathcal{P}(\mathcal{H})$, then for some n, there exist a polynomial p of degree n and $h_1, ..., h_n$ such that

$$f(\omega) = p(W(h_1)), ..., W(h_n))$$

If we use the Gram-Schmidt procedure to orthonormalize the h_i (letting $e_1 = \frac{h_1}{\|h_1\|}$, $e_2 = \frac{h_2 - \langle e_1, h_2 \rangle e_1}{\|h_2 - \langle e_1, h_2 \rangle e_1\|}$, ...) and then write the h_i in terms of the e_i and multiply out, we find that

$$f(\omega) = \hat{p}(W(e_1)), ..., W(e_n))$$

where $\hat{p}$ is another polynomial of degree n and $e_1 = \frac{h_1}{\|h_1\|}$. Hence we can assume where appropriate that the h_i are orthonormal in H. Note that if $e \in H$ is normal then

$$1 = \|e\|^2 = \int_0^T |e(t)|^2 dt = \text{var}(W(e))$$

so

$$W(e) \sim N(0, 1)$$

and

$$\mathbb{E}[p(W(e))] = \int_{\mathbb{R}} p(x) d\gamma(x).$$

More generally

$$\mathbb{E}[p(W(e_1,...,W(e_n)))] = \int_{\mathbb{R}^n} p(x)d\gamma(x)$$

where

$$d\gamma_n(x) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2} dx.$$

2.4.2 The Malliavin derivative on $\mathcal{P}(\mathcal{H})$

We have

$$\lim_{\epsilon\to 0} \frac{W(h)(\omega(t)+\epsilon\int_0^t \eta(s)ds) - W(h)\omega(t)}{\epsilon} = \frac{1}{\epsilon}\left(\int_0^T h(t)d(\omega(t)+\epsilon\int_0^t \eta(s)ds) - \int_0^T h(t)d\omega(t)\right)$$
$$= \int_0^T h(t)\eta(t)dt. \tag{2.11}$$

So if we define

$$W(H) \longrightarrow L^2([0,T]\times\Omega)$$

on $W(H)$ at $F = W(h)$ by

$$DF = h$$

then the directional derivative of F in the direction $\int_0^t h(s)\eta(s)ds$, $\eta \in H$ is

$$\langle h, \eta\rangle_H = \int_0^T h(t)\eta(t)dt.$$

A standard type of the calculation shows the following:
Define DF on $\mathcal{P}(\mathcal{H})$ at

$$F = f(W(h_1)),...,W(h_n)) \tag{2.12}$$

by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1),...,W(h_n))h_i. \tag{2.13}$$

Then the directional derivative in the above direction is

$$\langle DF, \eta\rangle_H = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1),...,W(h_n))\langle h_i, \eta\rangle_H. \tag{2.14}$$

This holds for all $\eta \in H$.
Although DF is defined only $a.e$, we will sometimes select a representative and speak of

$$D_tF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1),...,W(h_n))h_i(t).$$

An easy calculation for $F, G \in \mathcal{P}(\mathcal{H})$ yields

$$D(FG) = D(FG) + F(DG) \tag{2.15}$$

for $F, G \in \mathbb{D}^{1,2}$. Behind the derivative operator we have the following:

1.
$$D_t W_s = \mathbf{1}_{\{t \leq s\}}. \tag{2.16}$$

2.
$$\begin{aligned} D_t f(W_s) &= f'(W_s) D_t W_s \\ &= f'(W_s) \mathbf{1}_{\{t \leq s\}}. \end{aligned}$$

3.
$$\begin{aligned} D_t(\exp(W(S_1))) &= \exp(W(s_1) D_t(W(s_1))) \\ &= \exp(W(s_1) \mathbf{1}_{\{t \leq s_1\}}). \end{aligned}$$

Examples of Malliavin derivatives of random variables in $\mathcal{P}(\mathcal{H})$

Example 2.4.1

$$\begin{aligned} D(W(\mathbf{1}_{[0,t_1]})) &= D\left(\int_0^T \mathbf{1}_{[0,t_1](t)dW(t,\omega)}\right) = \mathbf{1}_{[0,t_1]} \\ &= D(W(t_1, \omega)) \\ &= D(W(t_1)) \end{aligned}$$

.

Example 2.4.2

$$D\left(\int_0^t s^2 dW(s)\right) = t^2 \mathbf{1}_{[0,T]}$$

The following gives an integration by parts formula.[22]

Theorem 2.4.3 *Let $h \in H$, $F \in \mathcal{P}(\mathcal{H})$. Then*

$$\begin{aligned} \mathbb{E}[\langle D_t F(\omega), h(t) \rangle_H] &= \mathbb{E}[F(\omega) \int_0^T h(t) dW(t, \omega)] \\ &= \mathbb{E}[F(\omega) W(h)(\omega)]. \end{aligned} \tag{2.17}$$

Proof

Since (2.17) is linear in h, we first normalise the equation and assume that the norm of h is one, that is $||h|| = 1$. Then there exist orthonormal elements $e_1, ..., e_n \in H$ such that $e_1 = h$ and $F \in \mathcal{P}$ of the form

$$F = f(W(e_1), ..., W(e_n))$$

where $f \in C_p^\infty(\mathbb{R}^n)$ and $n \geq 1$. We also recall the density of the standard normal distribution function defined in (2.4)

Then

$$\begin{aligned}
\mathbb{E}[\langle DF(\omega), h(\omega)\rangle_H] &= \mathbb{E}[\langle DF(\omega), e_1\rangle_H] \\
&= \mathbb{E}\left[\sum_i \frac{\partial}{\partial x_i} f(e_i, e_i)\right] \\
&= \mathbb{E}\left[\frac{\partial}{\partial x_1} f\right] \\
&= \int_{\mathbb{R}^n} \frac{\partial}{\partial x_1} f(x) d\gamma_n(x) \\
&= -\frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} f(x) \frac{\partial}{\partial x_1} (e^{-\frac{x^2}{2}}) dx \\
&= \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} f(x) x_1 e^{-\frac{x^2}{2}} dx \\
&= \mathbb{E}[F(\omega) W(e_1)] \\
&= \mathbb{E}[F(\omega) W(h(\omega))].
\end{aligned}$$

Thus, this complete the proof. □

Lemma 2.4.4 *Let $F, G \in \mathcal{P}(H)$ and $h \in H$. Then*

$$\mathbb{E}[G\langle DF, h\rangle_H] = \mathbb{E}[FGW(h)] - \mathbb{E}[F\langle DG, h\rangle_H]. \tag{2.18}$$

2.4.3 Extending the Domain of the derivative

Define

$$||F||_{1,2} = \|F\|_{L^2(\Omega)} + \|DF\|_{L^2([0,T]\times\Omega)}.$$

All the Malliavin derivatives that we have defined above have finite $\|.\|_{1,2}$.

Let $\mathbb{D}^{1,2}$ be a closure of $\mathcal{P}(\mathcal{H})$ in this norm. We shall show that the derivative on $P(H)$ can

be extended to a closed operator on $\mathbb{D}^{1,2}$, [24]
$A : H \longrightarrow K$ is a closable operator on a normed complete linear space if

$$F_n \longrightarrow F, AF_n \longrightarrow G_1, G_n \longrightarrow F, AG_n \longrightarrow G_2 \Longrightarrow G_1 = G_2.$$

If A is closable and we know F on a subset S of the space then we can extend A to an operator on the closure of S by defining $AF = G$ whenever there exist a sequence $F_n \longrightarrow F$ such that $AF_n \longrightarrow G$. The closability implies that G is uniquely defined by the latter two conditions. The extended operator is called the closure.
To prove closability, we need only to show that $F_n \longrightarrow 0$ and AF_n converges implies $AF_n \longrightarrow 0$.

Theorem 2.4.5 *The Malliavin derivative $D : \mathcal{P}(H) \longrightarrow L^2([0,T] \times \Omega)$ is closable.*

proof: Let $\{F_n : n \geq 1\}$ be a sequence of random variables in $\mathcal{P}(H)$ such that $F_n \longrightarrow 0$ in $L^2(\Omega)$ and the sequence of the derivative of F_n, $DF_n \longrightarrow \alpha$ in $L^2([0,T] \times \Omega)$, then for all $h \in H$ and $F \in \mathcal{P}(H)$ such that $FW(h)$ is bounded, we have that

$$\begin{aligned}
\mathbb{E}[F\langle \alpha, h\rangle_H] &= \lim_{n\to\infty} \mathbb{E}[F\langle DF_n, h\rangle_H] \\
&= \lim_{n\to\infty} \mathbb{E}[-F_n\langle DF, h\rangle_H + F_n FW(h)] \\
&= \lim_{n\to\infty} \mathbb{E}[F_n FW(h)] - \mathbb{E}[F_n\langle DF, h\rangle_H] \\
&= 0
\end{aligned}$$

because $F_n \longrightarrow 0$ as $n \longrightarrow \infty$ in $L^2(\Omega)$ and both $\langle DF, h\rangle_H$ and $FW(h)$ are bounded, this conclude that $\alpha = 0$. Thus, this complete the proof. □

Define $D : \mathbb{D}^{1,2} \longrightarrow L^2([0,T] \times \Omega)$ as the closure of our previously defined operator. In general it will not be defined on the whole $L^2(\Omega)$ and will not be continuous. However, F_n converges in $\mathbb{D}^{1,2}$ if and only if both F_n and DF_n converges. It follows that the domain of D is precisely $\mathbb{D}^{1,2}$ and the space is complete.

The following result is the chain rule:[24]

Proposition 2.4.6 *Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives. Suppose that $F = (F_1, ..., F_n)$ is a random vector whose components belong to the space $\mathbb{D}^{1,2}$. Then $\varphi(F) \in \mathbb{D}^{1,2}$, and*

$$D(\varphi(F)) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\varphi(F)DF_i. \tag{2.19}$$

Proof.
Let $\varphi_\epsilon(x) = \varphi * \psi_\epsilon$, where ψ_ϵ is an approximation of the identity for $\epsilon > 0 \quad, x \in \mathbb{R}$ where ψ is an infinitely continuous positive function with support in $[-1, 1]$ such that $\int_{\mathbb{R}} \psi(x)dx = 1$.
We observe that $\varphi_\epsilon \in \mathcal{C}^\infty$ and is bounded with partial derivatives.
On the other hand, because $F \in \mathbb{D}^{1,2}$, there exist a sequence $\{F_k\}_{k\geq 1}, F_k \in \mathcal{S}$ meaning from the definition

$$F_k = f_k(W(h_1), ..., w(h_n(k)))$$

where $f_k \in \mathcal{C}_p^\infty(\mathbb{R}^n)$ that converges to F in $L^2(\Omega)$ as $k \to \infty$ and the sequence DF_k converges to DF in $L^2(\Omega, H)$ as $k \to \infty$. By using the definition of the derivative, we have that

$$\begin{aligned} D(\varphi_\epsilon(F_k)) &= \sum_{i=1}^{n_k} \frac{\partial}{\partial x}(\varphi_\epsilon(F_k))(W(h_1), ..., W(h_n(k)))h_i \\ &= \varphi'_\epsilon(F_k)DF_k. \end{aligned}$$

On the hand , by using the triangle inequality we obtain

$$\begin{aligned} \|\varphi'_\epsilon(F_k)DF_k - \varphi'(F)DF\|_{L^2(\Omega,H)} \leq & \|\varphi'_\epsilon(F_k)(DF_k - DF)\|_{L^2(\Omega,H)} \\ & + \|(\varphi'_\epsilon(F_k) - \varphi'(F_k))DF\|_{L^2(\Omega,H)} \\ & + \|(\varphi'(F_k) - \varphi'(F))DF\|_{L^2(\Omega,H)}. \end{aligned}$$

We write the above triangle inequality as $Q \leq A + B + C$. We see that for any $\epsilon > 0$ and $k \geq 1, \varphi'_\epsilon(F_k)$ is bounded a.s by a constant which does not depend on ϵ and k, and hence (A) converges to zero as $k \to \infty$. On the other hand, by the dominated convergence theorem, we have that for any $k \geq 1, (B)$ converges to zero as $\epsilon \to 0$. In the same way, (C) converges to zero as $k \to \infty$. Thus $D(\varphi_\epsilon(F_k))$ converges to $\varphi'(F)DF$ in $L^2(\Omega, H)$ as $\epsilon \to 0$ and $k \to \infty$. On the other hand, $\varphi'_\epsilon(F_k)$ converges to $\varphi(F)$ in $L^2(\Omega)$ as $\epsilon \to 0$ and $k \to \infty$. Finally, the closability of the Malliavin derivative operator D from Theorem 2.4.5 implies that $\varphi(F) \in \mathbb{D}^{1,2}$ and that

$$D(\varphi(F)) = \varphi'(F)DF.$$

□

The chain rule can be extended to the case of a Lipschitz function, (see [24]).

Lemma 2.4.7 *Let $\{F_n : n \geq 1\}$ be a sequence of random variables in the space $\mathbb{D}^{1,2}$ which converges to F in the space $L^2(\Omega)$ and such that*

$$\sup_n \mathbb{E}[||DF_n||_H^2] < \infty. \tag{2.20}$$

Then $F \in \mathbb{D}^{1,2}$ and the sequence $\{DF_n : n \geq 1\}$ converges to DF in the weak topology of $L^2(\Omega; H)$.

Proof:
There exist a subsequence $\{F_{n(k)} : k \geq 1\}$ of the sequence $\{F_n : n \geq 1\}$ such that the sequence $\{DF_{n(k)} : k \geq 1\}$ converges in $L^2([0,T] \times \Omega)$, say to some element $\tau \in L^2(\Omega; H)$, then by (2.20), the projections of $DF_{n(k)}$ on any Wiener chaos converge in the weak topology of $L^2(\Omega)$ as $k \longrightarrow \infty$, to those of τ. Consequently, (2.20) implies that $F \in \mathbb{D}^{1,2}$ and $\tau = DF$. Thus, for any weakly convergent subsequence, the limit must be equals to τ and this implies the weak convergent of the entire sequence. Thus, this complete the proof. □

The next results is central in proving the existence of strong solutions.

Proposition 2.4.8 *Let $f_n \in \mathbb{D}1, 2$, $n = 1, 2, ...$ be a sequence of Malliavin differentiable random variables. Assume that there exist constants $\alpha > 0$ and $C > 0$ such that*

$$\sup_n \mathbb{E}[\mid f_n \mid^2] \leq C$$

$$\sup_n \mathbb{E}[\mid D_t f_n - D_{t'} f_n \mid^2] \leq C \mid t - t' \mid^\alpha, \qquad 0 \leq t' \leq t \leq T$$

$$\sup_n \sup_{0 \leq t \leq T} \mathbb{E}[\mid D_t f_n \mid^2] \leq C.$$

Then the sequence f_n, $n = 1, 2, ...$ is relatively compact in $L^2(\Omega)$

2.5 Skorohod integral

We now interoduce δ. the Skorohod integral ,defined as the adjoint operator of D.

Definition 2.5.1 *The Skorohod integral (δ), is a linear operator on $L^2([0,T] \times \Omega)$ with values in $L^2(\Omega)$ such that:*

1. *The domain of δ, (denoted by $Dom(\delta)$), is the set of processes $u \in L^2([0,T] \times \Omega)$ such that for any $F \in \mathbb{D}^{1,2}$*

$$|\mathbb{E}\left[\int_0^T D_t F u(t) dt\right] | \leq c_u ||F||_{L^2(\Omega)} \tag{2.21}$$

where c_u is a constant depending on u.

2. *If u belongs to $Dom(\delta)$, then*

$$\delta(u) = \int_0^T u(t)\delta W(t) \tag{2.22}$$

is the element in $L^2(\Omega)$ such that the integration by parts formula holds

$$\mathbb{E}[\langle DF, u\rangle_H] = \mathbb{E}\left[\int_0^T D_t F u(t) dt\right] = \mathbb{E}[F\delta(u)] \tag{2.23}$$

for any $F \in \mathbb{D}^{1,2}$.

In the following propositions, we sum up few properties of the Skorohod integral. [24]

Proposition 2.5.2 *If u is an adapted process belonging to $L^2([0,T]\times\Omega)$, then the Skorohod integral and the Itô integral coincides*

$$\delta(u) = \int_0^T u(t)dW(t). \tag{2.24}$$

Proposition 2.5.3 *If F belongs to $\mathbb{D}^{1,2}$ then for any $u \in Dom(\delta)$ such that*

$$\mathbb{E}[F^2 \int_0^T \|u_t\|^2 dt] < \infty,$$

one has

$$\delta(Fu) = F\delta(u) - \int_0^T D_t F.u_t dt \tag{2.25}$$

whenever the right hand side belongs to $L^2(\Omega)$. In particular, if u is in addition adapted, we have

$$\delta(Fu) = F\int_0^T u_t dW_t - \int_0^T D_t F_t.u_t dt$$

Proof:
For $G \in \mathbb{D}^{1,2}$ and using (2.15) and (2.23), we have

$$\begin{aligned}
\mathbb{E}[\langle\delta(Fu), G\rangle_H] &= \mathbb{E}[\langle Fu, DG\rangle_H] \\
&= \mathbb{E}[\langle u, FDG\rangle_H] \\
&= \mathbb{E}[\langle u, D(FG) - GDF\rangle_H] \\
&= \mathbb{E}[(\delta(u))FG - \langle u, DF\rangle_H G] \\
&= \mathbb{E}[(F\delta(u) - \langle u, DF\rangle_H)G].
\end{aligned}$$

Since $G \in \mathbb{D}^{1,2}$ is arbitrary, (2.25) follows. □

In general, the duality formula also known as the integration by parts formula is given by the following results:

Theorem 2.5.4 *(Duality formula). Let u be a square-integrable, adapted process and $F \in \mathbb{D}^{1,2}$. Then*

$$\mathbb{E}[\delta(Fu_t)] = \mathbb{E}\left[F\int_0^T u_t dW(t)\right] = \mathbb{E}\left[\int_0^T u_t D_t F dt\right]. \tag{2.26}$$

The following are the examples based on the Skorohod integral: For $\tau \in [0, T]$,

Example 2.5.5

$$\begin{aligned}\delta(W(\tau)) &= W(\tau)\delta(1) - \int_0^T D_t(W(\tau))dt \\ &= W(\tau)\int_0^T dW_t - \int_0^T \mathbf{1}_{\{t\leq\tau\}}dt \\ &= W(\tau)W_T - \tau.\end{aligned}$$

Example 2.5.6

$$\begin{aligned}\delta(W^2(\tau)) &= W(\tau)\delta(W(\tau)) - \int_0^T D_t(W(\tau))W(\tau)dt \\ &= W(\tau)[W(\tau)W_T - \tau] - \int_0^T \mathbf{1}_{\{t\leq\tau\}}W(\tau)dt \\ &= W^2(\tau)W_T - \tau W(\tau) - \tau W(\tau) \\ &= W^2(\tau)W_T - 2\tau W(\tau).\end{aligned}$$

The Malliavin derivative of a Skorohod integral is given by the following example:
For $\tau \in [0, T]$,

Example 2.5.7

$$\begin{aligned}D_t(\delta(W(\tau))) &= D_t(W(\tau)W_T - \tau) \\ &= W(\tau)D_t(W_T) + W_T D_t(W(\tau)) - D_t(\tau) \\ &= W(\tau)\mathbf{1}_{[0,T]}(t) + W_T\mathbf{1}_{[0,\tau]}(t) \\ &= W(\tau) + W_T\mathbf{1}_{[0,\tau]},\end{aligned}$$

where $\mathbf{1}_{(\cdot)}$ is an indicator function.

The Malliavin derivative is applied in order to simplify the calculation of the price sensitivities called the 'Greeks' which we will computer later on. The next proposition plays a huge role in the derivation of quantities where we have two random variables, say F, G and a continuously differentiable function f [24].

Proposition 2.5.8 *Let F and G be two random variables such that $F \in \mathbb{D}^{1,2}$. Consider an $H-$valued random variable u such that $DF = \langle DF, u\rangle_H \neq 0$ and $Gu(DF)^{-1} \in Dom(\delta)$. Then, for any continuously differentiable function f with bounded derivative, we have*

$$\mathbb{E}[f'(F)G] = \mathbb{E}[f(F)\delta(Gu(DF)^{-1})] \tag{2.27}$$

where $\delta(u)$ is the Skorohod integral of u and DF is the Malliavin derivative in the direction u.

Proof:
First of all we note that

$$\langle Df(F), u\rangle_H = \langle f'(F)DF, u\rangle_H = f'(F)\langle DF, u\rangle_H \tag{2.28}$$

which is obtained by applying the chain rule from prop...... Since we know that $\langle DF, u\rangle_H \neq 0$, by making $f'(F)$ the subject above, we obtain

$$f'(F) = \langle Df(F), u\rangle_H (\langle DF, u\rangle_H)^{-1}.$$

As a results, for a random variable G, we have

$$\begin{aligned}\mathbb{E}[f'(F)G] &= \mathbb{E}[\langle Df(F), u\rangle_H G(\langle DF, u\rangle_H)^{-1}] \\ &= \mathbb{E}[\langle Df(F), Gu((\langle DF, u\rangle_H)^{-1})\rangle_H].\end{aligned}$$

Now, since $Gu(DF)^{-1} \in Dom(\delta)$, an application of eqn(2.23) yields

$$\mathbb{E}[f'(F)G] = \mathbb{E}[f(F)\delta(Gu(DF)^{-1})] \tag{2.29}$$

which completes the proof. □

2.5.1 The Clark-Ocone formula

Suppose $W = \{W_t : t \in [0, T]\}$ is a 1-dimensional Brownian motion. The Itô representation theorem states that any $F \in L^2(\Omega)$ can be written as

$$F = \mathbb{E}[F] + \int_0^T \phi(t)dW_t \tag{2.30}$$

where ϕ is an adapted process in $L^2([0, T] \times \Omega)$. In addition, if $F \in \mathbb{D}^{1,2}$, it turns out that the process ϕ can be expressed as a Malliavin derivative of F. This is the Clark-Haussmann-Ocone formula (see [29] thereof).

Theorem 2.5.9 *Let $F \in \mathbb{D}^{1,2}$. Then*

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_t F|\mathcal{F}_t]dW_t \tag{2.31}$$

Proof:
Suppose that F can be written in the form (2.30) with $\phi \in L^2([0,T] \times \Omega)$. Then any $\varphi \in L^2([0,T] \times \Omega)$ using Itô isometry and that the expected value of an Itô integral is zero, we have

$$\begin{aligned}
\mathbb{E}[\delta(\varphi)F] &= \mathbb{E}\left[\int_0^T \varphi(t)dW_t \left(\mathbb{E}[F] + \int_0^T \phi dW_t\right)\right] \\
&= \mathbb{E}\left[\int_0^T \mathbb{E}[\varphi(t)\phi(t)]dt\right].
\end{aligned}$$

On the other hand, using integration by parts and taking into account that φ is adapted, we obtain

$$\begin{aligned}
\mathbb{E}[\delta(\varphi)F] &= \mathbb{E}\left[\int_0^T \varphi(t)D_t F dt\right] \\
&= \int_0^T \mathbb{E}[\varphi(t)\mathbb{E}[D_t F]]dt.
\end{aligned}$$

Comparing this, we get

$$\phi(t) = \mathbb{E}[D_t F|\mathcal{F}_t].$$

□

The above theorem shows that the Malliavin derivative provides an identification of the integrand in the martingale representation theorem in a Brownian motion framework. This plays a central role in financial mathematics. In particular, to obtain replicating portfolio strategies for options. Therefore the hedging portfolio is naturally related to the Malliavin derivative D of the terminal payoff.

Chapter 3

SDEs and Malliavin calculus

This chapter we discuss the existence, uniqueness and smoothness of the solutions to stochastic differential equations. We also show how to compute the Malliavin derivative of a stochastic process X_t.

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where the standard Brownian motion $\{W_t : 0 \leq t \leq T\}$ is defined. Let $\Omega = C([0,T], \mathbb{R})$ and $\mathbb{P}$ be the Wiener measure and $\mathcal{F}$ is the completion of the Borel $\sigma-$field of Ω with respect to $\mathbb{P}$. Let $H = L^2([0,T], \mathbb{R})$ be the underlying Hilbert space.

Let b and σ be measurable functions satisfying globally Lipschitz and boundedness conditions:

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq C|x-y| \tag{3.1}$$

for some $x, y \in \mathbb{R}$,

$$t \in [0,T].t \to b(t,0) \quad \text{and} \quad t \to \sigma(t,0) \tag{3.2}$$

are bounded on [0,T]. We denote by $X = \{X_t : 0 \leq t \leq T\}$ the solution of the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t \tag{3.3}$$

of which can be written in the integral form as

$$X_t = x + \int_0^t \sigma(X_s)dW_s + \int_0^t b(X_s)ds. \tag{3.4}$$

where $x \in \mathbb{R}$ is the initial value of the process X_t. We want to show that there is a unique continuous solution to (3.4) such that for all $t \in [0,T]$, the random variable X_t belongs to the space $\mathbb{D}^{1,2}$.

In addition, if the coefficients of (3.4) are infinitely differentiable and their partial derivatives of all orders are uniformly bounded, then the process X_t belongs to $\mathbb{D}^{1,2}$.

3.1 Existence and uniqueness of solutions

In this section, we establish the existence and uniqueness result that generalizes (3.4). Suppose that $\sigma : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ are measurable functions satisfying the following conditions: for a positive constant C:

$$|b(x,y) - b(x,y')| + |\sigma(x,y) - \sigma(x,y')| \leq C|y - y'| \tag{3.5}$$

for any $x \in \mathbb{R}$, $y, y' \in \mathbb{R}$.
The functions $x \to b(x,0)$ and $x \to \sigma(x,0)$ have at most polynomial growth order. i.e.

$$| \, b(x,0) \, | + | \, \sigma(x,0) \, | \leq C(1 + | \, x \, |). \tag{3.6}$$

With the above assumptions, we have the following results. [26]

Lemma 3.1.1 *Consider a continuous and adapted process $\alpha = \{\alpha(t) : 0 \leq t \leq T\}$ such that*

$$d_2 = \mathbb{E}\left[\sup_{0 \leq t \leq T} |\alpha(t)|^2\right] < \infty.$$

Then there exist a unique and continuous adapted process $X = \{X_t : 0 \leq t \leq T\}$ satisfying the stochastic differential equation

$$X_s = \alpha(t) + \int_0^t \sigma(s, X_s) dW_s + \int_0^t b(s, X_s) ds. \tag{3.7}$$

Moreover,

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t|^2\right] \leq C_1$$

where C_1 is a positive constant

Proof:
By using Picard's iteration scheme, We introduce the process $X_0(t) = \alpha(t)$ and

$$X_{n+1}(t) = \alpha(t) + \int_0^t b(s, X_n(s)) ds + \int_0^t \sigma(s, X_n(s)) dW_s \tag{3.8}$$

if $n \geq 0$. By recursive argument, one can show that X_n is a continuous and adapted process such that

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_n(t)|^2\right] < \infty. \tag{3.9}$$

By applying Doob's maximal inequality and Burkholder's inequality, and making use of conditions (3.1) and (3.2), we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|X_{n+1}(t)|^2\right] &\le c_2\left(d_2+\mathbb{E}\left[\left(\int_0^T|b(s,X_n(s))|ds\right)^2\right]+\mathbb{E}\left[|\int_0^T\sigma(s,X_n(s))dW_s|^2\right]\right)\\ &\le c_2\left[d_2+c_2'K^2T\int_0^T(1+\mathbb{E}[|s|^2])+\mathbb{E}[|X_n(s)|^2])ds\right]\\ &\le c_2\left[d_2+c_2'K^2T^2\left(1+\beta_2'+\sup_{0\le t\le T}\mathbb{E}[|X_n(t)|^2]\right)\right],\end{aligned}$$

where c_2 and c_2' are constants. Thus, (3.9) holds. Again by applying Doob's maximal inequality, Burkholder's inequality and making use of conditions (3.5) and (3.6), we obtain

$$\mathbb{E}\left[\sup_{0\le t\le T}|X_{n+1}(t)-X_n(t)|^2\right]\le c_2K^2T\int_0^T\mathbb{E}[|X_n(s)-X_{n-1}(s)|^2]ds.$$

It follows inductively that the preceding expression is bounded by

$$\frac{1}{n!}(c_2K^2T)^{n+1}\sup_{0\le s\le T}\mathbb{E}[|X_1(s)|^2].$$

As a results, we have

$$\sum_{n=0}^{\infty}\mathbb{E}\left[\sup_{0\le t\le T}|X_{n+1}(t)-X_n(t)|^2\right]<\infty,$$

which implies the existence of a continuous process X satisfying (3.7) such that

$$\mathbb{E}\left[\sup_{0\le t\le T}|X_t|^2\right]\le C_1.$$

The uniqueness of the solution is derived by means of a similar method.

3.2 Weak differentiability of the solution

We consider the case where the coefficients b and σ of the stochastic differential equation (3.4) are functions which satisfy globally Lipschitz and linear growth conditions. We want to compute the Malliavin derivative of (3.4).

Recall the identity on the 1-dimensional Gaussian Hilbert space $\partial\partial^*-\partial^*\partial=1$. Malliavin calculus and its adjoint, the Skorohod integral are sort of representation of these on the other spaces. We would expect this identity to be preserved:

$$D\delta-\delta D=1.$$

Theorem 3.2.1 *Suppose that $u \in L^2([0,T]\times\Omega)$ and that $\|u^2\|_{L^2([0,T]\times\Omega)} < \infty$ and $\|(\delta(D_t u))^2\|_{L^2([0,T]\times\Omega)} < \infty$. Then the above identity holds for u and*

$$D_t(\delta(u)) = \delta(D_t u) + u(t,\omega). \tag{3.10}$$

If in addition $u(t,\omega)$ is adapted, (3.10) can be written as

$$D_t\left(\int_0^T u(s)dW(s)\right) = \int_t^T D_t u(s)dW(s) + u(t,\omega).$$

Remark:The lower limit of the integral on the right is t since $D_t \int_0^T = \int_t^T$ if the integrand is adapted.

Let $u(s,\omega)$ be some $\mathcal{F}_s$-adapted process and let $r \leq t$. For the deterministic integral we have

$$D_r \int_0^t u(s)ds = \int_r^t D_r u(s)ds. \tag{3.11}$$

Proof: (Theorem 3.2.1)
For the stochastic integral, we restrict ourself to a simple adapted process of the form

$$u(t,\omega) = F(\omega)h(t)$$

with $h(t) = \mathbf{1}_{(s_1,s_2)}(t)$ and $\mathcal{F}_{s_1}$-measurable $F(\omega)$. Let again $r \leq t$. Then from the Fundamental Theorem of Calculus, we have

$$\begin{aligned}
D_r \int_0^t Fh(s)dB_s &= D_r\left(\int_{[0,r)} Fh(s)dW_s + \int_{[r,t]} Fh(s)dW_s\right) \\
&= 0 + D_r \int_0^1 Fh(s)\mathbf{1}_{[r,t]}(s)dW_s \\
&= D_r(FW(h\mathbf{1}_{[r,t]})) \\
&= (D_r F)W(h\mathbf{1}_{[r,t]}) + Fh(r) \\
&= \int_0^1 D_r Fh\mathbf{1}_{[r,t]}(s)dW_s + u(r) \\
&= u(r) + \int_r^t D_r u(s)dW_S. \qquad (3.12)
\end{aligned}$$

□

Let W be a scalar Brownian motion. Consider n-dimensional stochastic differential equation of the form (3.4) under the usual conditions on the coefficients. Then we have the following results:

Theorem 3.2.2 *Suppose that (3.11) and (3.12) holds and let $X_t \in \mathbb{D}^{1,2}$. Then the Malliavin derivative of (3.4) is given by*

$$D_r X_t = \sigma(X_r) + \int_r^t \sigma'(X_s) D_r X_s dW_s + \int_r^t b'(X_s) D_r X_s ds. \tag{3.13}$$

From Theorem 3.2.2, if we fix r and set $\hat{X} = D_r X$, we then have

$$\hat{X}_t = \sigma(X_r) + \int_r^t \sigma'(X_s)\hat{X}_s dW_s + \int_r^t b'(X_s)\hat{X}_s ds,$$

from which we obtain the linear stochastic differential equation of the form

$$d\hat{X}_t = \sigma'(X_t)\hat{X}_t dW_t + b'(X_t)\hat{X}_t dt, \qquad (t > r) \tag{3.14}$$

with the initial condition $\hat{X}_r = \sigma(X_r)$.

Solving (3.14) we obtain

$$\hat{X}_t = \hat{X}_r \exp\left\{\int_r^t [b'(X_s) - \frac{1}{2}(\sigma'(X_s))^2] ds + \int_r^t \sigma'(X_s) dW_s\right\}. \tag{3.15}$$

Since $\hat{X} = D_r X_t$, we have

$$D_r X_t = \sigma(X_r) \exp\left\{\int_r^t [b'(X_s) - \frac{1}{2}(\sigma'(X_s))^2] ds + \int_r^t \sigma'(X_s) dW_s\right\}. \tag{3.16}$$

The sensitivity of (3.4) with respect to the initial condition $X_0 = x$ is given by the following results:

Theorem 3.2.3 *Suppose that (3.1) and (3.6) holds, then the partial derivative of (3.4) with respect to the initial condition x (the first variational process) denoted by Y_t, that is,*

$$Y_t = \frac{\partial}{\partial x} X_t.$$

is the solution to the following stochastic differential equation

$$Y_t = I + \int_r^t \sigma'(X_s) Y_s dW_s + \int_r^t b'(X_s) Y_s ds. \tag{3.17}$$

where I is an identity matrix. We can write (3.17) in differential notation as

$$dY_t = \sigma'(X_t) Y_t dW_t + b'(X_t) Y_t dt, \qquad t > 0 \tag{3.18}$$

with $Y_0 = I$. Moreover, the inverse valued process $Z = Y_t^{-1}$ exist and satisfies

$$Z_t = I - \int_r^t Z_s \sigma_s' dW_s - \int_r^t Z_s (b_s' - (\sigma_s')^2) ds. \tag{3.19}$$

By making the application of the Itô's lemma, we see that the solution to stochastic differential (3.4) is given by

$$X_t = x \exp\left(\int_0^t \left(b'(X_s) - \frac{1}{2}(\sigma'(X_s))^2\right) ds + \int_0^t \sigma'(X_s) dW_s\right) \tag{3.20}$$

and the solution to (3.18) is given by

$$Y_t = \exp\left(\int_0^t \left(b'(X_s) - \frac{1}{2}(\sigma'(X_s))^2\right) ds + \int_0^t \sigma'(X_s) dW_s\right). \tag{3.21}$$

Proposition 3.2.4 *Suppose that the conditions (3.1) and (3.6) holds and let* $Y_t = \frac{\partial}{\partial x} X_t$. *Then one has*

$$D_s X_t = Y_t Y_s^{-1} \sigma(X_s) \mathbf{1}_{s \leq t}. \tag{3.22}$$

Proof:
The proof is obtained from comparing (3.14) and (3.18). □
From Proposition 3.22, we have the following result (see [22]).

Lemma 3.2.5 *Let* $a(t)$ *be a deterministic function of the form*

$$\int_0^T a(t) dt = 1 \tag{3.23}$$

and $X_{t_i} \in \mathbb{D}^{1,2}$. *Then*

$$\int_0^T D_t X_{t_i} a(t) \sigma^{-1}(t) Y_t dt = Y_{t_i}, \qquad i = 1, ..., n. \tag{3.24}$$

Proof:
By making the use of (3.22), we have

$$\begin{aligned} \int_0^T D_t X_{t_i} a(t) \sigma^{-1}(t) Y_t dt &= \int_0^T Y_{t_i} Y_t^{-1} \sigma_t \mathbf{1}_{t \leq t_k} a(t) \sigma^{-1}(t) Y_t dt \\ &= \int_0^T Y_{t_i} Y_t^{-1} \sigma(t) \sigma^{-1}(t) Y_t a(t) \mathbf{1}_{t \leq t_i} dt \\ &= \int_0^{t_i} Y_{t_i} a(t) dt \\ &= Y_{t_i} \int_0^{t_k} a(t) dt \\ &= Y_{t_i}. \end{aligned}$$

□

For a geometric Brownian motion

$$dX_t = bX_t dt + \sigma X_t dW_t, \qquad X_0 = x \in \mathbb{R}. \tag{3.25}$$

If the coefficients b and σ are constants then the process X_t in the integral form is given by

$$X_t = x + b\int_0^t X_s ds + \sigma \int_0^t X_s dW_s.$$

The first variational process $Y_t = \frac{\partial}{\partial x}(X_t)$ is given by

$$Y_t = I + b\int_0^t Y_s ds + \sigma \int_0^t Y_s dW_s. \tag{3.26}$$

It can be written in differential form as

$$dY_t = bY_t dt + \sigma Y_t dW_t \qquad Y_0 = I \tag{3.27}$$

where I is an identity matrix. By applying the Itô's lemma to (3.27), we obtain the solution

$$Y_t = e^{(b-\frac{1}{2}\sigma^2)t+\sigma W_t}. \tag{3.28}$$

The solution to (3.25) is given by

$$X_t = xe^{(b-\frac{1}{2}\sigma^2)t+\sigma W_t}. \tag{3.29}$$

By comparing (3.28) and (3.29), we see that

$$X_t = xY_t \tag{3.30}$$

which is equivalent to

$$Y_t = \frac{X_t}{x}.$$

Now we consider the one dimensional ordinary differential equation

$$dY = f(t)Y dt.$$

Clearly $Z = \frac{1}{Y}$ satisfies

$$dZ = -f(t)Z dt.$$

If we now consider

$$dX = A(X)dt, \quad X(0) = x$$

and

$$dY = \frac{\partial}{\partial x} A(X) Y dt, \quad Y(0) = I.$$

Consider the matrix ODE

$$dZ = -Z \frac{\partial}{\partial x} A(X) dt, \quad Z(0) = I.$$

By computing

$$\begin{aligned} d(ZY) &= \left(-Z\frac{\partial}{\partial x}A(X)dt\right) Y + Z\left(\frac{\partial}{\partial x}A(X)Y\right) dt \\ &= 0. \end{aligned}$$

We see that Y^{-1} exists for all times and $Z = Y^{-1}$ with Y defined as

$$dY = \frac{\partial}{\partial x} A(X) Y dB + \frac{\partial}{\partial x} A(X) Y dt, \quad Y(0) = I$$

and $Z = Y^{-1}$ exists for all times we have

$$dZ = -Z \frac{\partial}{\partial x} A(X) dB - Z \frac{\partial}{\partial x} A(X) Z dt, \quad Z(0) = I$$

Now we can generalize (3.22) as follows.

$$\begin{aligned} D_r X_t &= Y_t Y_r^{-1} \sigma(X_r) \\ &= Y_t Z_r \sigma(X_r). \end{aligned}$$

which implies that

$$Z = Y^{-1} D_r X_t \sigma^{-1}(X_t) \tag{3.31}$$

If we fix t and write $X = X_t$. The covariance matrix γ is given by

$$\begin{aligned} \gamma &= \int_0^1 D_r X [D_r X]^T dr \\ &= Y_t \left[\int_0^t Z_r \sigma(X_r) \sigma^T(X_r) Z_r^T dr\right] Y_t^T \end{aligned}$$

where a^T denote the transpose of a.

Example 3.2.6 *Let X solve the stochastic differential equation*

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \qquad X_0 = x$$

where the coefficients μ and σ are constants. The stochastic differential equation has exact solution

$$X_t = x \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right).$$

Then

$$D_t X_t = \sigma X_t \mathbf{1}_{\{t>s\}} \tag{3.32}$$

Chapter 4

The replicating portfolio

In this chapter, we give the application of Malliavin calculus in mathematical finance. We apply the Clack Ocone formula [30] to the computation of the replicating portfolios in the Malliavin calculus sense and give some examples based on the different types of payoff functions.

4.1 Representation of Hedging portfolio

From chapter 1, we recall (1.4) and (1.5):

$$X_t = x + \int_0^T b(X_t)dt + \int_0^T \sigma(X_t)dW_t \tag{4.1}$$

$$Z_t = g(X_T) - \int_0^T \phi_t dW_t. \tag{4.2}$$

We assume that the dimension of X is n and that of the Brownian motion W is d. We also assume that the drift coefficient b and the diffusion coefficient σ are uniformly bounded and that the function g is a measurable function and that there exist a constant $C > 0$ such that

$$|g(x)| \leq C(1 + |x|) \quad \text{for} \quad x \in \mathbb{R}.$$

The dimension of Z is assumed to be one. The following representation theorem for the hedging portfolio ϕ can be regarded as a special case of the Clark-Ocone formula [7]. Here we do not require the volatility matrix σ to be square matrix.

Theorem 4.1.1 *Suppose that the coefficients b and σ of (4.1) and their partial derivatives are uniformly bounded and that the function g is continuous. Denote $A = \{x \in \mathbb{R}^n : \frac{\partial}{\partial x} g(x)$ does not exist$\}$. In addition, we assume that $\mathbb{P}(X_T \in A) = 0$ and that $\frac{\partial}{\partial x} g(x)$ is uniformly bounded outside A. Then we have*

$$\phi_t = \mathbb{E}\left[\frac{\partial}{\partial x} g(X_T) Y_T \mathbf{1}_{\{X_T \notin A\}}\right] Y_t^{-1} \sigma(X_t) \tag{4.3}$$

where Y_T is the solution of the first variational process

$$Y_t = I + \int_0^T b'(X_t) Y_t dt + \int_0^T \sigma'(X_t) Y_t dW_t. \tag{4.4}$$

Here I denotes the $n \times n$ identity matrix.

Proof:
Let $\{g_n\}_{n>0}$ be a sequence approximating g. That is g_n are smooth functions such that the partial derivatives $\frac{\partial}{\partial x} g_n$ are uniformly bounded, $g_n \to g$ uniformly and $\frac{\partial}{\partial x} g_n(x) \to \frac{\partial}{\partial x} g(x)$ for all $x \notin A$ as $n \to \infty$. Since

$$g_n(X_T) \to g(X_T),$$

by the standard stability results of backward stochastic differential equations, one has

$$\mathbb{E}\left[\int_0^T |\phi_t^n - \phi_t|^2 dt\right] \to 0 \qquad \text{as} \quad n \to \infty. \tag{4.5}$$

On the other hand, we set

$$\hat{\phi}_t = \mathbb{E}\left[\frac{\partial}{\partial x} g(X_T) Y_T \mathbf{1}_{\{X_T \notin A\}}\right] Y_t^{-1} \sigma(X_t).$$

Then

$$\begin{aligned} |\phi_t^n - \hat{\phi}_t| \;\; \leq \;\; & \mathbb{E}\left[|\frac{\partial}{\partial x} g_n(X_T) - \frac{\partial}{\partial x} g(X_T)||Y_T| \mathbf{1}_{\{X_T \notin A\}}\right] |Y_t^{-1}||\sigma(X_T)| \\ & + \mathbb{E}\left[|\frac{\partial}{\partial x} g_n(X_T)||Y_T| \mathbf{1}_{\{X_T \in A\}}\right] |Y_t^{-1}||\sigma(X_t)|. \end{aligned} \tag{4.6}$$

where

$$|x| = [|x_1|, ..., |x_n|]^T \qquad \text{whenever} \qquad x = [x_1, ..., x_n]^T.$$

By noting that $\mathbb{P}(X_T \in A) = 0$ and that $\frac{\partial}{\partial x} g_n(X_T) \to \frac{\partial}{\partial x} g(X_T)$ as $n \to \infty$ for $X_T \notin A$, the application of the dominated convergence theorem yields

$$\mathbb{E}\left[\int_0^T |\phi_t^n - \hat{\phi}_t|^2 dt\right] \to 0 \qquad \text{as} \quad n \to \infty \tag{4.7}$$

We note that (4.5) and (4.7) imply that $\hat{\phi} = \phi$, $dt \times d\mathbb{P}$ a.s. $\hat{\phi}$ has a cádlág version since it is the product of a martingale and a continuous process. As a modification of $\hat{\phi}$, we conclude that ϕ has a cádlág version as well. This complete the proof □

We have to make use of the assumption that $\mathbb{P}(X_T \in A) = 0$. In practice, such an assumption is not easy to verify, especially in the case where the dimension $d < n$. Therefore the following theorem is useful [7].

Theorem 4.1.2 *Assume that the coefficients b and σ from (4.1) and their derivatives are uniformly bounded and that a function g is uniformly Lipschitz in all variables, and differentiable with respect to $(x_{d+1,...,x_n})$. In addition, assume that $det(\sigma_1(X_T)) \neq 0$. Then $\mathbb{P}(X_T \in A) = 0$. In particular (4.3) holds.*

Proof:
Set $\hat{X} = (x_1, ..., x_d)^T$. We first show that the law of $\hat{X}_T$ is absolutely continuous with Lebesgue measure on $\mathbb{R}^d$, denoted by $|.|_d$. Let $\hat{A} = \text{Proj}_{\mathbb{R}^d}(A)$ be the projection of A on $\mathbb{R}^d$, where A is defined in Theorem 4.1.1. That is,

$$\hat{A} = \{\hat{X} = (x_1, ..., x_d) : \exists (x_{d+1}, ..., x_n) \text{ such that } x = (x_1, ..., x_n) \in A\}.$$

Since a function g is Lipschitz continuous on $(x_1, ..., x_d)$ and differentiable on $(x_{d+1,...,x_n})$, we note that $|A|_d = 0$. We note that by the standard arguments, one can show that X_T is Malliavin differentiable as in (3.22) and (3.32), that is $X_T \in \mathbb{D}^{1,2}$ and

$$D_t X_T = Y_T Y_t^{-1} \sigma(X_t). \tag{4.8}$$

In particular,

$$D_T X_T = \sigma(X_T) \quad \text{and} \quad D_t \hat{X}_T = \sigma_1(X_T)$$

Now we define $\hat{\gamma} = \int_0^T D_t \hat{X}_T (D_t \hat{X}_T)^T dt$. From (4.8), we note that $D_t X_T$ is continuous in t and that

$$det(D_T \hat{X}_T) = det(\sigma_1(X_T)) \neq 0 \qquad a.s.$$

Therefore, for every $x \in \mathbb{R}^d - \{0\}$, $x D_t \hat{X}_T (D_t \hat{X}_T)^T x^T$ is nonnegative for every $t \in [0, T]$ and positive for t next to T. Hence we have

$$x \left\{ \int_0^T D_t X_T (D_t X_T)^T dt \right\} x^T > 0$$

which implies that the symmetric matrix $\hat{\gamma}$ has a positive determinant. Now we can conclude that the law of $\hat{X}_T$ is absolutely continuous with respect to $|.|_d$, which if we combine with the fact that $|\hat{A}|_d = 0$, we see that $\mathbb{P}(\hat{X}_T \in \hat{A}) = 0$. Since g is differentiable with respect to $x_{d+1}, ..., x_n$, we see that $\mathbb{P}(X_T \in A) = 0$, thus, the results from Theorem (4.1.1) follows. □

In chapter 1, we gave a digital option as an example where the payoff function is discontinuous. We will look at an option which is discontinuous at only one point. The results can be extended to the situation where we have many discontinuity points [7].

Theorem 4.1.3 *Suppose that $d = 1$, $\sigma_1(x) \geq c_0$ and $b, \sigma \in C^{0,2}$ with bounded first and second derivatives. Assume that a function g is uniformly Lipschitz continuous with respect to x_1, except for the point $x_1 = x_1^*$ and both $g(x_1^*+, x_2, ..., x_n)$ and $g(x_1^*-, x_2, ..., x_n)$ exist and are differentiable. Then for A as defined in Theorem (4.1.1) and $t \in [0, T]$, we have*

$$\phi_t = \mathbb{E}\left[\frac{\partial}{\partial x} g(X_T) Y_T u_t \mathbf{1}_{\{X_T^1 \notin A\}} + \mathbf{1}_{\{X_T^1 > x_1^*\}} \delta(F_t u)\right] \tag{4.9}$$

where $\hat{X}^2 = (x^2, ..., x^n)^T$, $Y = \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}$ is the solution of the first variational process (4.4), $\delta(\cdot)$ is the infinite Skorohod integral over $[t, T]$

$$F_t = \frac{\Delta g(x_1^*, \hat{X}_T^2)[Y_T^1 u_t] Y_T^1}{\|DX_T^1\|_{[t,T]}^2}, \qquad u_t = [Y_t]^{-1}\sigma(X_t), \qquad t \in [0, T] \tag{4.10}$$

and

$$\|DX_T^1\|_{[t,T]}^2 = \int_t^T |D_s X_T^1|^2 ds, \qquad \Delta g(x_1^*, \hat{X}_T^2) = g(x_1^*+, \hat{x}^2) - g(x_1^*-, \hat{x}^2). \tag{4.11}$$

4.2 Replicating portfolios (general case)

In this section we compute the replicating portfolio by considering the general case [24]. To compute the replicating portfolios, we assume that the process $\{S_t : 0 \leq t \leq T\}$ denote the underlying asset price process which satisfy the following stochastic differential equation

$$dS_t = b(S_t)S_t dt + \sigma(S_t)S_t dW_t, \qquad S_0 = x. \tag{4.12}$$

If we assume that the volatility matrix σ and the drift b are bounded and Lipschitz continuous then the stochastic differential equation (4.12) has a unique solution. We assume that the interest rate process is constant. We furthermore suppose that the volatility matrix σ is invertible (satisfy the uniform elliptic condition). The underlying asset price process S_t^1 and the associated discounted price process $\tilde{S}_t^1$

$$\tilde{S}_t^1 = e^{-rt} S_t^1 \tag{4.13}$$

evolve as

$$S_t^1 = S_0^1 + \int_0^t r_s S_s^1 ds + \int_0^s \sigma(S_s^1) S_s^1 dW_s, \tag{4.14}$$

$$\tilde{S}_t^1 = \tilde{S}_0^1 + \int_0^t \sigma(S_s^1) S_s^1 dW_s. \tag{4.15}$$

By making the use of the Itô's formula, we obtain

$$S_t^1 = x \exp\left(\int_0^t (r_s - \frac{1}{2}\sigma(S_s^1)^2) ds + \int_0^t \sigma(S_s^1) dW_s \right), \tag{4.16}$$

$$\tilde{S}_t^1 = x \exp\left(- \int_0^t (\frac{1}{2}\sigma(S_s^1)^2) ds + \int_0^t \sigma(S_s^1) dW_s \right). \tag{4.17}$$

Now we let (B,T) denote a European option where T is the expiry time and B is a positive $\mathcal{F}_t$-measurable random variable representing the payoff of the contingent claim. For such an option, a replicating portfolio V at time t is given by the process

$$V_t = \phi_t^0 e^{rt} + \phi_t^1 S_t^1 \tag{4.18}$$

such that the following holds. (see [1]).

1. Technical assumption:
   ϕ^0 and ϕ^1 are adapted processes such that $\phi_t^0 \in L^2([0,T])$ a.s and $\phi_t^1 \in L^2([0,T] \times \Omega)$.

2. Self-financing:
$$dV_t = r\phi_t^0 e^{rt} dt + \phi_t^1 dS_t^1 \qquad\qquad t < T.$$

3. Admissibility:
$$V_t \geq 0 \quad a.s \ \text{ for } \ a.e \ \ t < T.$$

4. Replicating:
$$V_T = B \qquad a.s.$$

Note that if the random variable B is square integrable, a replicating portfolio for such an option (B, T) exist and is given by

$$V_t = \mathbb{E}[e^{-r(T-t)} B | \mathcal{F}_t]. \tag{4.19}$$

V_t is a non-arbitrage option price of (B, T) as seen at time t. Since the issue of price is addressed, the question is, how can we determine the shares $\phi^i, \quad i = 1, 2$ (for hedging) to invest in order to replicate the option. The discounted process

$$\tilde{V}_t = e^{-rT} V_t \tag{4.20}$$

satisfies the stochastic differential equation

$$d\tilde{V}_t = \phi_t^1 d\tilde{S}_t^1 = \phi_t^1 \sigma(S_t^1) \tilde{S}_t^1 dW_t \tag{4.21}$$

where the last equation is obtained due to (4.15) into consideration. Moreover, the non-arbitrage option price of the discounted process is given by

$$\tilde{V}_t = \mathbb{E}[e^{-rt} B | \mathcal{F}_t]$$

which is a square integrable martingale in a Brownian motion sense and it can be expressed as

$$\tilde{V}_t = V_0 + \int_0^t \psi_s dW_s \tag{4.22}$$

which is obtained by integrating (4.21) both sides and ψ is an adapted process such that it belongs to the space $L^2([0, T] \times \Omega)$. Therefore from (4.19) and (4.22) we have

$$\psi_t = \phi_t^1 \sigma(S_t^1) \tilde{S}_t^1.$$

By rearranging and making ϕ_t^1 the subject we obtain

$$\phi_t^1 = \frac{1}{\tilde{S}_t^1} \psi_t \sigma^{-1}(S_t^1). \tag{4.23}$$

If ϕ^1 and V can be determined, then ϕ^0 can be calculated as follows:

$$\phi_t^0 = \tilde{V}_t - \phi_t^1 \tilde{S}_t^1. \tag{4.24}$$

The unsatisfactory issue is that (4.23) gives the replicating strategy ϕ^1 in terms of the process ψ but if the payoff function satisfies some regularity properties in the Malliavin sense, then the Clark-Ocone formula is concluded. We have the following result. (see [1] and [24])

Proposition 4.2.1 *Let $B \in \mathbb{D}^{1,2}$ then*

$$\phi_t^1 = \frac{e^{-r(T-t)}}{S_t^1} \mathbb{E}\left[D_t B | \mathcal{F}_t\right] \sigma^{-1}(S_t^1) \tag{4.25}$$

Proof:
The Clark-Ocone formula from Theorem 2.5.9 shows that

$$e^{-rT}B = \mathbb{E}\left[e^{-rT}B\right] + e^{-rT}\int_0^t \mathbb{E}[D_sB|\mathcal{F}_s]dW_s$$

so that

$$\begin{aligned}\tilde{V}_t &= \mathbb{E}\left[e^{-rT}B|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[e^{-rT}B\right] + e^{-rT}\int_0^t \mathbb{E}[D_sB|\mathcal{F}_s]dW_s.\end{aligned}$$

Then from (4.22), we have that

$$\psi_t = e^{-rT}\mathbb{E}[D_tB|\mathcal{F}_t]$$

and from (4.23) we have

$$\phi_t^1 = \frac{e^{-rT}}{\tilde{S}_t^1}\mathbb{E}\left[D_tB|\mathcal{F}_t\right]\sigma^{-1}(S_t^1). \tag{4.26}$$

By substituting (4.13) into (4.26), we obtain

$$\begin{aligned}\phi_t^1 &= \frac{e^{-rT}}{\tilde{S}_t^1}\mathbb{E}\left[D_tB|\mathcal{F}_t\right]\sigma^{-1}(S_t^1) \\ &= \frac{e^{-rT}}{e^{-rt}S_t^1}\mathbb{E}\left[D_tB|\mathcal{F}_t\right]\sigma^{-1}(S_t^1) \\ &= \frac{e^{-r(T-t)}}{S_t^1}\mathbb{E}\left[D_tB|\mathcal{F}_t\right]\sigma^{-1}(S_t^1).\end{aligned}$$

□

4.2.1 Examples

Example 4.2.2 *European call option*
Here we give some examples of hedging strategies of different payoff functions. The first one is the particular case of the European option where the payoff function B is given by

$$B = \Phi(S_T^1). \tag{4.27}$$

From Proposition 4.2.1, we note that

$$\phi_t^1 = \frac{e^{-r(T-t)}}{S_t^1}\mathbb{E}\left[D_tB|\mathcal{F}_t\right]\sigma^{-1}(S_t^1) = \frac{e^{-r(T-t)}}{S_t^1}\mathbb{E}\left[D_t\Phi(S_T^1)|\mathcal{F}_t\right]\sigma^{-1}(S_t^1). \tag{4.28}$$

By using the chain rule from Proposition 2.4.6, we can see that

$$D_t\Phi(S_t^1) = \Phi'(S_T^1)D_tS_T^1 = \Phi'(S_T^1)\sigma S_T^1$$

where the last equality is obtained by making the use of (3.32). Thus from (4.28)

$$\begin{aligned}
\phi_t^1 = \frac{e^{-r(T-t)}}{S_t^1}\mathbb{E}\left[D_t\Phi(S_T^1)|\mathcal{F}_t\right]\sigma^{-1}(S_t^1) &= \frac{e^{-r(T-t)}}{S_t^1}\mathbb{E}\left[\Phi'(S_T^1)\sigma S_T^1|\mathcal{F}_t\right]\sigma^{-1}(S_t^1) \\
&= \frac{e^{-r(T-t)}}{S_t^1}\mathbb{E}\left[\Phi'(S_T^1)\sigma S_T^1|\mathcal{F}_t\right]\frac{1}{\sigma S_t^1} \\
&= \frac{e^{-r(T-t)}}{S_t^1}\mathbb{E}\left[\Phi'(S_T^1\frac{S_t^1}{S_t^1})\frac{S_T^1}{S_t^1}|\mathcal{F}_t\right] \\
&= \frac{e^{-r(T-t)}}{S_t^1}\mathbb{E}\left[\Phi'(xS_{T-t}^1)S_{T-t}^1\right]|_{x=S_t^1}.
\end{aligned}$$

Thus, the replicating strategy ϕ_t is given by

$$\phi_t^1 = \frac{e^{-r(T-t)}}{S_t^1}\mathbb{E}\left[\Phi'(xS_{T-t}^1)S_{T-t}^1\right]|_{x=S_t^1}. \tag{4.29}$$

Example 4.2.3 *Asian option*
We consider an option whose payoff function is the average of the stock price given by

$$\bar{S}_T^1 = \frac{1}{T}\int_0^T S_t^1 dt. \tag{4.30}$$

The payoff function is this case is given by

$$B = \Phi(\bar{S}_T^1). \tag{4.31}$$

From Proposition 4.2.1, we note that

$$\begin{aligned}
\phi_t &= \frac{e^{-r(T-t)}}{S_t^1}\mathbb{E}\left[D_tB|\mathcal{F}_t\right]\sigma^{-1}(S_t^1) \\
&= \frac{e^{-r(T-t)}}{S_t^1}\mathbb{E}\left[D_t\Phi(\bar{S}_T^1)|\mathcal{F}_t\right]\sigma^{-1}(S_t^1) \\
&= \frac{e^{-r(T-t)}}{S_t^1}\mathbb{E}\left[\Phi'(\bar{S}_T^1)D_t\bar{S}_T^1|\mathcal{F}_t\right]\sigma^{-1}(S_t^1).
\end{aligned} \tag{4.32}$$

Now, from (3.32), we note that

$$\begin{aligned}
D_t\bar{S}_T^1 &= D_t\left(\frac{1}{T}\int_0^T S_t^1 dt\right) \\
&= \frac{1}{T}\int_0^T D_tS_t^1 dt \\
&= \frac{1}{T}\int_t^T \sigma S_r^1 dr \\
&= \frac{\sigma}{T}\int_t^T S_r^1 dr.
\end{aligned} \tag{4.33}$$

Now, from (4.32) and the fact that $\sigma(S_t^1) = \sigma S_t^1$, we note

$$\begin{aligned}\phi_t &= \frac{e^{-r(T-t)}}{S_t^1}\mathbb{E}\left[\Phi'(\bar{S}_T^1)D_t\bar{S}_T^1|\mathcal{F}_t\right]\sigma^{-1}(S_t^1)\\ &= \frac{e^{-r(T-t)}}{S_t^1}\mathbb{E}\left[\Phi'(\bar{S}_T^1)\frac{\sigma}{T}\int_t^T S_r^1 dr|\mathcal{F}_t\right]\frac{1}{\sigma S_t^1}.\end{aligned} \tag{4.34}$$

We can write the average stock price $\bar{S}_T^1$ as

$$\bar{S}_T^1 = \frac{t}{T}\bar{S}_t^1 + \frac{1}{T}\int_t^T S_r^1 dr \tag{4.35}$$

Equation (4.35) implies that

$$\frac{1}{T}\int_r^T S_r^1 dr = \bar{S}_T^1 - \frac{t}{T}\bar{S}_t^1 = \frac{T\bar{S}_T^1 - t\bar{S}_t^1}{T}.$$

At final time T, $\bar{S}_t^1 = \bar{S}_T^1$. Hence

$$\frac{T\bar{S}_T^1 - t\bar{S}_t^1}{T} = \frac{T\bar{S}_T^1 - t\bar{S}_T^1}{T} = \frac{(T-t)\bar{S}_T^1}{T}.$$

Now, from (4.34)

$$\begin{aligned}\Phi'(\bar{S}_T^1)\frac{1}{T}\int_t^T S_r^1 dr &= \Phi'\left(\frac{t}{T}\bar{S}_t^1 + \frac{1}{T}\int_t^T S_r^1 dr\right)\left(\frac{1}{T}\int_t^T S_r^1 dr\right)\\ &= \Phi'\left(\frac{t}{T}\bar{S}_t^1 + \frac{(T-t)\bar{S}_T^1}{T}\right)\left(\frac{(T-t)\bar{S}_T^1}{T}\right).\end{aligned} \tag{4.36}$$

Finally, by considering (4.34) and (4.36), we obtain

$$\begin{aligned}\Phi_t &= \frac{e^{-r(T-t)}}{S_t^1}\mathbb{E}\left[\Phi'(\bar{S}_T^1)\frac{1}{T}\int_t^T S_r^1 dr|\mathcal{F}_t\right]\frac{1}{S_t^1}\\ &= \frac{e^{-r(T-t)}}{S_t^1}\mathbb{E}\left[\Phi'\left(\frac{t}{T}\bar{S}_t^1 + \frac{(T-t)\bar{S}_T^1}{T}\right)\left(\frac{(T-t)\bar{S}_T^1}{T}\right)|\mathcal{F}_t\right]\frac{1}{S_t^1}\\ &= \frac{e^{-r(T-t)}}{S_t^1}\mathbb{E}\left[\Phi'\left(\frac{t}{T}\bar{S}_t^1 + \frac{S_t^1(T-t)\bar{S}_T^1}{S_t^1 T}\right)\left(\frac{(T-t)\bar{S}_T^1}{S_t^1 T}\right)|\mathcal{F}_t\right]\\ &= \frac{e^{-r(T-t)}}{y}\mathbb{E}\left[\Phi'\left(\frac{tx}{T} + \frac{y(T-t)\bar{S}_{T-t}^1}{T}\right)\left(\frac{(T-t)\bar{S}_{T-t}^1}{T}\right)\right]|x = \bar{S}_t^1, y = S_t^1.\end{aligned}$$

4.3 Construction of Hedging portfolio

In this section we consider the construction of a hedging portfolio for some derivative of financial instruments. We follow closely the work presented by [7] where we consider the consumption process denoted by c. We consider the market model consisting of a money market (or bond) S_t^0 and one stock S_t^1 whose dynamics are governed by

$$S_t^0 = S_0^0 + \int_0^t r_s S_s^0 ds. \tag{4.37}$$

$$S_t^1 = S_0^1 + \int_0^t \mu_s S_s^1 ds + \int_0^t \sigma_s S_s^1 dW_s \tag{4.38}$$

where processes r_t and μ_t are $L^1(\Omega\times[0,T])$-integrable, process σ_t is $L^2(\Omega\times[0,T])$-integrable, all defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$.

We define the discounted price process by

$$\beta_t = \frac{1}{S_t^1}. \tag{4.39}$$

Let $\eta_0(t)$ be the number of shares of bond and $\eta_1(t)$ be the number of shares of stock respectively, so the value of the investor's holdings at time t is $\pi_0(t) + \pi_1(t)$ where

$$\pi_0(t) = \eta_0(t) S_t^0, \qquad \pi_1(t) = \eta_1(t) S_t^1. \tag{4.40}$$

The process $\pi = (\pi_0, \pi_1) = \{\pi_0(t), \pi_1(t) : 0 \le t \le T\}$ with the values in $\mathbb{R}^2$ is called a portfolio process. We suppose that the portfolio $\pi = \pi(t)$ is adapted to the natural filtration $\mathcal{F}_t$. We define the gain process $G(t)$ by

$$dG(t) = \eta_0(t) r_t S_t^0 dt + \eta_1 [dS_t^1 + S_t^1 \varrho_t] dt$$

where ϱ_t denotes the dividend rate process and $G(0) = 0$ (there is no gain at initial time t=0). The wealth process $X = \{X^{x,\pi,c}(t) : 0 \le t \le T\}$ is given by

$$X_t = x - \int_0^t c(s) ds + G(t),$$

where $x > 0$ denote the initial value of an investment and $c(t)$ describe the consumption process. The wealth process satisfy

$$X_t = x + \int_0^t [r_s X_s - c(s)] ds + \int_0^t \pi_1(s)[r_s + \varrho_s - r_s] ds + \int_0^t \pi_1(s) \sigma_s dW_s.$$

We define a process called the market price of risk by

$$\theta_t = \frac{\mu_t + \varrho_t - r_t}{\sigma_t}.$$

In addition, we define

$$\tilde{W}_t = W_t + \int_0^T \theta_t dt, \tag{4.41}$$

$$Z_t = \mathbb{E}\left[\exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^s \theta_s^2 ds\right)\right]. \tag{4.42}$$

We assume that the process Z_t is a martingale. We introduce a probability measure on Ω, given by

$$\tilde{\mathbb{P}}(A) = \int_A Z_T d\mathbb{P} \quad \text{for} \;\; A \in \mathcal{F}_t.$$

Remark: The Girsanov theorem state that the process $\{\tilde{W}_t : 0 \leq t \leq T\}$ is a Brownian motion on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \tilde{\mathbb{P}})$.
By applying the Itô's formula and taking into consideration the discounting process $\beta = \beta_t$, for a wealth process $X = X_t$, we derive stochastic differential equation

$$\beta_t X_t = x - \int_0^t \beta_t c(s) ds + \int_0^t \beta_s \pi_1(s) \sigma_s d\tilde{W}_s$$

on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \tilde{\mathbb{P}})$. Finally, we introduce the state price density process which is given by

$$\tilde{H}_t = \beta_t \exp\left(- \int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \theta_s^2 ds \right). \tag{4.43}$$

Again by applying Itô's formula, we obtain the stochastic differential equation for the wealth process X_t which is given by

$$\tilde{H}_t X_t = x - \int_0^t \tilde{H}_s c(s) ds + \int_0^t \tilde{H}_s [\sigma_s \pi_1(s) \sigma_s - X_s \theta_s] dW_s$$

on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$.

4.3.1 Black-Scholes model

Here the aim is to compute the replicating portfolio for the Black-Scholes model based on the calculation of the Malliavin derivatives of appropriate stochastic processes. The following theorem ensures the existence of the replicating portfolio [5].

Theorem 4.3.1 *Let B denote a nonnegative $\mathcal{F}_t$-measurable random variable. If a consumption process $c(t)$ and initial wealth x satisfies the condition*

$$x = \mathbb{E}\left[\int_0^T \tilde{H}_t c(t) dt + \tilde{H}_T B \right],$$

then there exist a portfolio a portfolio $\pi = (\pi_0, \pi_1)$ such that the corresponding wealth process X_t which depends on x, c, π satisfies the following conditions

$$X_0 = x, \qquad X_T = B, \qquad X_t \geq 0 \;\; a.s. \qquad X_t = \pi_0(t) + \pi_0(t),$$

and can be described for all $\{0 \leq t \leq T\}$ by

$$\tilde{H}_t X_t = \mathbb{E}\left[\int_0^T \tilde{H} c(t) dt + \tilde{H}_T X_T \right].$$

Remark: There exists a stochastic process ϕ_t such that

$$\sigma_t \pi_1(t) = \frac{\phi_t}{\tilde{H}_t} + X_t \theta_t \tag{4.44}$$

This process can be derived from the relation

$$\mathbb{E}\left[\int_0^T \tilde{H} c(t) dt + \tilde{H}_T X_T\right] = x + \int_0^T \phi_t dW_t.$$

The main tool that we are going to use for the computation of the replicating portfolio is the Clark-Ocone formula from Theorem 2.5.9. We will consider the case when $c(t) = 0$ and the case when $B = 0$ a.s. For the case $c(t) = 0$, we have the following results [5]

Theorem 4.3.2 *We suppose that all the assumptions from Theorem 4.3.1 are satisfied. If* $c(t) = 0$*, then from the condition* $\tilde{H}_T B \in \mathbb{D}^{1,2}$*, it follows that the portfolio replicating a random variable* B *is given by*

$$\pi_1(t) = \frac{1}{\tilde{H}_t \sigma_t}\left(\mathbb{E}[\tilde{H}_T D_t B] - \mathbb{E}\left[\tilde{H}_T B\left(\int_t^T D_t r_s ds + \int_t^T D_t \theta_s d\tilde{W}_s\right)\right]\right)$$

If r_t and θ_t are deterministic functions, then the portfolio is given by

$$\begin{aligned} \pi_1(t) &= \exp\left(-\int_t^T r_s ds\right)\sigma_t^{-1}\tilde{\mathbb{E}}[D_t B] \\ &= \exp\left(-\int_t^T (r_s - \frac{1}{2}\theta_s^2) ds - \int_t^T \theta_s dW_s\right)\sigma_t^{-1}\mathbb{E}[D_t B]. \end{aligned} \tag{4.45}$$

Proof:
Since $c(t) = 0$, the process X_t is given by

$$X_t = \frac{1}{\tilde{H}_t}\mathbb{E}[F].$$

Let $F = \tilde{H}_T B$. If $F \in \mathbb{D}^{1,2}$ then by the Clark-Ocone formula from Theorem 2.5.9, $\phi_t = \mathbb{E}[D_t F]$. Hence from (4.44), we have

$$\begin{aligned} \pi_1(t) &= \frac{1}{\sigma_t}\left(\frac{\mathbb{E}[D_t F]}{\tilde{H}_t} + \frac{\mathbb{E}[F]}{\tilde{H}_t}\theta_t\right) \\ &= \frac{1}{\tilde{H}_t \sigma_t}\left(\mathbb{E}[D_t \tilde{H}_T B] + \mathbb{E}[\tilde{H}_T B]\theta_t\right) \\ &= \frac{1}{\tilde{H}_t \sigma_t}\left(\mathbb{E}[B D_t \tilde{H}_T] + \mathbb{E}[\tilde{H}_T D_t B] + \mathbb{E}[\tilde{H}_T B]\theta_t\right). \end{aligned} \tag{4.46}$$

where the last equality is obtained by making the use of (2.15). From (4.46), the process $D_t\tilde{H}_T$ is given by

$$\begin{aligned} D_t\tilde{H}_T &= D_t\left(\beta_t \exp\left(-\int_0^T \theta_t dW_t - \frac{1}{2}\int_0^T \theta_t^2 dt\right)\right) \\ &= -\tilde{H}_t\left(\theta_t + \int_t^s D_r r_u du + \int_t^s D_t\theta_u d\tilde{W}_u\right) 1_{\{0,s\}}(t). \end{aligned} \quad (4.47)$$

By substituting (4.47) into (4.46), we obtain

$$\begin{aligned} \pi_1(t) &= \frac{1}{\tilde{H}_t\sigma_t}\left(\mathbb{E}[\tilde{H}_T B]\theta_t + \mathbb{E}[\tilde{H}_T D_t B] - \mathbb{E}\left[\tilde{H}_T B\left(\theta_t + \int_t^s D_r r_u du + \int_t^s D_t\theta_u d\tilde{W}_u\right)\right]\right). \\ &= \frac{1}{\tilde{H}_t\sigma_t}\left(\mathbb{E}[\tilde{H}_T B]\theta_t + \mathbb{E}[\tilde{H}_T D_t B] - \mathbb{E}[\tilde{H}_T B]\theta_t - \mathbb{E}\left[\tilde{H}_T B\left(\int_t^s D_r r_u du + \int_t^s D_t\theta_u d\tilde{W}_u\right)\right]\right). \\ &= \frac{1}{\tilde{H}_t\sigma_t}\left(\mathbb{E}[\tilde{H}_T D_t B] - \mathbb{E}\left[\tilde{H}_T B\left(\int_t^s D_r r_u du + \int_t^s D_t\theta_u d\tilde{W}_u\right)\right]\right). \end{aligned} \quad (4.48)$$

□

For the case where $B = 0$, we have the following results.

Theorem 4.3.3 *We suppose that all the assumptions from Theorem 4.3.1 are satisfied. If $B = 0$ a.s, then for $\int_0^T \tilde{H}_t c(t)dt \in \mathbb{D}^{1,2}$, the replicating portfolio is given by*

$$\begin{aligned} \pi_1(t) &= \frac{1}{\tilde{H}_t\sigma_t}\mathbb{E}\left[\int_t^T \tilde{H}_s D_t c(s) ds\right] \\ &\quad -\frac{1}{\tilde{H}_t\sigma_t}\mathbb{E}\left[\int_t^T \tilde{H}_s c(s)\left(\int_t^S D_t r_u du + \int_t^S D_t\theta_u d\tilde{W}_u\right) ds\right]. \end{aligned} \quad (4.49)$$

Proof:
Since $B = 0$, The process X_t in this case is given by

$$X_t = \frac{1}{\tilde{H}_t}\mathbb{E}[F].$$

Now we let $F = \int_t^T \tilde{H}_s c(s)ds$. Similarly, if $F \in \mathbb{D}^{1,2}$, then $\phi_t = \mathbb{E}[D_t F]$. Hence from (4.44), we have

$$\begin{aligned} \pi_1(t) &= \frac{1}{\sigma_t}\left(\frac{\mathbb{E}[D_t F]}{\tilde{H}_t} + \frac{\mathbb{E}[F]}{\tilde{H}_t}\theta_t\right) \\ &= \frac{1}{\tilde{H}_t\sigma_t}\left(\mathbb{E}\left[D_t\int_t^T \tilde{H}_s c(s)ds\right] + \mathbb{E}\left[\int_t^T \tilde{H}_s c(s)ds\right]\theta_t\right) \\ &= \frac{1}{\tilde{H}_t\sigma_t}\left(\mathbb{E}\left[\int_t^T \tilde{H}_s D_t c(s)ds\right] + \mathbb{E}\left[\int_t^T c(s) D_t\tilde{H}_s ds\right] + \mathbb{E}\left[\int_t^T \tilde{H}_s c(s)ds\right]\theta_t\right) \end{aligned} \quad (4.50)$$

where the last equality is obtained by making the use of (2.15). From (4.50), by considering the Malliavin derivative $D_t\tilde{H}$ from (4.47), we obtain

$$\begin{aligned}
\pi_1(t) &= \frac{1}{\tilde{H}_t\sigma_t}\left(\mathbb{E}\left[\int_t^T \tilde{H}_s D_t c(s)ds\right] + \mathbb{E}\left[\int_t^T \tilde{H}_s c(s)ds\right]\theta_t\right) \\
&\quad -\frac{1}{\tilde{H}_t\sigma_t}\left(\mathbb{E}\left[\int_t^T c(s)\left(\tilde{H}_t\left(\theta_t + \int_t^s D_r r_u du + \int_t^s D_t\theta_u d\tilde{W}_u\right)\right)ds\right]\right). \\
&= \frac{1}{\tilde{H}_t\sigma_t}\left(\mathbb{E}\left[\int_t^T \tilde{H}_s D_t c(s)ds\right] + \mathbb{E}\left[\int_t^T \tilde{H}_s c(s)ds\right]\theta_t\right) - \frac{1}{\tilde{H}_t\sigma_t}\left(\mathbb{E}\left[\int_t^T \tilde{H}_s c(s)ds\right]\theta_t\right) \\
&\quad -\frac{1}{\tilde{H}_t\sigma_t}\left(\mathbb{E}\left[\int_t^T \tilde{H}_t c(s)\left(\int_t^s D_r r_u du + \int_t^s D_t\theta_u d\tilde{W}_u\right)ds\right]\right). \\
&= \frac{1}{\tilde{H}_t\sigma_t}\left(\mathbb{E}\left[\int_t^T \tilde{H}_s D_t c(s)ds\right]\right) \\
&\quad -\frac{1}{\tilde{H}_t\sigma_t}\left(\mathbb{E}\left[\int_t^T \tilde{H}_t c(s)\left(\int_t^s D_r r_u du + \int_t^s D_t\theta_u d\tilde{W}_u\right)ds\right]\right). \qquad (4.51)
\end{aligned}$$

□

4.3.2 Replication of European call option

Fir the Black-Scholes model, we consider the following stochastic differential equation

$$S_t^0 = S_0^0 + \int_0^t rS_s^0 ds. \qquad (4.52)$$

$$S_t^1 = S_0^1 + \int_0^t \mu S_s^1 ds + \int_0^t \sigma S_s^1 dW_s \qquad (4.53)$$

for $t \in [0, T]$ and given that $S_0^0 > 0$, $S_0^1 > 0$. A European call option for a stock price S_t given by stochastic differential equation (4.37) for $t \in [0, T]$ is described by a random variable

$$B = \max\{S_T - K, 0\} \qquad (4.54)$$

where K is the strike price. Note that for us to describe a portfolio $\pi(t) = (\pi_0(t), \pi_1(t))$ on a market with one stock of a stock price S_t, it is enough to compute $\eta_1(t)$ such that

$$\pi_1(t) = \eta_1(t)S_t^1$$

Hence, we have the following results for Black-Scholes model.

Theorem 4.3.4 *Let S_t^1 be a stochastic differential equation of the form (4.52). Let B be defined by (4.54). Then, we have*

$$\begin{aligned}\eta_1(t) &= \frac{\beta(T)}{\beta(t)}\frac{1}{\sigma S_t^1}\mathbb{E}[\sigma S_T^1 \mathbf{1}_{\{K,\infty\}}(S_T^1)] \\ &= \mathbb{E}\left[\frac{\beta(T)S_T^1}{\beta(t)S_t^1}\mathbf{1}_{\{K,\infty\}}(S_T^1)\right] \end{aligned} \tag{4.55}$$

Proof:
We can rearrange (4.40) such that

$$\eta_1(t) = \frac{\pi_1(t)}{S_t^1}. \tag{4.56}$$

From Theorem 4.3.2, $\pi_1(t)$ is given by

$$\pi_1(t) = \exp\left(-\int_t^T (r_s - \frac{1}{2}\theta_s^2)ds - \int_t^T \theta_s dW_s\right)\sigma_t^{-1}\mathbb{E}[D_t B] \tag{4.57}$$

We are required to take the Malliavin derivative of the process B given by (4.54). This is given as follows

$$D_t B = D_t(\max\{S_T - K, 0\}) = D_t S_T \mathbf{1}_{\{K,\infty\}}(S_T). \tag{4.58}$$

Now for $D_t S_T$, we take the Malliavin derivative of the solution of the stochastic differential equation given by (4.52). By making the use of Itô's formula, we obtain

$$S_t^1 = S_1(0)e^{(\mu - 1\frac{1}{2}\sigma^2)t - W_t}$$

and its Malliavin derivative is given by

$$D_t S_t^1 = \sigma_t S_t^1 \mathbf{1}_{[0,T]}(t). \tag{4.59}$$

Hence

$$\mathbb{E}[D_t B] = \mathbb{E}[\sigma_t S_T^1 \mathbf{1}_{\{K,\infty\}}(S_T^1)] \tag{4.60}$$

We note that

$$\exp\left(-\int_t^T (r_s - \frac{1}{2}\theta_s^2)ds - \int_t^T \theta_s dW_s\right) = \exp\left(-\int_t^T r_s ds\right)\exp\left(-\frac{1}{2}\theta_s^2 ds - \int_t^T \theta_s dW_s\right)$$

of which by taking (4.43) into consideration, we observe that

$$\frac{\beta_T}{\beta_t} = \exp\left(-\int_t^T r_s ds\right). \tag{4.61}$$

By substituting (4.60) and (4.61) into (4.56), we obtain the results. □

Chapter 5

Computation of price sensitivities

In this chapter we apply Malliavin calculus compute the price sensitivities (known as Greeks). The calculus is useful for discontinuous payoff function. We follow the work of Fournie *et al.* [9]. We also give few examples. We first define the option price $u(\cdot)$ as the probabilistic representation of the payoff function Φ given by

$$\Phi = \Phi(X_T)$$

which depend on the process $\{X_t : 0 \leq t \leq T\}$. We assume that Φ satisfy the integrability condition

$$\mathbb{E}[\Phi(X_T)^2] < \infty. \tag{5.1}$$

We will denote the option price by $u(x)$. From the arbitrary theory, the option price can be expressed in terms of the expectation as

$$u(x) = \mathbb{E}[\Phi(X_T)]. \tag{5.2}$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}$ is infinitely differentiable function of which all its partial derivatives have polynomial growth. Our focus is on the options of the European type which can be exercised only at maturity time T. The main interest of our study is on discontinuous payoff functionals, that is, we shall consider digital option. The aim is to take the partial derivative of the option price $u(\cdot)$ with respect to the underlying factor. That is:

$$\frac{\partial}{\partial \alpha}\mathbb{E}[\Phi(X_T)].$$

We will require the coefficient matrix σ to satisfy the following uniform elliptic condition:

$$\exists \eta > 0 : \ \xi^T \sigma(x)^T \sigma(x) \xi > \eta |\xi|^2 \ \text{ for all } \ \xi, x \in \mathbb{R}^n \ \text{ with } \ \xi \neq 0 \tag{5.3}$$

where ξ^T denotes the transpose of ξ. This condition ensures that $\sigma(X_t)^{-1}Y_t$ belongs to the space $L^2(\Omega \times [0,T])$ where Y_t is the first variational process (3.21) of the stochastic process X_t for $t \geq 0$ given by (3.4).
The next results allows us to assume infinite smoothness of the payoff function when deriving the price sensitivity formulas. Let L^2 denote the class of locally integrable functions such that the set of discontinuous payoff functions has Lebesgue measure zero and satisfy (5.1).
The following Lemma verifies some quite standard but useful result. It justifies the differentiation under the expectation operator [29].

Lemma 5.0.1 *Let Φ be a real valued random variable depending on a parameter $x \in \mathbb{R}$. Suppose further that, for almost every $\omega \in \Omega$, the mapping $x \to \Phi(\omega)$ is continuously differentiable in $[a,b]$ and that*

$$\mathbb{E}\left[\sup_{x\in[a,b]} |\frac{\partial}{\partial x}\Phi(X_T^x)|\right] < \infty.$$

Then the mapping $x \to \mathbb{E}[\Phi(X_T^x)]$ is differentiable in (a,b) and for every $x \in (a,b)$, we have

$$\frac{\partial}{\partial x}\mathbb{E}[\Phi(X_T^x)] = \mathbb{E}\left[\frac{\partial}{\partial x}\Phi(X_T^x)\right].$$

Proof:
Since a function Φ is continuously differentiable with bounded derivatives, we have

$$\frac{\Phi(X_T^{x+h}) - \Phi(X_T^x)}{\|h\|} - \frac{\langle \frac{\partial}{\partial x}\Phi, h\rangle}{\|h\|} \to 0 \quad \text{as} \quad h \to 0.$$

The second term is uniformly integrable in h since the partial derivative of the payoff function Φ are assumed to be bounded. In addition, by the mean value theorem

$$\| \frac{\Phi(X_T^{x+h}) - \phi(X_T^x)}{\|h\|} \| \leq M \sum_{i=1}^{m} \frac{\| X_{T_i}^{x+h} - X_{T_i}^x \|}{\|h\|}.$$

Since

$$\sum_{i=1}^{m} \frac{\| X_{T_i}^{x+h} - X_{T_i}^x \|}{\|h\|}$$

is uniformly integrable in h leads to the uniform integrability in h of

$$\| \frac{\Phi(X_T^{x+h}) - \Phi(X_T^x)}{\|h\|} \| .$$

This in turn tells us that

$$\frac{\Phi(X_T^{x+h}) - \Phi(X_T^x)}{\|h\|} - \frac{\langle \frac{\partial}{\partial x}\Phi, h\rangle}{\|h\|}$$

is uniformly integrable in h. Since it converges to 0, the dominated convergence theorem tells us that it also converges to 0 in L^1 and hence, the results follows. □

Lemma 5.0.2 *Suppose that*

$$\frac{\partial}{\partial x}\mathbb{E}[\Phi(X_T)] = \mathbb{E}[\Phi(X_T)\pi] \tag{5.4}$$

holds for $\Phi \in C_0^\infty(\mathbb{R})$, $\pi \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. Suppose also that $x \to \pi$ is continuous, almost surely. Then (5.4) holds for $\Phi \in L^2$.

Proof:
Let Φ satisfy (5.1) and approximate it by a sequence $\{\Phi_n\}_{n>0}$ of infinitely differentiable functions each with bounded derivatives and compact support such that $\Phi_n \to \Phi$ Lebesgue almost everywhere as $n \to \infty$. Since x has transition probability that are absolutely continuous with respect to Lebesque measure and discontinuous of Φ have measure zero, we have

$$\Phi_n(X_T) \to \Phi(X_T) \qquad a.s$$

Furthermore, the family $\Phi_n(X_t)^2$ is uniformly integrable so

$$\Phi_n(X_T) \to \Phi(X_T) \qquad a.s$$

in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and turns also in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ as $n \to \infty$.
Define the option price

$$u(x) = \mathbb{E}[\Phi(X_T)]$$

and

$$u_n(x) = \mathbb{E}[\Phi_n(X_T)]$$

and note that

$$U_n(x) \to u(x) \qquad \text{for every } x \in [a, b].$$

Furthermore, let

$$g(x) = \mathbb{E}[\Phi(X_T)\pi].$$

By Cauchy-Schwartz inequality

$$\mid \frac{\partial}{\partial x}u_n(x) - g(x) \mid \leq \varepsilon_n(x)\psi(x)$$

where

$$\varepsilon_n(x) = \left(\mathbb{E}\left[(\Phi_n(X_T) - \Phi(X_T))^2\right]\right)^{1/2}$$

and

$$\psi(x) = \left(\mathbb{E}[|\pi|^2]\right)^{1/2}.$$

From the assumption, it follows that ψ and ε are continuous. Thus for arbitrary compact set $K \subset \mathbb{R}$, we have

$$\sup_{x\in K} \mid \frac{\partial}{\partial x} u_n(x) - g(x) \mid \leq C_n \sup_{x\in K} \varepsilon_n(x)$$

with

$$C_n = \sup_{x\in K} \psi(x).$$

Since $\sup_{x\in K} \varepsilon_n(x) \to 0$ as $n \to \infty$, it follows that

$$\frac{\partial}{\partial x} u_n(x) \to g(x),$$

uniformly on compact subsets of $\mathbb{R}$, proving the results. □

We are now ready to derive the formulas for the price sensitivities.

Theorem 5.0.3 *Let $a(t)$ be a continuous deterministic function of the form (3.23). Let X_t be a stochastic differential equation of the form (3.4) and $\Phi : \mathbb{R} \to \mathbb{R}$ be a function of polynomial growth. Then the Delta of the option is given by*

$$\Delta = \mathbb{E}\left[\Phi(X_T)\int_0^T a(t)\sigma^{-1}(X_t)Y_t dW_t\right]. \tag{5.5}$$

Proof:
We assume that the payoff function Φ is continuously differentiable with bounded gradient.

$$\Delta = u'(x) = \frac{\partial}{\partial x}\mathbb{E}[\Phi(X_T)].$$

Using Lemma 5.0.1, we have

$$\frac{\partial}{\partial x}\mathbb{E}[\Phi(X_T)] = \mathbb{E}\left[\frac{\partial}{\partial x}\Phi(X_T)\right] = \mathbb{E}\left[\Phi'(X_T)\frac{\partial}{\partial x}(X_T)\right]. \tag{5.6}$$

Recall that $Y_t = \frac{\partial}{\partial x}(X_t)$, we have

$$\frac{\partial}{\partial x}\mathbb{E}[\Phi(X_T)] = \mathbb{E}[\Phi'(X_T)Y_T]. \tag{5.7}$$

From Lemma 3.2.5, we see that

$$Y_{t_k} = \int_0^T D_t X_{t_k} a(t)\sigma^{-1}(t) Y_t dt.$$

Therefore

$$\mathbb{E}[\Phi'(X_T)Y_T] = \mathbb{E}\left[\int_0^T \Phi'(X_T) D_r X_T a(t)\sigma^{-1}(X_t) Y_t dt\right]$$

By chain rule (Proposition 2.4.6), we obtain

$$\mathbb{E}\left[\int_0^T \Phi'(X_T) D_r X_T a(s)\sigma^{-1}(X_s) Y_s ds\right] = \mathbb{E}\left[\int_0^T D_r(\Phi(X_T)) a(s)\sigma^{-1}(X_s) Y_s ds\right].$$

By applying the integration by parts formula (Theorem 2.5.4), we deduce that

$$\mathbb{E}\left[\int_0^T D_r(\Phi(X_T)) a(s)\sigma^{-1}(X_s) Y_s ds\right] = \mathbb{E}\left[\Phi(X_T)\int_0^T a(s)(\sigma^{-1}(X_s) Y_s) dW_s\right] \tag{5.8}$$

which complete the proof. □

Remarks:

i. There is no differentiation of the payoff function Φ.

ii. There is no need to know the density of the density function, but the diffusion.

iii. The Malliavin weight function π does not depend on the payoff function Φ.

Example 5.0.4 *Now for a geometric Brownian motion given by the stochastic differential equation of the following type*

$$dX_t = bX_t dt + \sigma X_t dW_t, \qquad X_0 = x \in \mathbb{R} \tag{5.9}$$

where the coefficients b and σ are constants and the process $\{W_t : 0 \le t \le T\}$ is the standard Brownian motion. From Theorem 5.0.3, we need to calculate

$$\int_0^T a(t)(\sigma^{-1}(X_t) Y_t) dW_t.$$

The fact that

$$\sigma(X_t) = \sigma X_t \tag{5.10}$$

implies that

$$\sigma^{-1}(X_t) = \frac{1}{\sigma X_t}.$$

We recall that

$$Y_t = \frac{\partial}{\partial x}(X_t) = \frac{X_t}{x}.$$

Therefore, choosing $a(t) = \frac{1}{T}$*, we have*

$$\begin{aligned}\int_0^T a(t)(\sigma^{-1}(X_t)Y_t)dW_t &= \frac{1}{T}\int \frac{1}{\sigma X_t}\frac{X_t}{x}dW_t \\ &= \frac{1}{T}\int_0^T \frac{1}{\sigma x}dW_t \\ &= \frac{W_T}{\sigma x T}.\end{aligned}$$

Therefore from Theorem 5.0.3, we have

$$\Delta = \mathbb{E}\left[\Phi(X_T)\frac{W_T}{\sigma x T}\right].$$

The second price sensitivity to be computed is Gamma (Γ). This Gamma measures the change in Delta. This is actually the second derivative of the option price with respect to the initial price x and it is given by the following result [22]

Proposition 5.0.5 *Let* $a(\cdot)$ *be a deterministic function of the form (3.23),* $\Phi : \mathbb{R} \to \mathbb{R}$ *be a function with polynomial growth and* Y_t *be the first variational process of the form (3.21). Assume that* $u = a(t)(\sigma^{-1}(X_t)Y_t)$ *and* $\delta(u) = \int_0^T a(t)(\sigma^{-1}(X_t)Y_t)dW_t$ *then Gamma is given by*

$$\Gamma = \mathbb{E}\left[\Phi(X_T)\left(\delta(u)\delta(u) + \frac{\partial}{\partial x}(\delta(u))\right)\right]. \tag{5.11}$$

Proof:
First we suppose that a function $\Phi(\cdot)$ is a continuously differentiable with bounded derivatives, by the definition of Gamma, we have

$$\Gamma = \frac{\partial^2}{\partial x^2}\mathbb{E}[\Phi(X_T)] = \frac{\partial}{\partial x}\mathbb{E}[\Phi(X_T)\delta(u)]. \tag{5.12}$$

The expectation (5.12) is the same as the Delta we computed from Theorem 5.0.3 where we made the use of both the integration by parts formula and the chain rule. So to take the partial derivative of that expectation, we note that the product $\Phi(X_T)\delta(u)$ is a function which depend on both the process X_T and the initial value x. Hence by making the use of the product rule, we obtain

$$\begin{aligned}\Gamma = \frac{\partial}{\partial x}\mathbb{E}[\Phi(X_T)\delta(u)] &= \mathbb{E}\left[\frac{\partial}{\partial x}\Phi(X_T)\delta(u)\frac{\partial}{\partial x}(X_T) + \Phi(X_T)\frac{\partial}{\partial x}(\delta(u))\right] \\ &= \mathbb{E}\left[\frac{\partial}{\partial x}\Phi(X_T)\delta(u)Y_T + \Phi(X_T)\frac{\partial}{\partial x}(\delta(u))\right] \\ &= \mathbb{E}\left[\frac{\partial}{\partial x}\Phi(X_T)\delta(u)Y_T\right] + \mathbb{E}\left[\Phi(X_T)\frac{\partial}{\partial x}(\delta(u))\right].\end{aligned}$$

Again we note that the first expectation on from the last equality is the same as the one from the computation of Delta, so in the same way we obtain

$$\begin{aligned}\Gamma &= \mathbb{E}\left[\Phi(X_T)\delta(u)\delta(u)\right] + \mathbb{E}\left[\Phi(X_T)\frac{\partial}{\partial x}(\delta(u))\right] \\ &= \mathbb{E}\left[\Phi(X_T)\delta(u)\delta(u) + \Phi(X_T)\frac{\partial}{\partial x}(\delta(u))\right] \qquad (5.13)\\ &= \mathbb{E}\left[\Phi(X_T)\left(\delta(u)\delta(u) + \frac{\partial}{\partial x}(\delta(u))\right)\right].\end{aligned}$$

□

Example 5.0.6 *From (5.13), we have*

$$\Gamma = \mathbb{E}[\Phi(X_T)\delta(u)\delta(u)] + \mathbb{E}\left[\Phi(X_T)\frac{\partial}{\partial x}(\delta(u))\right] \qquad (5.14)$$

Recall that

$$\begin{aligned}\delta(u) &= \delta(a(t)(\sigma^{-1}(X_t)Y_t)) \\ &= \int_0^T a(t)(\sigma^{-1}(X_t)Y_t))dW_t.\end{aligned}$$

From Example (5.0.4), we see that

$$\int_0^T a(t)(\sigma^{-1}(X_t)Y_t))dW_t = \frac{W_T}{\sigma x T}. \qquad (5.15)$$

For the first term in (5.13), we have

$$\delta(u)\delta(u) = \frac{1}{x\sigma T}\delta\left(\frac{W_T}{\sigma x T}\right).$$

If we let $F = W_T$ and $u = \frac{1}{\sigma x T}$ from Proposition 2.25, we obtain

$$\begin{aligned}\frac{1}{x\sigma T}\delta\left(\frac{W_T}{\sigma x T}\right) &= \frac{1}{\sigma x T}\left(\frac{W_T}{\sigma x T}\int_0^T dW_t - \int_0^T D_t W_T.\frac{1}{\sigma x T}dt\right) \\ &= \frac{1}{\sigma x T}\left(\frac{W_T^2}{\sigma x T} - \frac{1}{\sigma x}\right) \\ &= \frac{1}{\sigma x^2 T}\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma}\right).\end{aligned}$$

Note that from property (2.16), $D_t W_s = \mathbf{1}_{\{t \leq s\}}$. Therefore the first term in (5.13) reduces to

$$\mathbb{E}[\Phi(X_T)\delta(u)\delta(u)] = \mathbb{E}\left[\Phi(X_T)\frac{1}{\sigma x^2 T}\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma}\right)\right]. \tag{5.16}$$

From the second term we have

$$\frac{\partial}{\partial x}(\delta(u)) = \frac{\partial}{\partial x}\left(\frac{W_T}{\sigma x T}\right) = -\frac{W_T}{\sigma x^2 T}.$$

Therefore the second term in (5.13) reduces to

$$\mathbb{E}\left[\Phi(X_T)\frac{\partial}{\partial x}(\delta(u))\right] = -\mathbb{E}\left[\Phi(X_T)\frac{W_T}{\sigma x^2 T}\right]. \tag{5.17}$$

Combining (5.16) and (5.17), we obtain

$$\begin{aligned} \Gamma &= \mathbb{E}\left[\Phi(X_T)\frac{1}{\sigma x^2 T}\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma}\right)\right] - \mathbb{E}\left[\Phi(X_T)\frac{W_T}{\sigma x^2 T}\right] \\ &= \mathbb{E}\left[\Phi(X_T)\frac{1}{\sigma x^2 T}\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\right)\right]. \end{aligned}$$

5.1 Variations in the diffusion coefficient

The last price sensitivity to be computed is the so called Vega ($\mathcal{V}$) which is defined as the partial derivative of the option price with respect to the diffusion coefficient (volatility) σ. Since the drift coefficient $\underline{\ }(\cdot)$ and the diffusion coefficient $\sigma(\cdot)$ from (3.4) are functions of the underlying asset price, Vega and Rho quantify the impact of small perturbation in a specified direction on both the drift and the diffusion coefficients $b(\cdot)$ and $\sigma(\cdot)$. The payoff function Φ is assumed to be path dependent and has finite L^2 norm. First we introduce a set of deterministic functions [9]

$$\hat{\Lambda}_m = \left\{ a \in L^2([0,T]) : \int_{t_{i-1}}^{t_i} a(t)dt = 1 \quad \text{for} \quad i = 1, 2, ..., m \right\}. \tag{5.18}$$

Let $\tilde{\sigma} : \mathbb{R} \to \mathbb{R}^n$ be a continuously differentiable with bounded derivatives. We suppose that for every small perturbation $\epsilon \in [0, T]$, $(\sigma + \epsilon\tilde{\sigma})(\cdot)$ is continuously differentiable with bounded derivatives. The functions σ and $\tilde{\sigma}$ are assumed to satisfy the following uniform ellipticity conditions:

$$\exists \eta > 0 : \; \xi^*(\sigma + \tilde{\sigma})^*(x)(\sigma + \tilde{\sigma})(x)\xi \geq \eta|\xi|^2 \; \text{ for any } \; \xi, x \in \mathbb{R} \; \text{ with } \; \xi \neq 0 \tag{5.19}$$

In order to compute the partial derivative of the option price $u(x)$ with respect to the diffusion coefficient matrix σ in the direction $\tilde{\sigma}$, we consider the perturbed stochastic process $\{X_t^\epsilon : 0 \leq t \leq T\}$ defined by

$$dX_t^\epsilon = b(X_t^\epsilon) + [\sigma(X_t^\epsilon) + \epsilon\tilde{\sigma}](X_t^\epsilon)dW_t, \qquad X_0^\epsilon = x \in \mathbb{R} \tag{5.20}$$

where the process $\{W_t : 0 \leq t \leq T\}$ is the standard Brownian motion and ϵ is a very small parameter. For the payoff function Φ, we define ta perturbed option price $u^\epsilon(x)$ by

$$u^\epsilon(x) = \mathbb{E}[\Phi(X_T^\epsilon)] \tag{5.21}$$

of the perturbed stochastic process X_t^ϵ. We also introduce the first variational process Z_t^ϵ. This is actually the partial derivative of the process X_t^ϵ with respect to ϵ given by

$$\begin{aligned} dZ_t^\epsilon &= b'(X_t^\epsilon)Z_t^\epsilon dt + \tilde{\sigma}(X_t^\epsilon)dW_t + [\sigma' + \epsilon\tilde{\sigma}'](X_t^\epsilon)Z_t^\epsilon dW_t \\ Z_0^\epsilon &= 0 \end{aligned}$$

where 0 is the zero column vector of $\mathbb{R}^n$. Now we define the generalized Vega ($\mathcal{V}$) as follows:

Definition 5.1.1 *:$\mathcal{V}$ is defined as the partial derivative of the perturbed option price with respect to the small parameter ϵ in the direction $\tilde{\sigma}$ given by*

$$\mathcal{V} = \frac{\partial}{\partial\epsilon} u^\epsilon(x) \mid_{\epsilon=0} .$$

Next we consider the process $\{\beta_t : 0 \leq t \leq T\}$ defined by

$$\beta_t = Z_t Y^{-1}, \qquad 0 \leq t \leq T. \tag{5.22}$$

This process satisfies the following regularity results:

Lemma 5.1.2 *The process $\{\beta_t : 0 \leq t \leq T\}$ belongs to $\mathbb{D}^{1,2}$.*
The process $\{Y_t^{-1} : 0 \leq t \leq T\}$ satisfies

$$\begin{aligned} dY_t^{-1} &= Y_t^{-1}[-b'(X_t) + [\sigma'(X_t)]^2]dt - \sigma'(X_t)Y_t^{-1}dW_t \\ Y_0^{-1} &= I. \end{aligned}$$

The process $\{Y_t^{-1} : 0 \leq t \leq T\}$ belongs to $\mathbb{D}^{1,2}$ and also the process $\{Z_t : 0 \leq t \leq T\}$ belongs to $\mathbb{D}^{1,2}$. The following proposition gives the partial derivative of the perturbed option price $u^\epsilon(x)$ with respect to the small parameter ϵ at $\epsilon = 0$ given by

Proposition 5.1.3 *Assume that the matrix σ is uniformly elliptic. Then for any payoff function Φ of polynomial growth, the function $\epsilon \to u^\epsilon(x)$ is differentiable at $\epsilon = 0$ for any $x \in \mathbb{R}$. For any $a \in \hat{\Lambda}$ we have*

$$\frac{\partial}{\partial \epsilon} u^\epsilon(x) \mid_{\epsilon=0} = \mathbb{E}[\Phi(X_T)\delta(\sigma^{-1}(X_t)Y_t\tilde{\beta}_a(T))]$$

where

$$\tilde{\beta}_a(t) = \sum_{i=1}^{m} a(t)(\beta(t_i) - \beta(t_{i-1}))\mathbf{1}_{\{t_{i-1} \le t \le t_i\}} \tag{5.23}$$

and $\delta(\sigma^{-1}(X_t)Y_t\tilde{\beta}_a(t))$ is the Skorohod integral of the anticipating process

$$\{\sigma^{-1}(X_t)Y_t\tilde{\beta}_a(t) : 0 \le t \le T\}.$$

Proof:
In the same way from the proof of Theorem 5.0.3, we assume that Φ is a continuously differentiable function with bounded derivatives. From Lemma 5.0.1, we note that the partial derivative of the option price $u^\epsilon(x)$ with respect to the small parameter ϵ is actually obtained by differentiating inside the expectation operator. If we choose versions of the process $\{X_t^\epsilon : 0 \le t \le T\}$ which are continuously differentiable with respect to ϵ. Since Φ is continuously differentiable, we have

$$\begin{aligned}
\mathcal{V} = \frac{\partial}{\partial \epsilon} u^\epsilon(x) \mid_{\epsilon=0} &= \mathbb{E}\left[\frac{\partial}{\partial \epsilon}\Phi(X_T^\epsilon)\right] \\
&= \mathbb{E}\left[\Phi'(X_T^\epsilon)\frac{\partial}{\partial \epsilon}(X_T^\epsilon)\right] \\
&= \mathbb{E}[\Phi'(X_T^\epsilon)Z_{t_i}].
\end{aligned}$$

From equation (3.22), we have

$$D_t X_t = Y_{t_i} Y_t^{-1} \sigma_t \mathbf{1}_{\{t \le t_i\}} \quad \text{for any} \quad t \in [0, T].$$

Hence we have

$$\begin{aligned}
\int_0^T D_t X_{t_i} \sigma^{X_t} Y_t \mathbf{1}_{\{t \le t_i\}} \tilde{\beta}_a(T) dt &= \int_0^T Y_{t_i} Y_t^{-1} \sigma(X_t) \sigma^{-1}(X_t) Y_t \tilde{\beta}_a(T) \mathbf{1}_{\{t_{i-1} \le t \le t_i\}} dt \\
&= \int_0^T Y_{t_i} \tilde{\beta}_a(T) \mathbf{1}_{\{t_{i-1} \le t \le t_i\}} dt \\
&= Y_{t_i} \int_0^T \tilde{\beta}_a(T) \mathbf{1}_{\{t_{i-1} \le t \le t_i\}} dt.
\end{aligned}$$

Considering (5.26), we obtain

$$
\begin{aligned}
Y_{t_i}\int_0^T \tilde{\beta}_a(T)\mathbf{1}_{\{t\le t_i\}}dt &= Y_{t_i}\sum_{j=1}^{i}\int_0^T a(t)(\beta_{t_i}-\beta_{t_{i-1}})\mathbf{1}_{\{t_{i-1}\le t\le t_i\}}dt \\
&= Y_{t_i}\sum_{j=1}^{i}\int_{t_{i-1}}^{t_i} a(t)(\beta_{t_i}-\beta_{t_{i-1}})dt.
\end{aligned}
$$

Using the fact that $a(\cdot)$ is a deterministic function such that

$$
\int_{t_{i-1}}^{t_i} a(t)dt = 1 \qquad \text{and} \qquad \beta_{t_0} = 0.
$$

It follows that from (5.22)

$$
\begin{aligned}
Y_{t_i}\sum_{j=1}^{i}\int_{t_{i-1}}^{t_i} a(t)(\beta_{t_i}-\beta_{t_{i-1}})dt &= Y_{t_i}\beta_{t_i} \\
&= Z_{t_i}. \qquad (5.24)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{\partial}{\partial\epsilon}u^\epsilon(x)\mid_{\epsilon=0} &= \mathbb{E}[\Phi'(X_T^\epsilon)Z_{t_i}] \\
&= \mathbb{E}\left[\int_0^T \Phi'(X_t^\epsilon)D_tX_{t_i}\sigma^{-1}(X_t)Y_t\tilde{\beta}_t dt\right].
\end{aligned}
$$

By making the use of the chain rule from Proposition 2.4.6, we obtain

$$
\mathbb{E}\left[\int_0^T \Phi'(X_t^\epsilon)D_tX_{t_i}\sigma^{-1}(X_t)Y_t\tilde{\beta}_t dt\right] = \mathbb{E}\left[\int_0^T D_t(\Phi(X_t^\epsilon))\sigma^{-1}(X_t)Y_t\tilde{\beta}_t dt\right].
$$

Finally, since the process $\{\sigma^{-1}(X_t)Y_t : 0 \le t \le T\}$ belongs to $L^2(\Omega\times[0,T])$ and is $\mathcal{F}_t$-adapted by the uniform elliptic conditions (5.3) and by Lemma 5.1.2, $\tilde{\beta}_a(T)$ belongs to $\mathbb{D}^{1,2}$ where $a(\cdot)$ is a deterministic function. By making the use of the integration by parts formula from proposition 2.5.4 we conclude that

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T D_t(\Phi(X_t^\epsilon))\sigma^{-1}(X_t)Y_t\tilde{\beta}_t dt\right] &= \mathbb{E}\left[\Phi(X_t^\epsilon)\delta(\sigma^{-1}(X_t)Y_t\tilde{\beta}_t)\right] \\
&= \mathbb{E}\left[\Phi(X_t^\epsilon)\int_0^T \sigma^{-1}(X_t)Y_t\tilde{\beta}_t dW_t\right].
\end{aligned}
$$

□

Remark:
Note that the Mallivin weight $\int_0^T \sigma^{-1}(X_t)Y_t\tilde{\beta}_t dW_t$ does not depend on the payoff function $\Phi(\cdot)$ but on the deterministic function $a(\cdot)$.

Example 5.1.4 *The last example of the price sensitivity is Vega ($\mathcal{V}$). The general formula for Vega from Proposition 5.26 is given by*

$$\mathcal{V} = \frac{\partial}{\partial \epsilon} u^{\epsilon}(x) \mid_{\epsilon=0} = \mathbb{E}[\Phi(X_T)\delta(\sigma^{-1}(X_t)Y_t\tilde{\beta}_a(T))] \tag{5.25}$$

where

$$\tilde{\beta}_a(t) = \sum_{i=1}^{m} a(t)(\beta(t_i) - \beta(t_{i-1}))\mathbf{1}_{\{t_{i-1}\le t\le t_i\}} \tag{5.26}$$

Now, the Malliavin weight $\delta(\sigma^{-1}(X_t)Y_t\tilde{\beta}_a(T))$ can be computed by letting $F = \tilde{\beta}_a(T))$ and $u = \sigma^{-1}(X_t)Y_t$ from Proposition 2.25 such that

$$\delta(\sigma^{-1}(X_t)Y_t\tilde{\beta}_a(T)) = \tilde{\beta}_a(T)\int_0^T \sigma^{-1}(X_t)Y_t dW_t - \int_0^T D_t\tilde{\beta}_a(T)(\sigma^{-1}(X_t)Y_t)dt. \tag{5.27}$$

By expanding the summation $\tilde{\beta}_a(T)$ from (5.26), we obtain $a(t)\beta_{t_m}$ where $\beta_{t_0} = \beta_0 = 0$. Since β_{t_m} is the last term of the summation, we can set $t_m = T$ so that we have

$$a(t)\beta_{t_m} = a(t)\beta_T.$$

From equation (5.22),

$$a(t) = \frac{1}{T} \quad \text{and} \quad \beta_T = Z_T Y_T^{-1} \tag{5.28}$$

where Z_T is the variational process with respect to the diffusion coefficient σ. The solution to (5.9) is given by

$$X_T = xe^{(b-\frac{1}{2}\sigma^2)T+\sigma W_T} \tag{5.29}$$

Therefore

$$\frac{\partial}{\partial \sigma} X_T = (-\sigma T + W_T)xe^{(b-\frac{1}{2}\sigma^2)T+\sigma W_T} = (W_T - \sigma T)X_T. \tag{5.30}$$

Y_T^{-1} is the inverse of the first variational process (3.21) given by

$$Y_T^{-1} = \frac{x}{X_T}. \tag{5.31}$$

From (5.28), we have

$$\begin{aligned} a(t)\beta_T = \frac{Z_T Y_T^{-1}}{T} &= \frac{(W_T - \sigma T)X_T}{T}.\frac{x}{X_T} \\ &= \frac{xW_T}{T} - \sigma x. \end{aligned} \tag{5.32}$$

The Malliavin derivative of (5.32) is given by

$$D_t\left(\frac{xW_T}{T} - \sigma x\right) = D_t\left(\frac{xW_T}{T}\right) - \sigma x D_t(1) = \frac{x}{T} \tag{5.33}$$

By substituting (5.32) and (5.33) into (5.27), we obtain

$$\tilde{\beta}_a(T)\int_0^T \sigma^{-1}(X_t)Y_t dW_t - \int_0^T D_t\tilde{\beta}_a(T)(\sigma^{-1}(X_t)Y_t)dt = \left(\frac{xW_T}{T} - \sigma x\right)\int_0^T \sigma^{-1}(X_t)Y_t dW_t - \int_0^T \frac{x}{T}(\sigma^{-1}(X_t)Y_t)dt. \quad (5.34)$$

By making the use of (5.10), we obtain

$$\begin{aligned}\left(\frac{xW_T}{T} - \sigma x\right)\int_0^T \sigma^{-1}(X_t)Y_t dW_t - \int_0^T \frac{x}{T}(\sigma^{-1}(X_t)Y_t)dt &= \left(\frac{xW_T}{T} - \sigma x\right)\int_0^T \frac{1}{\sigma T}dW_t - \int_0^T \frac{x}{T}\frac{1}{\sigma x}dt \\ &= \left(\frac{xW_T}{T} - \sigma x\right)\left(\frac{W_T}{\sigma x}\right) - \left(\frac{1}{\sigma T}\right)(T) \\ &= \frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}. \quad (5.35)\end{aligned}$$

Thus

$$\delta(\sigma^{-1}(X_t)Y_t\tilde{\beta}_a(T)) = \frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}.$$

Therefore Vega is given by (From Proposition 5.1.3)

$$\mathcal{V} = \mathbb{E}\left[\Phi(X_T)\left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right)\right]. \quad (5.36)$$

Remark:

We note that the relationship between Γ and $\mathcal{V}$ can be summarized as follows:

$$\begin{aligned}\Gamma &= \frac{1}{\sigma x^2 T}\mathbb{E}\left[\Phi(X_T)\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\right)\right] \\ &= \frac{\mathcal{V}}{\sigma x T}. \quad (5.37)\end{aligned}$$

The results we have shown can be extended to the case where we consider a random variable G that is Malliavin differentiable and is independent of the initial value x (see [1]). Thus the extension is given by the following results

Theorem 5.1.5 *Suppose that the the diffusion coefficient σ is uniform elliptic and that $\mathbb{E}[\int_0^T \mid \sigma^{-1}(X_s)Y_s \mid^2 ds] < \infty$, in which Y denotes the first variational process. Let $G \in \mathbb{D}^{1,2}$ be a random variable which does not depend on x. Then for any measurable function Φ with polynomial growth one has*

$$\Delta = \mathbb{E}\left[\Phi(X_T)\pi\right]$$

where

$$\pi = \frac{1}{T}\left(G\int_0^T \sigma^{-1}(X_t)Y_t dW_t - \int_0^T D_t G(\sigma^{-1}(X_t)Y_t)dt\right). \tag{5.38}$$

Proof:
We assume that the payoff function Φ is continuously differentiable with bounded gradient. By Lemma 5.0.1, Delta is given by

$$\begin{aligned}\Delta &= \mathbb{E}\left[\Phi'(X_T)\frac{\partial}{\partial x}(X_T)G\right] \\ &= \mathbb{E}[\Phi'(X_T)Y_T G]\end{aligned}$$

This from Lemma 3.2.5 yields

$$\begin{aligned}\mathbb{E}[\Phi'(X_T)Y_T G] &= \mathbb{E}\left[\int_0^T (\Phi'(X_T)D_r X_T)a(t)\sigma^{-1}(X_t)Y_t G dt\right] \\ &= \mathbb{E}\left[\int_0^T D_r(\Phi(X_T))a(t)\sigma^{-1}(X_t)Y_t G dt\right].\end{aligned}$$

The last equality is obtained by using the Chain rule from (2.4.6). The random variable $\sigma^{-1}(X_t)Y_t \in L^2(\Omega \times [0,T])$ by Elliptic conditions (5.3), we let $a(t) = \frac{1}{T}$ and apply the duality formula (2.5.4) to obtain

$$\begin{aligned}\Delta &= \mathbb{E}\left[\Phi(X_T)\frac{1}{T}\int_0^T \sigma^{-1}(X_t)Y_t G dt\right] \\ &= \mathbb{E}\left[\Phi(X_T)\delta\left(\frac{1}{T}\sigma^{-1}(X_t)Y_t G\right)\right].\end{aligned}$$

Now we let $\pi = \delta\left(\frac{1}{T}\sigma^{-1}(X_t)Y_t G\right)$, since $\sigma^{-1}(X_t)Y_t$ is adapted, we can let $F = G$ and $u = \sigma^{-1}(X_t)Y_t$ from Proposition 2.25 such that

$$\begin{aligned}\pi &= \frac{1}{T}\left(G\int_0^T \tau_s dW_s - \int_0^T D_s G(\tau_s)ds\right) \\ &= \frac{1}{T}\left(G\int_0^T \sigma^{-1}(X_s)Y_s dW_s - \int_0^T D_s G(\sigma^{-1}(X_s)Y_s)ds\right).\end{aligned} \tag{5.39}$$

This complete the proof for the payoff function Φ being continuously differentiable. In the very same fashion, the general case is proven by the density argument. □

In the Black-Scholes model (where we consider the drift coefficient and the diffusion coefficient as constants), we obtain the following result.

Corollary 5.1.6 *Suppose $b(x) = \mu x$ and $\sigma(x) = \sigma x$ with σ being invertible. Then π is given by*

$$\pi = \frac{1}{Tx}\left((\sigma^{-1})(GW_T - \int_0^T D_t G dt)\right). \tag{5.40}$$

proof:
Since $\sigma(X_t)$ is assumed to be invertible, we have

$$\sigma^{-1}(X_t) = \frac{1}{\sigma x}.$$

We recall that $Y_t = \frac{X_t}{x}$, Therefore from (5.39), we have

$$\begin{aligned}
\pi &= \frac{1}{T}\left(G\int_0^T \sigma^{-1}(X_t)Y_t dW_t - \int_0^T D_t G(\sigma^{-1}(X_t)Y_t)dt\right) \\
&= \frac{1}{T}\left(G\int_0^T \frac{1}{\sigma X_t}\frac{X_t}{x}dW_t - \int_0^T D_t G(\frac{1}{\sigma X_t}\frac{X_t}{x})dt\right) \\
&= \frac{1}{T}\left(G\int_0^T \frac{1}{\sigma x}dW_t - \int_0^T D_t G(\frac{1}{\sigma x})dt\right) \\
&= \frac{1}{Tx}\left(\sigma^{-1}\left(G\int_0^T dW_t - \int_0^T D_t G dt\right)\right) \\
&= \frac{1}{Tx}\left(\sigma^{-1}\left(GW_T - \int_0^T D_t G dt\right)\right).
\end{aligned}$$

□

Proposition 5.1.7 *Suppose $b(x) = \mu x$ and $\sigma(x) = \sigma x$ with σ invertible. Then for any Φ with polynomial growth and a random variable $G = 1$, one has*

$$\Delta = \frac{\partial}{\partial x}\mathbb{E}[\Phi(X_T)] = \mathbb{E}[\Phi(X_T)\pi^\Delta], \quad \Gamma = \frac{\partial^2}{\partial x^2}\mathbb{E}[\Phi(X_T)] = \mathbb{E}[\Phi(X_T)\pi^\Gamma] \tag{5.41}$$

where

$$\pi^\Delta = \frac{1}{xT\sigma}W_T \tag{5.42}$$

$$\pi^\Gamma = (\pi^\Delta)^2 - \frac{1}{Tx^2\sigma^2} - \frac{\pi^\Delta}{x}. \tag{5.43}$$

Note that the above result is a special case for a random variable $G = 1$ and yields the same results as in example (5.0.4) and (5.0.6)

5.2 Greeks w.r.t the correlation in a stochastic volatility model

Here we do the computation of price sensitivities (Greeks) using the Malliavin calculus under assumptions that the underlying asset and interest rate both evolve from a stochastic volatility model and the interest rate model respectively. We suppose that the process S evolve as the following stochastic differential equation of the Black-Scholes dynamics (see [1]).

$$dS_t = rS_t dt + \lambda_t S_t dW_t^1, \tag{5.44}$$

$$d\lambda_t = k(\theta - \lambda_t)dt + \beta dW_t^2 \tag{5.45}$$

where the processes W_t^1 and W_t^1 are correlated Brownian motions with

$$d\langle W_t^1, W_t^2\rangle = \rho dt, \qquad \rho \in [-1, 1].$$

We consider a digital option with a payoff $\mathbf{1}_{[k,\infty)}$. We are interested in computing the sensitivity of the option with respect to ρ. First we set

$$W_t^1 = \sqrt{1-\rho^2}B_t^1, \qquad W_t^2 = B_t^2 \tag{5.46}$$

where B_t^1 and B_t^2 are independent Brownian motion and $B = (B_t^1, B_t^2)$ is a 2-dimensional Brownian motion. By substituting (5.46) into (5.44) and (5.45), we obtain the stochastic differential equation

$$dS_t = rS_t dt + \lambda_t S_t(\sqrt{1-\rho^2}dB_t^1 + \rho dB_t^2), \tag{5.47}$$

$$d\lambda_t = k(\theta - \lambda_t)dt + \beta dB_t^2. \tag{5.48}$$

By applying Itô's formula to (5.47), we obtain the solution

$$S_T = \exp\left(\int_0^t (r - \frac{1}{2}\lambda_s^2)ds + \sqrt{1-\rho^2}\int_0^t \lambda_t dB_t^1 + \rho\int_0^t \lambda_t dB_t^2\right). \tag{5.49}$$

For a smooth function Φ, one has

$$\frac{\partial}{\partial\rho}\mathbb{E}[\Phi(S_T)] = \mathbb{E}\left[\Phi'(S_T)\frac{\partial}{\partial\rho}(S_T)\right].$$

By considering (5.49), we obtain

$$\begin{aligned}
\frac{\partial}{\partial\rho}(S_T) &= S_T\frac{\partial}{\partial\rho}\left(\int_0^t (r - \frac{1}{2}\lambda_s^2)ds + \sqrt{1-\rho^2}\int_0^t \lambda_t dB_t^1 + \rho\int_0^t \lambda_t dB_t^2\right) \\
&= S_T\left(\int_0^T \lambda_t dB_t^2 - \frac{2\rho}{\sqrt{1-\rho^2}}\int_0^T \lambda_t dB_t^1\right) \\
&= S_T R
\end{aligned} \tag{5.50}$$

where

$$R = \int_0^T \lambda_t dB_t^2 - \frac{2\rho}{\sqrt{1-\rho^2}} \int_0^T \lambda_t dB_t^1. \qquad (5.51)$$

From (5.49), by making the use of chain rule from Proposition 2.4.6, we obtain

$$\begin{aligned} D_s^1 S_T &= S_T D_s^1 \left(\sqrt{1-\rho^2} \int_0^T \lambda_t dB_t^1 + \rho \int_0^T \lambda_t dB_t^2 \right) \\ &= S_T (\sqrt{1-\rho^2} \times \lambda_t) \\ &= \frac{\partial}{\partial \rho}(S_T) \frac{\lambda_s \sqrt{1-\rho^2}}{R} \qquad (5.52) \end{aligned}$$

so that

$$D_s^1 \Phi(S_T) = \Phi'(S_T) D_s^1 S_T = \Phi'(S_T) \frac{\partial}{\partial \rho}(S_T) \frac{\lambda_s \sqrt{1-\rho^2}}{R}.$$

Therefore

$$\Phi'(S_T) \frac{\partial}{\partial \rho}(S_T) = \frac{R}{\lambda_s \sqrt{1-\rho^2}} D_s^1 \Phi(S_T) = \frac{1}{T\sqrt{1-\rho^2}} \int_0^T D_s^1 \Phi(S_T) \frac{R}{\lambda_s} ds.$$

By applying the duality formula from Theorem 2.5.4 with respect to the Brownian motion B_t^1, we obtain

$$\begin{aligned} \mathbb{E}\left[\Phi'(S_T) \frac{\partial}{\partial \rho}(S_T) \right] &= \frac{1}{T\sqrt{1-\rho^2}} \mathbb{E}\left[\int_0^T D_s^1 \Phi(S_T) \frac{R}{\lambda_s} ds \right] \\ &= \mathbb{E}\left[\Phi(S_T) \frac{1}{T\sqrt{1-\rho^2}} \delta_1 \left(\frac{R}{\lambda} \right) \right]. \qquad (5.53) \end{aligned}$$

Now by applying the properties of the Skorohod integral from (2.25) for adapted processes, we obtain

$$\begin{aligned} \delta_1 \left(\frac{R}{\lambda} \right) &= R\delta_1 \left(\frac{1}{\lambda} \right) - \int_0^T D_s^1 R \frac{1}{\lambda_1} ds \\ &= \int_0^T \lambda_s^{-1} dB_s^1 - \int_0^T D_s^1 R \frac{1}{\lambda_1} ds. \qquad (5.54) \end{aligned}$$

Moreover, we note that the Malliavin derivative of R in (5.51) is given by

$$D_s^1 R = -\frac{2\rho}{\sqrt{1-\rho^2}} \lambda_s.$$

Hence, we can conclude that

$$\frac{\partial}{\partial \rho} \mathbb{E}[\Phi(S_T)] = \mathbb{E}\left[\Phi(S_T) \left(R \int_0^T \lambda_s^{-1} dB_s^1 + \frac{2\rho T}{\sqrt{1-\rho^2}} \right) \right]. \qquad (5.55)$$

Remark: We note that from (5.55), there is no need to differentiate the function Φ and the Malliavin weight function $R\int_0^T \lambda_s^{-1} dB_s^1 + \frac{2\rho T}{\sqrt{1-\rho^2}}$ does not depend on Φ.

Next we consider a 3-dimensional Brownian motion $\{W_t^j : 0 \leq t \leq T\}$ $j = 1, 2, 3$ is defined. The dynamics of the underlying asset price and the interest rate evolves according to the stochastic differential equation system. (see [5]).

$$\begin{cases} dS_t = r_t S_t dt + S_t \sigma(V_t) dZ_t^1 \\ dV_t = u(V_t)dt + v(V_t)dZ_t^2 \\ dr_t = f(r_t)dt + g(r_t)dZ_t^3, \end{cases} \tag{5.56}$$

where $\{Z_t\}_{0\leq t\leq T}^j$ are correlated Brownian motions with correlation coefficients $\rho_{ij} \in (-1, 1)$ for $i, j = 1, 2, 3$. The solution to the stochastic differential equation system (5.56) is given by S_t which represent the underlying asset, V_t which represent the volatility and r_t which represent the interest rate process. This processes has the initial values S_0, V_0 and r_0 respectively. These correlated Brownian motions may be written as the combination of three independent Brownian motions $\{W_t^j : 0 \leq t \leq T\}$ as

$$\begin{cases} dZ_t^1 = dW_t^1 \\ dZ_t^2 = \rho_{12} dW_t^1 + \mu_1 dW_t^2 \\ dZ_t^3 = \rho_{13} dW_t^1 + \mu_2 dW_t^2 + \mu_3 dW_t^3, \end{cases} \tag{5.57}$$

where

$$\mu_1 = \sqrt{1 - \rho_{12}^2}, \qquad \mu_2 = \frac{\rho_{23} - \rho_{12}\rho_{13}}{\mu_1},$$

$$\mu_3 = \frac{\sqrt{1 - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 + 2\rho_{13}\rho_{12}\rho_{13}}}{\mu_1}.$$

We assume that the correlation coefficients ρ_{ij} are chosen in a way that μ_3 is a real number. It is also assumed that $\sigma(\cdot), u(\cdot), v(\cdot), f(\cdot)$ and $g(\cdot)$ are continuously differentiable functions with bounded derivatives. We also assume that σ, v and g are adapted and non-zeros.

Now if we substitute (5.57) into (5.56) we obtain

$$\begin{cases} dS_t = r_t S_t dt + S_t \sigma(V_t) dW_t^1 \\ dV_t = u(V_t)dt + v(V_t)(\rho_{12} dW_t^1 + \mu_1 dW_t^2) \\ dr_t = f(r_t)dt + g(r_t)(\rho_{13} dW_t^1 + \mu_2 dW_t^2 + \mu_3 dW_t^3) \end{cases} \tag{5.58}$$

which can be represented in a matrix form as

$$d\begin{pmatrix} S_t \\ V_t \\ r_t \end{pmatrix} = \begin{pmatrix} r_t S_t \\ u(V_t) \\ f(r_t) \end{pmatrix} dt + \begin{pmatrix} S_t\sigma(V_t) & 0 & 0 \\ \rho_{12}v(V_t) & \mu_1 v(V_t) & 0 \\ \rho_{13}g(r_t) & \mu_2 g(r_t) & \mu_3 g(r_t) \end{pmatrix} d\begin{pmatrix} W_t^1 \\ W_t^2 \\ W_t^3 \end{pmatrix}. \tag{5.59}$$

If we let

$$X_t = \begin{pmatrix} S_t \\ V_t \\ r_t \end{pmatrix}, \qquad A(X_t) = \begin{pmatrix} r_t S_t \\ u(V_t) \\ f(r_t) \end{pmatrix},$$

and

$$B(X_t) = \begin{pmatrix} S_t\sigma(V_t) & 0 & 0 \\ \rho_{12}v(V_t) & \mu_1 v(V_t) & 0 \\ \rho_{13}g(r_t) & \mu_2 g(r_t) & \mu_3 g(r_t) \end{pmatrix} \tag{5.60}$$

and

$$x = \begin{pmatrix} S_0 \\ V_0 \\ r_0 \end{pmatrix}, \qquad W_t = \begin{pmatrix} W_t^1 \\ W_t^2 \\ W_t^3 \end{pmatrix}$$

where the process $\{W_t : 0 \leq t \leq T\}$ is a standard three-dimensional Brownian motion, then we obtain a stochastic differential equation given by

$$dX_t = A(X_t)dt + B(X_t)dW_t, \qquad X_0 = x \in \mathbb{R}. \tag{5.61}$$

Note that the stochastic differential equation (5.61) is similar to (3.4) but it is in 3-dimension whereas (3.4) is in 1-dimension. We assume that the drift coefficient $A(\cdot)$ and the diffusion coefficient $B(\cdot)$ are bounded with partial derivatives and satisfy linear growth and the Lipschitz conditions:
There exist a constant $C < \infty$ such that

$$\mid A(x) - A(y) \mid + \mid B(x) - B(y) \mid \leq C \mid x - y \mid;$$

$$\mid A(x) \mid + \mid B(x) \mid \leq C(1 + \mid x \mid).$$

These conditions ensures the existence and uniqueness of the strong solution of (5.61). We also require the coefficient matrix to B to satisfy the following uniform Elliptic condition:

$$\text{There exist } \ \eta > 0 \ \text{ such that } \ (\xi^T B(x)^T)(B\xi) \geq \eta|\xi|^2 \ \text{ for all } \ \xi, x \in \mathbb{R}^3 \ \text{ with } \ \xi \neq 0$$

where ξ^T denotes the transpose of ξ. These conditions ensures the existence of the inverse of the coefficient matrix B. The first variational process in this is given by

$$\begin{aligned} dY_t &= A'(X_t)Y_t dt + \sum_{j=0}^{3} B'_j(X_t)Y_t dW_t^j, \qquad (5.62)\\ Y_0 &= 1_{3\times 3}. \end{aligned}$$

Here $A'(\cdot)$ is the Jacobian of $A(\cdot)$ and $B'_j(\cdot)$ is the j^{th} column vector of the matrix $B(\cdot)$ with respect to x respectively and $1_{3\times 3}$ is the identity matrix of $\mathbb{R}^3$.
Recall that the first variational process is given by $Y_t = \frac{\partial}{\partial x} X_t$. Thus it follows that if $(Y_t Y_s^{-1} B) \in L^2([0,T]\times\Omega)$ for all $s,t\in[0,T]$, then the process X_t is Malliavian differentiable and its Malliavin derivative is given by

$$D_s X_t = Y_t Y_s^{-1} B(X_t)\mathbf{1}_{s\le t}, \qquad s \ge 0, \quad as. \qquad (5.63)$$

The components of the first variational process Y_t can be calculated as in the following results [5].

Proposition 5.2.1 *Let X_t and Y_t be defined by (5.61) and (5.62) respectively, Then $Y_t^{23} = 0$ and $Y_t^{ij} = 0$ for $i > j$ where $ij = 1, 2, 3$. Thus the diagonal entries Y_t^{11}, Y_t^{22} and Y_t^{33} have the following solutions.*

$$Y_t^{11} = \exp\left(\int_0^T \left(r_t - \frac{1}{2}\sigma^2(V_t)\right)dt + \int_0^T \sigma(V_t)dZ_t^1\right), \qquad (5.64)$$

$$Y_t^{22} = \exp\left(\int_0^T \left(u'(V_t) - \frac{1}{2}v'(V_t)^2\right)dt + \int_0^T v'(V_t)dZ_t^2\right), \qquad (5.65)$$

$$Y_t^{33} = \exp\left(\int_0^T \left(f'(r_t) - \frac{1}{2}g'(r_t)^2\right)dt + \int_0^T g'(r_t)dZ_t^3\right). \qquad (5.66)$$

Furthermore, Y_t^{12} and Y_t^{13} satisfy

$$dY_t^{12} = r_t Y_t^{12} dt + [\sigma(V_t)Y_t^{12} + S_t\sigma'(V_t)Y_t^{22}]dW_t^1, \qquad (5.67)$$

$$dY_t^{13} = [r_t Y_t^{13} + S_t Y_t^{33}]dt + \sigma(V_t)Y_t^{13}dW_t^1 \qquad (5.68)$$

with the initial values $Y_0^{12} = Y_0^{13} = 0$.

Note that if we apply Itô lemma to (5.64), (5.65) and (5.66), we obtain

$$\begin{aligned} dY_t^{11} &= r_t Y_t^{11} dt + \sigma(V_t)Y_t^{11}dZ_t^1,\\ dY_t^{22} &= u'(V_t)Y_t^{22}dt + v'(V_t)Y_t^{22}dZ_t^2,\\ dY_t^{33} &= f'(r_t)Y_t^{33}dt + g'(r_t)Y_t^{33}dZ_t^3 \end{aligned}$$

with the initial values $Y_0^{11} = Y_0^{22} = Y_0^{33} = 1$
Remark: The first variational process Y_t^{11} from (5.64) satisfy

$$Y_t^{11} = \frac{1}{S_0} S_t. \tag{5.69}$$

We consider the payoff function Φ which depends on the process X_T given by

$$\Phi = \Phi(X_T) \tag{5.70}$$

where T is the maturity time. The option price u is the probabilistic representation of the payoff function (5.70) given by

$$u(x) = \mathbb{E}[\Phi(X_T)].$$

To obtain a valid computation result, we avoid the degeneracy of the Malliavin weights with probability one [see [5]], we introduce the set of deterministic functions given by

$$\tilde{\Gamma} = \left\{ a(t) \in L^2([0,T]) : \int_0^{t_i} a(t)dt = 1 \quad \forall i = 1, 2, ..., n \right\} \quad \text{in} \quad \mathbb{R}. \tag{5.71}$$

As in Theorem 5.0.3. The general formula for Delta (Δ) in 3-dimensional case is given by

$$\Delta = u'(x) = \mathbb{E}\left[\Phi(X_T)\pi^{\Delta}\right] \tag{5.72}$$

where

$$\pi^{\Delta} = \int_0^T a(t)(B^{-1}(X_t)Y_t)^T dW_t. \tag{5.73}$$

To compute the inverse of matrix $B(X_t)$ given in (5.60), we use the adjoint method. The determinant of the matrix $B(X_t)$ is given by

$$\det(B(X_t)) = \mu_1\mu_3 S_t \sigma(V_t) v(V_t) g(r_t). \tag{5.74}$$

The adjoint (the transpose of the cofactor matrix) of $B(X_t)$ is given by

$$\text{Adj}(B(X_t)) = \begin{pmatrix} \mu_1\mu_3 v(V_t)g(r_t) & 0 & 0 \\ -\rho_{12}\mu_3 v(V_t)g(r_t) & \mu_3 S_t\sigma(V_t)g(r_t) & 0 \\ \rho_{12}\mu_2 v(V_t)g(r_t) - \rho_{13}\mu_1 v(V_t)g(r_t) & -\mu_2 S_t\sigma(V_t)g(r_t) & \mu_1 S_t\sigma(V_t)v(V_t) \end{pmatrix} \tag{5.75}$$

where $Adj(\cdot)$ is the adjoint of $(\cdot)$. The inverse of the volatility matrix $B(X_t)$ is given by

$$\begin{aligned} B^{-1}(X_t) &= \frac{1}{\det(B(X_t))} Adj(B(X_t)) \\ &= \begin{pmatrix} \frac{1}{S_t\sigma(V_t)} & 0 & 0 \\ \frac{-\rho_{12}}{\mu_1 S_t\sigma(V_t)} & \frac{1}{\mu_1 v(V_t)} & 0 \\ \frac{\rho_{12}\mu_2 - \rho_{13}\mu_1}{\mu_1\mu_3 S_t\sigma(V_t)} & \frac{-\mu_2}{\mu_1\mu_3 v(V_t)} & \frac{1}{\mu_3 g(r_t)} \end{pmatrix}. \end{aligned} \tag{5.76}$$

We calculate the product $B^{-1}(X_t)Y_t$ as follows:

$$\begin{aligned} B^{-1}(X_t)Y_t &= \begin{pmatrix} \frac{1}{S_t\sigma(V_t)} & 0 & 0 \\ \frac{-\rho_{12}}{\mu_1 S_t\sigma(V_t)} & \frac{1}{\mu_1 v(V_t)} & 0 \\ \frac{\rho_{12}\mu_2-\rho_{13}\mu_1}{\mu_1\mu_3 S_t\sigma(V_t)} & \frac{-\mu_2}{\mu_1\mu_3 v(V_t)} & \frac{1}{\mu_3 g(r_t)} \end{pmatrix} \begin{pmatrix} Y_t^{11} & Y_t^{12} & Y_t^{13} \\ 0 & Y_t^{22} & 0 \\ 0 & 0 & Y_t^{33} \end{pmatrix} \\ &= \begin{pmatrix} \frac{Y_t^{11}}{S_t\sigma(V_t)} & \frac{Y_t^{12}}{S_t\sigma(V_t)} & \frac{Y_t^{13}}{S_t\sigma(V_t)} \\ \frac{-\rho_{12}Y_t^{11}}{\mu_1 S_t\sigma(V_t)} & \frac{Y_t^{22}}{\mu_1 v(V_t)} - \frac{\rho_{12}Y_t^{12}}{\mu_1 S_t\sigma(V_t)} & -\frac{\rho_{12}Y_t^{13}}{\mu_1 S_t\sigma(V_t)} \\ \frac{(\rho_{12}\mu_2-\rho_{13}\mu_1)Y_t^{11}}{\mu_1\mu_3 S_t\sigma(V_t)} & \frac{(\rho_{12}\mu_2-\rho_{13}\mu_1)Y_t^{12}}{\mu_1\mu_3 S_t\sigma(V_t)} - \frac{\mu_2 Y_t^{22}}{\mu_1\mu_3 v(V_t)} & \frac{(\rho_{12}\mu_2-\rho_{13}\mu_1)Y_t^{13}}{\mu_1\mu_3 S_t\sigma(V_t)} - \frac{Y_t^{33}}{\mu_3 g(r_t)} \end{pmatrix}. \end{aligned}$$

The transpose of the above matrix is given by

$$(B^{-1}(X_t)Y_t)^T = \begin{pmatrix} \frac{Y_t^{11}}{S_t\sigma(V_t)} & \frac{-\rho_{12}Y_t^{11}}{\mu_1 S_t\sigma(V_t)} & \frac{(\rho_{12}\mu_2-\rho_{13}\mu_1)Y_t^{11}}{\mu_1\mu_3 S_t\sigma(V_t)} \\ \frac{Y_t^{12}}{S_t\sigma(V_t)} & \frac{Y_t^{22}}{\mu_1 v(V_t)} - \frac{\rho_{12}Y_t^{12}}{\mu_1 S_t\sigma(V_t)} & \frac{(\rho_{12}\mu_2-\rho_{13}\mu_1)Y_t^{12}}{\mu_1\mu_3 S_t\sigma(V_t)} - \frac{\mu_2 Y_t^{22}}{\mu_1\mu_3 v(V_t)} \\ \frac{Y_t^{13}}{S_t\sigma(V_t)} & -\frac{\rho_{12}Y_t^{13}}{\mu_1 S_t\sigma(V_t)} & \frac{(\rho_{12}\mu_2-\rho_{13}\mu_1)Y_t^{13}}{\mu_1\mu_3 S_t\sigma(V_t)} - \frac{Y_t^{33}}{\mu_3 g(r_t)} \end{pmatrix}. \tag{5.77}$$

If we choose the deterministic function $a(t) = \frac{1}{T}$ from (5.71), we see that

$$\begin{aligned} &\int_0^T a(t)(B^{-1}(X_t)Y_t)^T dW_t \\ &= \frac{1}{T}\int_0^T \begin{pmatrix} \frac{Y_t^{11}}{S_t\sigma(V_t)} & \frac{-\rho_{12}Y_t^{11}}{\mu_1 S_t\sigma(V_t)} & \frac{(\rho_{12}\mu_2-\rho_{13}\mu_1)Y_t^{11}}{\mu_1\mu_3 S_t\sigma(V_t)} \\ \frac{Y_t^{12}}{S_t\sigma(V_t)} & \frac{Y_t^{22}}{\mu_1 v(V_t)} - \frac{\rho_{12}Y_t^{12}}{\mu_1 S_t\sigma(V_t)} & \frac{(\rho_{12}\mu_2-\rho_{13}\mu_1)Y_t^{12}}{\mu_1\mu_3 S_t\sigma(V_t)} - \frac{\mu_2 Y_t^{22}}{\mu_1\mu_3 v(V_t)} \\ \frac{Y_t^{13}}{S_t\sigma(V_t)} & -\frac{\rho_{12}Y_t^{13}}{\mu_1 S_t\sigma(V_t)} & \frac{(\rho_{12}\mu_2-\rho_{13}\mu_1)Y_t^{13}}{\mu_1\mu_3 S_t\sigma(V_t)} - \frac{Y_t^{33}}{\mu_3 g(r_t)} \end{pmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \\ dW_t^3 \end{pmatrix}. \end{aligned} \tag{5.78}$$

We observe that the derivative of the option price is denoted by

$$\frac{\partial}{\partial x}u(x) = \left(\frac{\partial u}{\partial S_0}, \frac{\partial u}{\partial V_0}, \frac{\partial u}{\partial r_0}\right)^T$$

where T denote the transpose. We note that the first row of the solution from matrix (5.78) correspond to Delta which the change in option price with respect to the initial price S_0. The second row correspond to Vega ($\mathcal{V}$) which is the change in option price with respect to the initial volatility V_0. The last row correspond to Rho (ρ) which is the change in option price with respect to the initial interest rate r_0. Hence the Malliavin weight of Delta is given by

$$\pi^{\Delta} = \frac{1}{T}\left(\int_0^T \frac{Y_t^{11}}{S_t\sigma(V_t)}dW_t^1 - \int_0^T \frac{\rho_{12}Y_t^{11}}{\mu_1 S_t\sigma(V_t)}dW_t^2 + \int_0^T + \frac{(\rho_{12}\mu_2 - \rho_{13}\mu_1)Y_t^{11}}{\mu_1\mu_3 S_t\sigma(V_t)}dW_t^3\right) \tag{5.79}$$

Since

$$Y_t^{11} = \frac{1}{S_0}S_t.$$

We obtain

$$\pi^{\Delta} = \frac{1}{S_0 T}\left(\int_0^T \frac{dW_t^1}{\sigma(V_t)} - \frac{\rho_{12}}{\mu_1}\int_0^T \frac{dW_t^2}{\sigma(V_t)} + \frac{(\rho_{12}\mu_2 - \rho_{13}\mu_1)}{\mu_1\mu_3}\int_0^T \frac{dW_t^3}{\sigma(V_t)}\right). \tag{5.80}$$

The Malliavin weight for Vega is given by

$$\begin{aligned}\pi^{\mathcal{V}} &= \int_0^T \frac{Y_t^{12}}{S_t\sigma(V_t)}dW_t^1 + \int_0^T \left(\frac{Y_t^{22}}{\mu_1 v(V_t)} - \frac{\rho_{12}Y_t^{12}}{\mu_1 S_t\sigma(V_t)}\right)dW_t^2 \\ &\quad + \int_0^T \left(\frac{(\rho_{12}\mu_2 - \rho_{13}\mu_1)Y_t^{12}}{\mu_1\mu_3 S_t\sigma(V_t)} - \frac{\mu_2 Y_t^{22}}{\mu_1\mu_3 v(V_t)}\right)dW_t^3\end{aligned} \tag{5.81}$$

and the Malliavin weight for Rho is given by

$$\begin{aligned}\pi^{\rho} &= \int_0^T \frac{Y_t^{13}}{S_t\sigma(V_t)}dW_t^1 - \int_0^T \frac{\rho_{12}Y_t^{13}}{\mu_1 S_t\sigma(V_t)}dW_t^2 \\ &\quad + \int_0^T \left(\frac{(\rho_{12}\mu_2 - \rho_{13}\mu_1)Y_t^{13}}{\mu_1\mu_3 S_t\sigma(V_t)} - \frac{Y_t^{33}}{\mu_3 g(r_t)}\right)dW_t^3.\end{aligned} \tag{5.82}$$

The computation of Rho (ρ) and Vega ($\mathcal{V}$) is not straight forward as in the computation of Delta (Δ) since the drift and the diffusion coefficients are not constants. Vega and Rho quantify the impact of small perturbation as in Proposition 5.1.3, (see [5] thereof). It can be again seen that all the Malliavin wight functions $\pi^{\Delta}, \pi^{\mathcal{V}}$ and π^{ρ} does not depend on the payoff function Φ but on the deterministic function $a(\cdot)$.

5.2.1 The independent case

We consider the follwing system

$$\begin{cases} dS_t = r_t S_t dt + S_t\sigma(V_t)dW_t^1 \\ dV_t = u(V_t)dt + v(V_t)dW_t^2 \\ dr_t = f(r_t)dt + g(r_t)dW_t^3. \end{cases} \tag{5.83}$$

which can be represented in a matrix form as

$$d\begin{pmatrix} S_t \\ V_t \\ r_t \end{pmatrix} = \begin{pmatrix} r_t S_t \\ u(V_t) \\ f(r_t) \end{pmatrix} dt + \begin{pmatrix} S_t\sigma(V_t) & 0 & 0 \\ 0 & v(V_t) & 0 \\ 0 & & g(r_t) \end{pmatrix} d\begin{pmatrix} W_t^1 \\ W_t^2 \\ W_t^3 \end{pmatrix} \tag{5.84}$$

where $\{W_t^i : 0 \leq t \leq T\}$ for $i = 1, 2, 3$ are uncorrelated independent Brownian motions. If we let

$$X_t = \begin{pmatrix} S_t \\ V_t \\ r_t \end{pmatrix}, \quad A(X_t) = \begin{pmatrix} r_t S_t \\ u(V_t) \\ f(r_t) \end{pmatrix},$$

and

$$B(X_t) = \begin{pmatrix} S_t\sigma(V_t) & 0 & 0 \\ 0 & v(V_t) & 0 \\ 0 & 0 & g(r_t) \end{pmatrix} \tag{5.85}$$

and

$$x = \begin{pmatrix} S_0 \\ V_0 \\ r_0 \end{pmatrix}, \qquad W_t = \begin{pmatrix} W_t^1 \\ W_t^2 \\ W_t^3 \end{pmatrix}$$

where the process $\{W_t : 0 \leq t \leq T\}$ is a standard three-dimensional Brownian motion, then we obtain a stochastic differential equation given by

$$dX_t = A(X_t)dt + B(X_t)dW_t, \qquad X_0 = x \in \mathbb{R}. \tag{5.86}$$

We assume that the drift coefficient $A(\cdot)$ and the diffusion coefficient $B(\cdot)$ are bounded with partial derivatives and satisfy linear growth and the Lipschitz conditions. From the general formula from Theorem 5.0.3, since in this case our matrix $B(X_t)$ from (5.85) is a strictly diagonal matrix, its inverse is given by

$$B^{-1}(X_t) = \begin{pmatrix} \frac{1}{S_t\sigma(V_t)} & 0 & 0 \\ 0 & \frac{1}{v(V_t)} & 0 \\ 0 & 0 & \frac{1}{g(r_t)} \end{pmatrix} \tag{5.87}$$

The first variation process Y_t for Delta is given by

$$Y_t = \begin{pmatrix} \frac{S_t}{S_0} \\ 0 \\ 0 \end{pmatrix}$$

Thus

$$\begin{aligned} (B^{-1}(X_t)Y_t)^T &= Y_t^T (B^{-1}(X_t))^T \\ &= \begin{pmatrix} \frac{S_t}{S_0} & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{S_t\sigma(V_t)} & 0 & 0 \\ 0 & \frac{1}{v(V_t)} & 0 \\ 0 & 0 & \frac{1}{g(r_t)} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{S_0\sigma(V_t)} & 0 & 0 \end{pmatrix}. \end{aligned} \tag{5.88}$$

where T denote the transpose. If we multiply (5.88) by a 3-dimensional Browinan motion column matrix we obtain

$$\begin{aligned}(B^{-1}(X_t)Y_t)^T dW_t &= \begin{pmatrix}\frac{1}{S_0\sigma(V_t)} & 0 & 0\end{pmatrix}\begin{pmatrix}dW_t^1\\ dW_t^2\\ dW_t^3\end{pmatrix}.\\ &= \frac{1}{S_0\sigma(V_t)}dW_t^1. \qquad (5.89)\end{aligned}$$

If we let the deterministic function $a(t) = \frac{1}{T}$. Then the Malliavin weight function for Delta (Δ) is given by

$$\pi^{\Delta} = \int_0^T a(t)(B^{-1}(X_t)Y_t)^T dW_t = \frac{1}{TS_0}\int_0^T \frac{1}{\sigma(V_t)}dW_t^1.$$

For the computation of Vega, the first variational process Y_t is given by

$$Y_t = Y_t = \begin{pmatrix}\frac{S_t}{S_0}\\ \frac{V_t}{V_0}\\ 0\end{pmatrix}. \qquad (5.90)$$

Similarly to Delta, we have

$$\begin{aligned}(B^{-1}(X_t)Y_t)^T dW_t &= Y_t^T(B^{-1}(X_t))^T dW_t\\ &= \begin{pmatrix}\frac{S_t}{S_0} & \frac{V_t}{V_0} & 0\end{pmatrix}\begin{pmatrix}\frac{1}{S_t\sigma(V_t)} & 0 & 0\\ 0 & \frac{1}{v(V_t)} & 0\\ 0 & 0 & \frac{1}{g(r_t)}\end{pmatrix}\begin{pmatrix}dW_t^1\\ dW_t^2\\ dW_t^3\end{pmatrix}\\ &= \begin{pmatrix}\frac{1}{S_0\sigma(V_t)} & \frac{V_t}{V_0 v(V_t)} & 0\end{pmatrix}\begin{pmatrix}dW_t^1\\ dW_t^2\\ dW_t^3\end{pmatrix}\\ &= \frac{1}{S_0\sigma(V_t)}dW_t^1 + \frac{V_t}{V_0 v(V_t)}dW_t^2. \qquad (5.91)\end{aligned}$$

If we again let a deterministic function $a(t) = \frac{1}{T}$, then the Maliavin weight function for Vega ($\mathcal{V}$) is given by

$$\pi^{\mathcal{V}} = \int_0^T a(t)(B^{-1}(X_t)Y_t)^T dW_t = \frac{1}{TS_0}\int_0^T \frac{1}{\sigma(V_t)}dW_t^1 + \frac{1}{TV_0}\int_0^T \frac{V_t}{v(V_t)}dW_t^2.$$

Lastly for the computation of Rho, the first variational process Y_t is given by

$$Y_t = \frac{\partial}{\partial r_0}S_t = \begin{pmatrix}\frac{S_t}{S_0}\\ 0\\ \frac{r_t}{r_0}\end{pmatrix} \qquad (5.92)$$

. In the very same way, we have

$$
\begin{aligned}
(B^{-1}(X_t)Y_t)^T dW_t &= Y_t^T (B^{-1}(X_t))^T dW_t \\
&= \begin{pmatrix} \frac{S_t}{S_0} & 0 & \frac{r_t}{r_0} \end{pmatrix} \begin{pmatrix} \frac{1}{S_t\sigma(V_t)} & 0 & 0 \\ 0 & \frac{1}{v(V_t)} & 0 \\ 0 & 0 & \frac{1}{g(r_t)} \end{pmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \\ dW_t^3 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{S_0\sigma(V_t)} & 0 & \frac{r_t}{r_0 g(r_t)} \end{pmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \\ dW_t^3 \end{pmatrix} \\
&= \frac{1}{S_0\sigma(V_t)} dW_t^1 + \frac{r_t}{r_0 g(r_t)} dW_t^3. \qquad (5.93)
\end{aligned}
$$

If we again let a deterministic function $a(t) = \frac{1}{T}$, then the Maliavin weight function for Rho (ρ) is given by

$$\pi^\rho = \int_0^T a(t)(B^{-1}(X_t)Y_t)^T dW_t = \frac{1}{TS_0}\int_0^T \frac{1}{\sigma(V_t)} dW_t^1 + \frac{1}{Tr_0}\int_0^T \frac{r_t}{g(r_t)} dW_t^3.$$

We note that in all the cases where we computed the price sensitivities, there are no direct computations of the derivative of the payoff functions. All the results we got indicates that the efficiency of the Malliavin calculus in the computation of price sensitivities does not depend on the type/nature of the payoff function. By making use of the integration by parts formula, we have indeed seen that a Greek can be represented as the expectation of the product of the payoff function and the Malliavin weight function.

Chapter 6

Conclusion

This study was based on the computation of the hedging portfolios and the price sensitivities, known as Greeks, in the case where we have the discontinuous payoff functions using the Malliavin calculus approach. We introduced the importance and the background of Malliavin calculus and also what have been done before. We developed the Wiener's construction of Brownian motion and the stochastic integral. We discussed some important properties of Malliavin calculus which includes essential tools such as the integration by parts formula. This formula avoid the direct derivation of the functional, instead result in the product of the functional and the so called Malliavin weight function. The integration by parts formula plays a huge role in the computation of the price sensitivities. We only restricted ourselves to one dimensional case. The Clark-Ocone formula is used for the computation of hedging portfolios. We showed the Malliavin derivative of stochastic differential equation where the focus is on the diffusion process. As a result, we constructed the first variational process which is the partial derivative of the stochastic differential equation with respect to the initial condition.

For the application of the Malliavin calculus to mathematical finance, we used the Clark-Ocone formula to obtain the general representation formula of the replicating strategy. The general formula was applied to different types of payoff functions of the European type where we realised that the hedging portfolio is naturally related to the Malliavin derivative of the terminal payoff. In addition, we computed the price sensitivities in the Malliavin sense. The Malliavin calculus properties are used to compute the general representation formula of price sensitivities which include the Delta (Δ), Gamma (Γ) and Vega ($\mathcal{V}$). We considered the geometric Brownian motion case as an example.

Further we computed the price sensitivities with respect to the correlation in a stochastic volatility model. We considered the 2-dimensional correlated Brownian motion where we computed the sensitivity of the option with respect to ρ. We considered also a 3-dimensional Brownian motion and compute the price sensitivity which includes the Delta, Vega and Rho. We conclude our study by considering a 3-dimensional uncorrelated Brownian motion and compute again the Delta, Vega and Rho of the option price. We hope to extend the diffusion case to processes that include jumps. We would also like to apply similar concepts to option price of American type where the exercise of the option take place on or before the maturity time T.

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