APPLICATION OF ADOMIAN DECOMPOSITION METHOD TO SOLVING NONLINEAR DIFFERENTIAL EQUATIONS

by

Nkhoreng Hazel Sekgothe

DISSERTATION

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Abstract

Modelling with differential equations is of paramount importance as it provides pertinent insight into the dynamics of many engineering and technological devices and/or processes. Many such models, however, involve differential equations that are inherently nonlinear and difficult to solve. Many numerical methods have been developed to solve a variety of differential equations that cannot be solved analytically. Most numerical methods, however, require discretisation, linearisation of the nonlinear terms and other simplifying approximations that may inhibit the accuracy of the solution. Further, in some methods high computational complexity is involved. Due to the importance of differential equations in modelling real life phenomena and these stated shortfalls, continuous pursuit of more efficient solution techniques by the scientific community is ongoing. Industrial and technological advancement are to a larger extent dependent upon efficient and accurate solution techniques.

In this work, we investigate the use of Adomian decomposition method in solving nonlinear ordinary and partial differential equations. One advantage of Adomian decomposition method that has been demonstrated in literature is that it achieves a rapidly convergent infinite series solution. The method is also advantageous in that it does not require one to linearise and discretise the equations as is done with other numerical methods. In our investigation, among other important examples, we will apply the Adomian decomposition method to solve selected fluid flow and heat transfer problems. Fluid flow and heat transfer models have pertinent applications in engineering and technology. The Adomian decomposition method will be compared with other series solution methods, namely the differential transform method and the homotopy analysis method. The desirable attributes of the Adomian decomposition method that are stated in literature have been ascertained in this work and it has also been demonstrated that the Adomian decomposition method compares favourably with the other series solution methods. It has also been demonstrated that in some cases nonlinear complexity results in slow convergence rate of the Adomian decomposition method.

Declaration

I, **Hazel Sekgothe**, declare that this research is the result of my own work that Dr L Rundora was guiding me to do, except for the materials referred to as cited in the reference section and has not been submitted in any other institution or university for the award of any other degree or diploma.

Signature:.......*NHS*......Date:...22 January 2021.....

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Dedication

To my beloved parents (Motubatse Abram Sekgothe and Maatlou Dinah Sekgothe) and Family

Contents

List of Figures

List of Tables

Chapter 1

Introduction

In this chapter an introduction to the research problem is provided. The aim, objectives of the study and the significance of the study are also stated. The chapter concludes by giving an outline of the dissertation.

1.1   Background of study

Quantitative descriptions for many models in physical, biological, and even social sciences are provided through the use of differential equations. The descriptions are normally in terms of unknown functions of one, two or more independent variables, and the relation between derivatives with respect to those variables. If two or more independent variables are involved, the differential equation is a partial differential equation (PDE). Otherwise, it is an ordinary differential equation (ODE). If the relations between the unknown functions and their derivatives involved in the equation are nonlinear, the ODE or PDE is said to be nonlinear. Nonlinear differential equations are difficult to handle primarily because of difficulties in answering questions to do with uniqueness and stability of solutions. The breakdown of the superposition principle due to the nonlinear terms means that only numerical solutions can be established.

The rapid advancement of technology in the contemporary era has seen a growing need for scientific computation in handling and analysing big data manifesting from a plethora of real life modelling phenomena. Numerical methods for solving nonlinear ODEs and PDEs are at the heart of many of these scientific computations [1]. Differential equations have become a useful tool for describing these natural phenomena of science and engineering models. Therefore, it becomes important to be familiar with all traditional and recently developed methods for solving differential equations, and the implementation of these methods. Although many standard methods for solving differential equations exist, more efficient methods still need to be developed or investigated [2]. Adomian decomposition method (ADM) is one of the methods for solving differential equations. With ADM, nonlinear problems can be solved elegantly without resorting to linearised approximations of the equations. Another important advantage of ADM is that it is capable of greatly reducing the size of computational work while still maintaining high accuracy of the numerical solution [3].

1.2   Problem statement

The inability of analytical techniques to solve most nonlinear differential equations has led to the tremendous growth in research in numerical methods. Many researchers have solved important nonlinear models (differential equation models) using numerical techniques. However, further refinement of solution techniques to achieve computational efficiency is still an on-going process in the scientific community. Traditional perturbation methods provide the most versatile tools for tackling nonlinear problems but their dependence on small parameters bears a negative limitation on practicality. Several solution techniques that have been developed for solving nonlinear differential equation models include Runge-Kutta integration schemes, finite difference schemes, finite element methods, variational iteration, the standard homotopy method, ADM, the homotopy perturbation method (HPM) and the differential transform method (DTM). Although, like the other methods, the ADM does not assure on its own the existence and uniqueness of the solution, it has proved to be a powerful semi-analytic approximation technique that achieves rapidly convergent series with less volume of computational work [4]. It achieves solutions without any discretization, linearization or restrictive assumptions and is mostly free from round-off errors. ADM has thus proved to be a pertinent tool for untangling complex nonlinear problems. Of particular interest in this study are nonlinear ODEs and PDEs arising from fluid flow phenomena. Physical phenomena are often dominated by fluids, and as such, life would not exist without fluids and without the behaviour that fluids exhibit. Fluid flow in varied geometries find applications in many engineering processes and devices. Solutions to such fluid flow models contribute to the improvement in technology that leads to the general advancement in the quality of life.

This work seeks to provide a detailed description of ADM, apply it to solve some nonlinear ordinary and partial differential equations and to compare with other series solution methods like the homotopy analysis method (HAM) and DTM in solving nonlinear differential equations, and ultimately culminating in solving contemporary fluid flow problems.

1.3   Aim of the Study

The aim of this study is to investigate the application of ADM to solving some nonlinear differential equations and to compare it with other methods like the HAM, DTM and perturbation methods.

1.4   Objectives of the study

The objectives of the study are to:

(i) Review some numerical methods for solving nonlinear differential equations.

(ii) Describe the ADM.

(iii) Apply the ADM to solve nonlinear ODEs and PDEs.

(iv) Apply ADM to solve contemporary fluid flow problems.

(v) Compare ADM with other methods like the HAM, DTM and perturbation methods.

1.5 Significance of the study

Mathematical modelling, in particular modelling with differential equations, is a pertinent tool for engineering solutions. Mathematical modelling thus contributes significantly to advancement of technology and human civilisation. Any engineering system can be studied either experimentally (testing and taking measurements) or theoretically (by analysis or calculations). The experimental approach has the advantage that it deals with the actual physical system, and the desired quantity is determined by measurement, within the limits of experimental error. However this approach is expensive, time consuming and often impractical. The theoretical approach (including the numerical approach) has the advantage that it is fast and inexpensive, but the results obtained are subjected to the accuracy of the assumption, approximations, and idealisations made in the analysis. Theoretically, differential equations are used to investigate a wide variety of problems in science and engineering. Modelling with differential equations and solving the equations give rise to solutions that can be interpreted to provide answers to the real life problems concerned. Since most of the models inherently give rise to nonlinear differential equations, it becomes pertinent to have a working knowledge of numerical methods. Numerical methods may generally be expensive in terms of computational time and complexity. Some methods may simply fail to converge, while some may require some discretization, linearisation and restrictive simplifying assumptions that significantly reduce the accuracy of the solution. In this project, ADM is chosen because it is a direct semi-analytic approach that provides solution in the form of a fast converging series. The investigations in the application of ADM and the other methods in solving nonlinear differential equations will contribute to the body of knowledge in applied mathematics in general and numerical methods in particular.

1.6 Structure of the dissertation

Chapter 1

Chapter one contains the introduction, the aim and objectives of the study and the significance of this study.

Chapter 2

This chapter contains the literature review.

Chapter 3

Perturbation methods, DTM and HAM are fully described and applied to solve selected examples to show the effectiveness of the methods. The first objective is achieved in this chapter.

Chapter 4

In chapter 4, a detailed description of the ADM is presented and the method is applied to solve the examples that were solved in chapter 3 using DTM and HAM. The second, third and fifth objectives are achieved in this chapter.

Chapter 5

In chapter 5, the DTM, ADM and HAM are applied to solve selected fluid flow and heat transfer problems, and the results are compared. The fourth and fifth objectives are achieved in this chapter.

Chapter 6

This chapter concludes the dissertation with a general discussion, conclusions, recommendations, limitations of the study and possible future research work.

Chapter 2

Literature Review

In this chapter we provide a succinct literature review of some of the work that has been done by several researchers in the methods that are covered in the scope of this work.

2.1 Literature Review

Differential equations arising from fluid flow problems are mostly nonlinear and complex, and inherent varied flow geometries add to the complexity maze. Yet successful traction provides a rewarding fulfilment as the solutions help unlock avenues to advancement in technology and modernity. Owing to the ubiquitous nature of fluid mechanics, applications are diverse [5]. They include biological flow systems such as the flow of blood in warm bodies, fluid flow models in nanotechnology, geothermal engineering, thermal insulation of buildings, chemical catalytic reactors, petroleum reservoirs, direct contact heat exchangers, nuclear waste repositories, and heat pipe technology. For instance, blood flow constitutive models coupled with nanotechnology advancement has led to revolutionising the medical industry, thus contributing to longevity [6].

It thus remains relevant to upscale the pursuit of solution techniques to nonlinear differential equations. Makinde and Rundora [7] used a semi-discretisation finite difference scheme to investigate unsteady mixed convection flow of a reactive Casson fluid in a permeable wall channel filled with a porous medium. Rundora and Makinde [8] used the same method to solve nonlinear partial differential equations modelling buoyancy effects on unsteady reactive variable properties fluid flow in a channel filled with a porous medium. Adesanya *et al.* [9] used the ADM to investigate natural convection flow of heat generating hydromagnetic couple stress fluid with time periodic boundary conditions. Biazar [10] used the ADM to solve systems of differential equations by converting the system into a system of ordinary differential equations, and it was demonstrated that the ADM has the ability of solving systems of both linear and nonlinear differential equations. Tomaizeh [11] presented some modifications of the ADM and demonstrated that the ADM solves the problem without using linearization, perturbation or any other preferable assumptions that may change the size of the computational work while still maintaining high accuracy of the numerical solution.

Jebari *et al.* [12] used the ADM to solve a nonlinear equation with exponential nonlinearity and showed that the ADM is a very powerful and efficient method for solving different kinds of problems arising in various fields of science and engineering and present a rapid convergence of the solution. Odibat [13] proposed the optimized decomposition method to design a new optimal construction of the series solution based on a linear approximation of the nonlinear equation. They developed an efficient adaptation of the optimized decomposition method to expand the application of the method to nonlinear PDEs. A comparison between the suggested method and

the Adomian decomposition method was made which is carried out through numerical simulation of some test problems. Shah *et al.* [14] related the analytical solution of the fractional order dispersive partial differential equations, using the Laplace-Adomian decomposition method. They used Caputo operator to define the derivative of fractional-order. Laplace-Adomian decomposition method solutions for both fractional and integer orders are obtained in series form, showing higher convergence of the method they proposed. Li and Pang [15] studied the ADM including its iterative scheme and convergence analysis. The algebraic equations and fractional differential equations were taken as applications to illustrate ADM's efficiency.

The differential transform method (DTM) was first proposed by Zhou [16] in solving linear and nonlinear initial value problems in electrical circuit analysis. Since then, several researchers applied the method to different types of equations. For example, Chen *et al.* [17] applied DTM to solve second order eigenvalue problems and the transverse vibration of a twisted beam under axial loading. A semi-analytical solution in the form of a polynomial was obtained. Gepreel *et al.* [18] studied the approximate solutions for some of nonlinear Biomathematics models via the e-epidemic SI1I2R model characterizing the spread of viruses in a computer network and SIR childhood disease model. They applied the reduced differential transform method (RDTM) to discuss the analytic approximate solutions to the SI1I2R model for the spread of virus HCV-subtype and SIR childhood disease model. Kharrat and Toma [19] presented a general framework of the DTM for solving strongly nonlinear initial value problems represented by ordinary differential equations. They found that DTM does not require any discretization, linearization or small perturbation and therefore it reduces significantly the numerical computation.

Liao [20] and Abbasbandy *et al.* [21] developed the HAM and proved its convergence. Since then, many researchers have used this method to solve certain classes of differential equations. Abbasbandy [22] compared HAM and HPM in the heat transfer field and found that HAM provides a convenient way of controling the convergence of the approximation series because it contains the auxiliary parameter $\hbar$, which is a fundamental qualitative difference in analysis between HAM and other methods. Tan *et al.* [23] solved the quadratic Riccati differential equation using HAM and compared the method with ADM , HPM and the exact solution. The authors discovered that HPM and ADM are special cases of HAM. The auxiliary parameter $\hbar$ in HAM provided them with a convenient way to control and adjust the convergence region. This is one pertinent advantage of HAM. Hashim *et al.* [24] applied HAM to derive exact and approximate analytical solutions of both linear and nonlinear fractional initial value problems. The authors demonstrated that ADM is a special case of HAM when they did some applications. Abbasbandy [25] applied HAM to solve the generalised Hirota-Satsuma coupled KdV equation and compared the results with those of ADM and HPM. It was also found that HAM provides a convenient way to control the convergence of approximation series by adapting $\hbar$. Hamrelaine *et al.* [26] solved the 3rd-order magnetohydrodynamic Jeffery-Hamel flow with injection or suction using HAM. They compared HAM and Runge-Kutta method of order 4 using Mathematica software. Ghiasi [27] used HAM to study Sakiadis flow of thixotropic fluid with the inclined Lorentz force and Newtonian heating.

DTM, ADM and HAM are all series solution methods that differ from other numerical methods in that they are direct without requiring linearisation approximation and other simplifying assumptions that normally affect accuracy and efficiency. As related methods, it is interesting to compare ADM, DTM and HAM in order to determine if there is significant difference in the

accuracy and the efficiency of the three methods.

Chapter 3

Review of some numerical methods

In this chapter, the perturbation methods, the DTM and the HAM are described and applied to solve some selected examples.

3.1 Perturbation methods

3.1.1 Perturbation theory. Perturbation theory is commonly used to investigate the physical systems that can be solved exactly but contain small perturbation parameters [28]. When solving such systems by using perturbation theory, expansion around the perturbation parameter is involved and approximates are expressed as power series of these parameters. There are two types of perturbation methods, namely, regular perturbation method and singular perturbation method. A regular perturbation method is one for which the perturbed problem for small, nonzero values of ε is qualitatively the same as the unperturbed problem for $\varepsilon = 0$ and a singular perturbation method is one for which the perturbed problem is qualitatively different from the unperturbed problem [29].

3.1.2 Asymptotic expansion of perturbation method. Following the presentation in [29], consider a nonlinear differential equation of the form

$$L(x) + \epsilon N(x) = 0 \tag{3.1.1}$$

where x is assumed to be a function of one variable $x = x(t), L(x)$ is a linear operator which contains derivatives in terms of t, $N(x)$ is a nonlinear operator and ϵ is a small parameter. Considering the nonlinear term in (3.1.1) to be a small perturbation and assuming that its solution can be written as a power series in the small parameter ϵ, we write

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots . \tag{3.1.2}$$

Substituting (3.1.2) into (3.1.1) and equating terms having identical powers of ϵ, we obtain a number of differential equations that can be integrated recursively to find the values for the functions

$$x_0(t), x_1(t), x_2(t), \cdots$$

3.1.3 Solving nonlinear ordinary differential equations using the perturbation methods..

Example 3.1.4. Obtain a two-term expansion for the solution of the initial value problem

$$\frac{dy}{dx} - y = \epsilon y^2 e^{-x}, \qquad y(0) = 1. \tag{3.1.3}$$

Solution 3.1.4.1. The solution $y(x; \epsilon)$ depends on the independent variable x and the parameter ϵ. We will assume that the parameter ϵ dependence is of a standard power series form

$$y(x; \epsilon) = y_0 + \epsilon y_1(x),$$

where the coefficient functions $y_0(x)$ and $y_1(x)$ are independent of ϵ. Substituting the expansion into the differential equation leads to

$$\frac{dy_0}{dx} + \epsilon \frac{dy_1}{dx} - y_0 - \epsilon y_1 = \epsilon (y_0^2 + 2\epsilon y_0 y_1 + \epsilon^2 y_1^2) e^{-x}.$$

Equating coefficients of powers of ϵ on the left-hand and right-hand sides of the equation leads to the set of equations

$$0(\epsilon^0): \frac{dy_0}{dx} - y_0 = 0, \qquad y_0(0) = 1$$
$$, 0(\epsilon^1): \frac{dy_1}{dx} - y_1 = \epsilon(y_0^2) e^{-x}, \qquad y_1(0) = 0.$$

Solving the obtained differential equations we have

$$\frac{dy_0}{dx} - y_0 = 0,$$
$$I = e^{\int -1 dx} = e^{-x},$$
$$e^{-x}\left(\frac{dy_0}{dx} - y_0\right) = 0,$$

9

$$e^{-x}\frac{dy_0}{dx} - e^{-x}y_0 = 0,$$

$$\frac{d}{dx}(e^{-x}y_0) = 0,$$

$$\int \frac{d}{dx}(e^{-x}y_0) = \int 0dx,$$

$$\implies e^{-x}y_0 = K,$$

$$\implies y_0 = Ke^x.$$

Using the initial condition we find that $K = 1$. Thus we have

$$y_0 = e^x.$$

Solving the second obtained differential equation

$$\frac{dy_1}{dx} - y_1 = (y_0^2)e^{-x}, \qquad y_1(0) = 0,$$

we have

$$\frac{dy_1}{dx} - y_1 = (e^x)^2 e^{-x},$$

$$\frac{dy_1}{dx} - y_1 = e^x,$$

$$I = e^{\int -1dx} = e^{-x},$$

$$e^{-x}(\frac{dy_1}{dx} - y_1) = e^{-x}(e^x),$$

$$\implies e^{-x}\frac{dy_1}{dx} - e^{-x}y_1 = 1,$$

$$\frac{d}{dx}(e^{-x}y_1) = 1,$$

$$\int \frac{d}{dx}(e^{-x}y_1) = \int 1dx,$$

$$\implies e^{-x}y_1 = x + D,$$

$$\implies y_1 = (x + D)e^x.$$

By the initial condition we find that $D = 0$. Then we have

$$y_1 = xe^x.$$

Thus the two-term expansion for the solution of (3.1.3) is

$$y(x) = y_0 + \epsilon y_1$$
$$= e^x + \epsilon x e^x.$$

Example 3.1.5. Obtain a two-term expansion for the solution of

$$\frac{d^2y}{dx^2} + y = \epsilon \frac{dy}{dx}, \quad y(0) = 1, \quad \frac{dy}{dx}(0) = 0.$$

Solution 3.1.5.1. The solution $y(x; \epsilon)$ depends on the independent variable x and the parameter ϵ. We will assume that the parameter ϵ dependence is of a standard power series form

$$y(x; \epsilon) = y_0 + \epsilon y_1(x),$$

where the coefficient functions $y_0(x)$ and $y_1(x)$ are independent of ϵ. Substituting the expansion into the differential equation leads to

$$\frac{d^2y_0}{dx^2} + \epsilon \frac{d^2y_1}{dx^2} + y_0 + \epsilon y_1 = \epsilon \frac{dy_0}{dx} + \epsilon^2 \frac{dy_1}{dx}$$

Equating coefficients of powers of ϵ on the left and right hand sides of the equation leads to the set of equations

$$0(\epsilon^0) : \frac{d^2y_0}{dx^2} + y_0 = 0, \quad y_0(0) = 1, \quad \frac{dy_0}{dx}(0) = 0,$$
$$0(\epsilon^1) : \frac{d^2y_1}{dx^2} + y_1 = \frac{dy_0}{dx}, \quad y_1(0) = 0, \quad \frac{dy_1}{dx}(0) = 0.$$

11

Solving these differential equations we have

$$\frac{d^2 y_0}{dx^2} + y_0 = 0,$$

$$m^2 + 1 = 0,$$

$$\implies m = \pm i$$

The general solution of the $0(\epsilon^0)$ equation is thus

$$y_0 = A\cos x + B\sin x,$$

and the initial conditions yield $A = 1$ and $B = 0$. Therefore

$$y_0 = \cos x.$$

The $0(\epsilon^1)$ equation is

$$\frac{d^2 y_1}{dx^2} + y_1 = -\sin x$$

Solving this differential equation using the method of undetermined coefficients , we first solve the related homogeneous equation

$$\frac{d^2 y_1}{dx^2} + y_1 = 0.$$

to find y_h. As in the previous equation, we have

$$y_{1h} = A\sin x + B\cos x.$$

For the particular solution we have

$$y_{1p} = Ax \sin x + Bx \cos x$$

$$y_{1p}' = A \sin x + Ax \cos x + B \cos x - Bx \sin x$$

$$y_{1p}'' = 2A \cos x - Ax \sin x - 2B \sin x - Bx \cos x$$

Now,

$$2A \cos x - Ax \sin x - 2B \sin x - Bx \cos x + Ax \sin x + Bx \cos x = -\sin x$$

$$\implies 2A \cos x - Bx \sin x = -\sin x$$

$$\implies A = 0, \quad , -2B = -1, \quad B = \frac{1}{2}$$

Therefore

$$y_{1p} = \frac{x}{2} \cos x.$$

Thus the two-term expansion for the solution is

$$y(x) = y_0 + \epsilon y_1.$$

$$= cosx + \epsilon \frac{x}{2} cosx$$

$$= cosx \left(1 + \epsilon \frac{x}{2} \right).$$

3.2   Differential transform method

For illustration of DTM, we follow [16,17]. We first consider $u(x, y)$ which is analytic at (x, y) and differentiated continuously in the domain of interest to give

$$U_k(x) = \frac{1}{k!} \left(\frac{\partial^k u(x, y)}{\partial y^k} \right), \tag{3.2.1}$$

13

where $U_k(x)$ is the transformed function, which is called the T-function in brief. The differential transform of $U_k(x)$ is defined as

$$u(x, y) = \sum_{k=0}^{\infty} U_k(x)(t - t_0)^k. \tag{3.2.2}$$

Combining (3.2.1) and (3.2.2), we obtain

$$u(x, y) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\partial^k u(x, y)}{\partial y^k} \right)_{t=0} (t - t_0)^k. \tag{3.2.3}$$

When $t_0 = 0$, (3.2.3) reduces to

$$u(x, y) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\partial^k u(x, y)}{\partial y^k} \right)_{t=0} t^k$$

and (3.2.2) is expressed as

$$u(x, y) = \sum_{k=0}^{\infty} U_k(x) t^k. \tag{3.2.4}$$

In real application, the infinite series in (3.2.4) is truncated at $k = n$ so that $u(x, y)$ is approximated by the finite series

$$u(x, y) = \sum_{k=0}^{n} U_k(x) t^k.$$

Usually, the values of n are decided by convergence of the series coefficients. The theorems that can be deduced from (3.2.1) and (3.2.4) follow here under.

Theorem 3.2.1. *If the original function is*

$$u(x, y) = w(x, y) \pm v(x, y),$$

then the transformed function is

$$U_k(x) = W(x, y) \pm V(x, y).$$

Theorem 3.2.2. *If the original function is*

$$u(x, y) = \beta v(x, y),$$

then the transformed function is

$$U_k(x) = \beta V(x, y).$$

14

Theorem 3.2.3. *If the original function is*

$$u(x, y) = \frac{\partial^m w(x, y)}{\partial y^m},$$

then the transformed function is

$$U_k(x) = \frac{(k+m)!}{k!} W_k(x).$$

Theorem 3.2.4. *If the original function is*

$$u(x, y) = \frac{\partial w(x, y)}{\partial y},$$

then the transformed function is

$$U_k(x) = \frac{\partial W_k(x)}{\partial x}.$$

Theorem 3.2.5. *If the original function is*

$$u(x, y, t) = \frac{\partial w(x, y, t)}{\partial y},$$

then the transformed function is

$$U_k(x, y) = \frac{\partial W_k(x, y)}{\partial x}.$$

Theorem 3.2.6. *If the original function is*

$$u(x, y, t) = g(x, y, t)h(x, y, t),$$

then the transformed function is

$$U_k(x, y, t) = \sum_{r=0}^{k} G(r)H(k - r).$$

3.3 Solving nonlinear ordinary differential equations using the differential transform method.

Example 3.3.1. Solve the first order nonlinear differential equation

$$y' + y - y^2 = 0, \tag{3.3.1}$$

with the initial condition

$$y(0) = 2$$

using DTM. The exact solution for this problem is $y(x) = \frac{-2}{-2+e^x}$.

Solution 3.3.1.1. Taking the differential transform of Equation (3.3.1) and the initial condition respectively, we obtain

$$(k+1)Y(k+1) + Y(k) - \sum_{r=0}^{k} Y(r)Y(k-r) = 0,$$

$$(k+1)Y(k+1) = -Y(k) + \sum_{r=0}^{k} Y(r)Y(k-r).$$

Using the initial condition, we have

$$Y(0) = 2$$

$$If \quad k = 0, \quad Y(1) = -Y(0) + \sum_{r=0}^{0} Y(r)Y(0-r)$$

$$= -Y(0) + Y(0)Y(0)$$

$$= -2 + 4$$

$$= 2.$$

$$If \quad k = 1, \quad 2Y(2) = -Y(1) + \sum_{r=0}^{1} Y(r)Y(1-r)$$

$$= -Y(1) + Y(0)Y(1) + Y(1)Y(0)$$

$$= -2 + 4 + 4$$

$$= 6.$$

$$Y(2) = 3.$$

16

$$If \quad k = 2, \quad 3Y(3) = -Y(2) + \sum_{r=0}^{2} Y(r)Y(2-r)$$

$$= -Y(2) + Y(0)Y(2) + Y(1)Y(1) + Y(2)Y(0)$$

$$= -3 + 6 + 4 + 6$$

$$= 13.$$

$$Y(3) = \frac{13}{3}$$

$$If \quad k = 3, \quad 4Y(4) = -Y(3) + \sum_{r=0}^{3} Y(r)Y(3-r)$$

$$= -Y(3) + Y(0)Y(3) + Y(1)Y(2) + Y(2)Y(1) + Y(3)Y(0)$$

$$= -\frac{13}{3} + \frac{26}{3} + 6 + 6 + \frac{26}{3}$$

$$= 25.$$

$$Y(4) = \frac{25}{4},$$

and so on. Finally, the differential inverse transform of $Y_k(t)$ gives

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^k$$

$$= 2 + 2t + 3t^2 + \frac{13}{3}t^3 + \frac{25}{4}t^4 + \cdots$$

Example 3.3.2. Use DTM to solve the second order nonlinear ode

$$y'' + y^2 = 0 \tag{3.3.2}$$

with the initial conditions

$$y(0) = 1 \quad and \quad y'(0) = 0.$$

Solution 3.3.2.1. Taking the differential transform of (3.3.2) and the initial condition respec-

17

tively, we obtain the following

$$(k+1)(k+2)Y(k+2) + \sum_{r=0}^{k} Y(r)Y(k-r) = 0,$$

$$(k+1)Y(k+1) = -\sum_{r=0}^{k} Y(r)Y(k-r).$$

From the initial conditions, we have

$$Y(0) = 1,$$

$$Y(1) = 0.$$

$$If \quad k = 0, \quad 2Y(2) = -\sum_{r=0}^{0} Y(r)Y(0-r)$$

$$= -Y(0)Y(0)$$

$$= -1.$$

$$Y(2) = -\frac{1}{2}.$$

$$If \quad k = 1, \quad 6Y(3) = -\sum_{r=0}^{1} Y(r)Y(1-r)$$

$$= -Y(0)Y(1) - Y(1)Y(0)$$

$$= 0.$$

$$Y(3) = 0.$$

$$If \quad k = 2, \quad 12Y(4) = -\sum_{r=0}^{2} Y(r)Y(2-r)$$

$$= -Y(0)Y(2) - Y(1)Y(1) - Y(2)Y(0)$$

$$= \frac{1}{2} - 0 + \frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{2}.$$

$$Y(4) = \frac{1}{12}.$$

$$If \quad k = 3, \quad 20Y(5) = -\sum_{r=0}^{3} Y(r)Y(3-r)$$

$$= -Y(0)Y(3) - Y(1)Y(2) - Y(2)Y(1) - Y(3)Y(0)$$

$$= 0.$$

$$Y(5) = 0.$$

$$If \quad k = 4, \quad 30Y(6) = -\sum_{r=0}^{4} Y(r)Y(4-r)$$

$$= -Y(0)Y(4) - Y(1)Y(3) - Y(2)Y(2) - Y(3)Y(1) - Y(4)Y(0)$$

$$= -\frac{1}{12} - 0 + \frac{1}{4} - 0 - \frac{1}{12}$$

$$= -\frac{5}{12}.$$

$$Y(6) = -\frac{1}{72},$$

and so on. The differential inverse transform of $Y_k(t)$ gives

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^k$$
$$= 1 - \frac{1}{2}t^2 + \frac{1}{12}t^4 - \frac{1}{72}t^6 + \cdots$$

3.4 Solving nonlinear partial differential equations using the differential transform method.

Example 3.4.1. Solve the first order nonlinear partial differential equation

$$u_t + u_x + u^2 = 0, \tag{3.4.1}$$

with the initial condition

$$u(x, 0) = \frac{1}{2x},$$

using DTM.

Solution 3.4.1.1. Taking the differential transform of Equation (3.4.1) and the initial condition respectively, we obtain

$$(k+1)U(k+1) + \frac{\partial U(k)}{\partial x} + \sum_{r=0}^{k} U(r)U(k-r) = 0,$$
$$(k+1)U(k+1) = -\frac{\partial U(k)}{\partial x} - \sum_{r=0}^{k} U(r)U(k-r).$$

From the initial condition, we have

$$U(0) = \frac{1}{2x}.$$

$$If \quad k = 0, \quad U(1) = -\frac{\partial U(0)}{\partial x} - \sum_{r=0}^{k} U(r)U(0-r)$$

$$= -\frac{\partial U(0)}{\partial x} - U(0)U(0)$$

$$= \frac{1}{2x^2} - \frac{1}{4x^2}$$

$$U(1) = \frac{1}{4x^2}.$$

$$If \quad k = 1, \quad 2U(2) = -\frac{\partial U(0)}{\partial x} - \sum_{r=0}^{k} U(r)U(0-r)$$

$$= -\frac{\partial U(1)}{\partial x} - \sum_{r=0}^{k} U(r)U(1-r)$$

$$= -\frac{\partial U(1)}{\partial x} - U(0)U(1) - U(1)U(0)$$

$$= \frac{1}{2x^3} - \frac{1}{8x^3} - \frac{1}{8x^3}$$

$$= \frac{1}{4x^3}.$$

$$U(2) = \frac{1}{8x^3}.$$

$$If \quad k = 2, \quad 3U(3) = -\frac{\partial U(2)}{\partial x} - \sum_{r=0}^{k} U(r)U(2-r)$$

$$= -\frac{\partial U(2)}{\partial x} - U(0)U(2) - U(1)U(1) - U(2)U(0)$$

$$= \frac{3}{8x^4} - \frac{1}{64x^4} - \frac{1}{64x^4} - \frac{1}{64x^4}$$

$$= \frac{3}{16x^4}.$$

$$U(3) = \frac{1}{16x^4}.$$

$$If \quad k = 3, \quad 4U(4) = -\frac{\partial U(3)}{\partial x} - \sum_{r=0}^{k} U(r)U(3-r)$$

$$= -\frac{\partial U(3)}{\partial x} - U(0)U(3) - U(1)U(2) - U(2)U(1) - U(3)U(0)$$

$$= \frac{1}{4x^5} - \frac{1}{32x^5} - \frac{1}{32x^5} - \frac{1}{32x^5} - \frac{1}{32x^5}$$

$$= \frac{1}{8x^5}.$$

$$U(4) = \frac{1}{32x^5}.$$

$$If \quad k = 4, \quad 5U(5) = -\frac{\partial U(4)}{\partial x} - \sum_{r=0}^{k} U(r)U(4-r)$$

$$= -\frac{\partial U(4)}{\partial x} - U(0)U(4) - U(1)U(3) - U(2)U(2) - U(3)U(1) - U(4)U(0)$$

$$= \frac{5}{32x^6} - \frac{1}{64x^6} - \frac{1}{64x^6} - \frac{1}{64x^6} - \frac{1}{64x^6} - \frac{1}{64x^6}$$

$$= \frac{5}{64x^6}.$$

$$U(5) = \frac{1}{64x^6},$$

and so on. The differential inverse transform of $U_k(x)$ gives

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k$$

$$= \frac{1}{2x} + \frac{1}{4x^2}t + \frac{1}{8x^3}t^2 + \frac{1}{16x^4}t^3 + \frac{1}{32x^5}t^4 + \frac{1}{64x^6}t^5 + \cdots$$

3.5 Homotopy analysis method

Following [20, 21, 24, 30], we consider the nonlinear differential equation

$$N[u(x,t)] = 0, \tag{3.5.1}$$

where N is a nonlinear operator, x and t denote the independent variables and $u(x,t)$ is the unknown function. By means of homotopy analysis method, we first construct the so-called

zeroth-order deformation equation

$$(1-q)L[U(x,t;q) - u_0(x,t)] = q\hbar H(x,t)[N[U(x,t)]] \qquad (3.5.2)$$

where $q\varepsilon[0,1]$ is the homotopy embedding parameter. L is an auxiliary linear operator, $\hbar \neq 0$ is an auxiliary parameter used to control the error and $H(x,t)$ is an auxiliary function. $U(x,t;q)$ is an unknown function, $u_0(x,t)$ is an initial guess. When $q = 0$ and $q = 1$, equation (3.5.2) becomes

$$U(x,t;0) = u_0(x,t) \quad and \quad U(x,t;1) = u(x,t),$$

So if q varies from 0 to 1, the solution will change from $u_0(x,t)$ to the solution $u(x,t)$. Expanding $U(x,t;q)$ in Taylor series with respect to q gives

$$U(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m U(x,t;q)}{\partial q^m}|_{q=0}q^m. \qquad (3.5.3)$$

If $L, \hbar, H(x,t)$ and $u_0(x,t)$ are properly chosen, then the series (3.5.3) converges at $q = 1$, then we have

$$U(x,t;1) = u_0(x,t) + \sum_{m=0}^{\infty} u_m(x,t),$$

where

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m U(x,t;q)}{\partial q^m}|_{q=0},$$

which must be one of the solutions to the original equation as proved by Liao [43]. Differentiating (3.5.2) m times with respect to q, then setting $q = 0$ and diving by $m!$, gives the so-called m^{th}-order deformation equation as,

$$L[U_m(x,t) - \chi_m u_{m-1}(x,t)] = qH(x,t)R_m[u_{m-1}(x,t)],$$

where

$$\chi_m = \begin{cases} 0 \text{ if } m \leq 1 \\ 1 \text{ if } m > 1, \end{cases}$$

and

$$R_m(x) = \frac{1}{(m-1)!} \frac{\partial^{m-1} U(x,t;q)}{\partial q^{m-1}} [N_2 U(x,t;q) - \varphi(x)]_{q=0}.$$

3.6 Solving nonlinear ordinary differential equations using the homotopy analysis method.

Example 3.6.1. Solve the following first order nonlinear ode

$$y' + y - y^2 = 0,$$

with the initial condition

$$y(0) = 2.$$

Solution 3.6.1.1. Let $L(y) = y'$. Then $y_0 = 2$ is an initial guess.

By the so-called zeroth-order deformation equation

$$(1 - q)L[U(x, t; q) - u_0(x, t)] = q\bar{h}[N[U(x, t)] + L[U(x, t)],$$

we have

$$(1 - q)[y'(x) - y_0'(x)] = q\bar{h}[y'(x) + y(x) - y^2(x)]. \qquad (3.6.1)$$

We will assume that the solution is of a standard power series form

$$y = y_0 + y_1 q + y_2 q^2 + y_3 q^3 + \cdots$$

Substituting this into (3.6.1) we have

$$(1 - q)[y_0' + y_1'q + y_2'q^2 + y_3'q^3 + \cdots - y_0'] = q\bar{h}[y_0' + y_1'q + y_2'q^2 + y_3'q^3 + \cdots + (y_0 + y_1 q + y_2 q^2 + y_3 q^3 + \cdots) - (y_0 + y_1 q + y_2 q^2 + y_3 q^3 + \cdots)^2].$$

By the first derivative, we have

$$(1 - q)(y_1' + 2y_2'q + 3y_3'q^2 + \cdots) + (-1)(y_1'q + y_2'q^2 + y_3'q^3 + \cdots)$$

$$= \hbar[y_0' + y_1'q + y_2'q^2 + y_3'q^3 + \cdots + (y_0 + y_1 q + y_2 q^2 + y_3 q^3 + \cdots) - (y_0 + y_1 q + y_2 q^2 + y_3 q^3 + \cdots)^2]$$

$$+ \hbar q[y_1' + 2y_2'q + 3y_3'q^2 + \cdots + (y_1 + 2y_2 q + 3y_3 q^2 + \cdots) - 2(y_0 + y_1 q + y_2 q^2 + y_3 q^3 + \cdots)$$

$$(y_1 + 2y_2 q + 3y_3 q^2 + \cdots)].$$

Taking $q = 0$, we obtain

$$y_1' = \hbar y_0' + \hbar y_0 - \hbar y_0^2$$
$$= \hbar(0) + \hbar(2) - \hbar(2)^2$$
$$= 2\hbar - 4\hbar$$
$$= -2\hbar$$
$$y_1 = -2 \int_0^x \hbar \, dx \, dx$$
$$= -2\hbar x.$$

From the second derivative, we have

$$(1-q)(2y_2' + 6y_3'q + \cdots) + (-1)(y_1' + 2y_2'q + 3y_3'q^2 + \cdots) + (-1)(y_1' + 2y_2'q + 3y_3'q^2 + \cdots)$$

$$= \hbar[y_1' + 2y_2'q + 3y_3'q^2 + \cdots + (y_1 + 2y_2q + 3y_3q^2 + \cdots) - 2(y_0 + y_1q + y_2q^2 + y_3q^3 + \cdots)$$

$$(y_1 + 2y_2q + 3y_3q^2 + \cdots)] + \hbar[y_1' + 2y_2'q + 3y_3'q^2 + \cdots + (y_1 + 2y_2q + 3y_3q^2 + \cdots)$$

$$- 2(y_0 + y_1q + y_2q^2 + y_3q^3 + \cdots)(y_1 + 2y_2q + 3y_3q^2 + \cdots)] + \hbar q[2y_2' + 6y_3'q + \cdots$$

$$+ (2y_2 + 6y_3q + \cdots) - 2(y_0 + y_1q + y_2q^2 + y_3q^3 + \cdots)(2y_2 + 6y_3q + \cdots).$$

Taking $q = 0$, we have

$$2y_2' - y_1' - y_1' = 2\hbar y_1' + 2\hbar y_1 - 4\hbar y_0 y_1$$
$$y_2' = y_1' + \hbar y_1' + \hbar y_1 - 2\hbar y_0 y_1$$
$$= -2\hbar + \hbar(-2\hbar) + 2\hbar(2)(-2\hbar x)$$
$$= -2\hbar - 2\hbar^2 + 6\hbar^2 x$$
$$y_2 = -2 \int_0^x \hbar \, dx - 2 \int_0^x \hbar^2 \, dx + 6 \int_0^x \hbar^2 x \, dx$$
$$= -2\hbar x - 2\hbar^2 x + 3\hbar^2 x^2.$$

By the third derivative, we have

$$(1-q)(6y_3' + \cdots) + (-1)(2y_2' + 6y_3'q + \cdots) + (-1)(2y_2' + 6y_3'q + \cdots) + (-1)(2y_2' + 6y_3'q + \cdots)$$

$$= \hbar[2y_2' + 6y_3'q + \cdots + (2y_2 + 6y_3q + \cdots) - 2(y_0 + y_1q + y_2q^2 + y_3q^3 + \cdots)(2y_2 + 6y_3q + \cdots)$$

$$- 2(y_1 + 2y_2q + 3y_3q^2 + \cdots)(y_1 + 2y_2q + 3y_3q^2 + \cdots)]$$

$$+ \hbar[2y_2' + 6y_3'q + \cdots + (2y_2 + 6y_3q + \cdots) - 2(y_0 + y_1q + y_2q^2 + y_3q^3 + \cdots)(2y_2 + 6y_3q + \cdots)$$

$$- 2(y_1 + 2y_2q + 3y_3q^2 + \cdots)(y_1 + 2y_2q + 3y_3q^2 + \cdots)]$$

$$+ \hbar[2y_2' + 6y_3'q + \cdots + (2y_2 + 6y_3q + \cdots) - 2(y_0 + y_1q + y_2q^2 + y_3q^3 + \cdots)(2y_2 + 6y_3q + \cdots)$$

$$- 2(y_1 + 2y_2q + 3y_3q^2 + \cdots)(y_1 + 2y_2q + 3y_3q^2 + \cdots)]$$

$$+ \hbar q[6y_3' + \cdots$$

Taking $q = 0$, gives

$$6y_3' - 2y_2' - 2y_2' - y_2' = 6\hbar y_2' + 6\hbar y_2 - 12\hbar y_0 y_2 - 6\hbar y_1^2$$

$$y_3' = y_2' + \hbar y_2' + \hbar y_2 - 2\hbar y_0 y_2 - \hbar y_1^2$$

$$= -2\hbar - 2\hbar^2 + 6\hbar^2 x + \hbar(-2\hbar - 2\hbar^2 + 6\hbar^2 x) - 2\hbar(2)(-2\hbar x - 2\hbar^2 x + 3\hbar^2 x^2)$$

$$- \hbar(-2\hbar x)^2$$

$$= -2\hbar - 2\hbar^2 + 6\hbar^2 x - 2\hbar^2 - 2\hbar^3 + 6\hbar^3 x - 2\hbar^2 x - 2\hbar^3 x + 3\hbar^2 x^2 + 8\hbar^2 x + 8\hbar^3 x$$

$$- 12\hbar^3 x^2 - 4\hbar^3 x^3$$

$$y_3 = -2\int_0^x \hbar dx - 2\int_0^x \hbar^2 dx + 6\int_0^x \hbar^2 x dx - 2\int_0^x \hbar^2 dx - 2\int_0^x \hbar^3 dx + 6\int_0^x \hbar^3 x dx$$

$$- 2\int_0^x \hbar^2 x dx - 2\int_0^x \hbar^3 x dx + 3\int_0^x \hbar^2 x^2 dx + 8\int_0^x \hbar^2 x dx + 8\int_0^x \hbar^3 x dx$$

$$- 12\int_0^x \hbar^3 x^2 dx dx - 4\int_0^x \hbar^3 x^3 dx$$

$$= -2\hbar - 2\hbar^2 x + 3\hbar^2 x^2 - 2\hbar^2 x - 2\hbar^3 x + 3\hbar^3 x^2 - \hbar^2 x^2 - \hbar^3 x^2 + \hbar^3 x^3$$

$$+ 4\hbar^2 x^2 + 4\hbar^3 x^2 - 4\hbar^2 x^3 - \frac{4}{3}\hbar^3 x^3.$$

We proceed in the same manner to get y_4, y_5, $\cdots$ Thus the solution is given as

$$y(x) = y_0 + y_1 + y_2 + y_3 + \cdots$$

$$= 2 - 2\hbar x - 2\hbar x - 2\hbar^2 x + 3\hbar^2 x^2 - 2\hbar - 2\hbar^2 x + 3\hbar^2 x^2 - 2\hbar^2 x - 2\hbar^3 x + 3\hbar^3 x^2 - \hbar^2 x^2$$

$$- \hbar^3 x^2 + \hbar^3 x^3 + 4\hbar^2 x^2 + 4\hbar^3 x^2 - 4\hbar^2 x^3 - \frac{4}{3}\hbar^3 x^3 + \cdots$$

For $\hbar = -1$, we obtain

$$y(x) = 2 + 2x + 2x - 2x + 3x^2 + 2 - 2x + 3x^2 + 2x - 2x - 3x^2 - x^2 + x^2 - x^3 + 4x^2 - 4x^2$$

$$- 4x^3 + \frac{4}{3}x^3 + \cdots$$

$$= 2 + 2x + 3x^2 + \frac{13}{3}x^3 + \cdots$$

which coincides with the one obtained by DTM.

Example 3.6.2. Consider the following second order nonlinear ode

$$y'' + y^2 = 0,$$

with the initial conditions

$$y(0) = 1 \quad and \quad y'(0) = 0.$$

Solution 3.6.2.1. Let $L(y) = y''$. From the initial conditions, the initial guess is found to be $y_0(x) = 1$. By the so-called zeroth-order deformation equation

$$(1-q)L[U(x,t;q) - u_0(x,t)] = q\bar{h}[N[U(x,t)] + L[U(x,t)]],$$

we have

$$(1-q)[y''(x) - y_0''(x)] = q\bar{h}[y''(x) + y^2(x)]. \qquad (3.6.2)$$

We will assume that the solution is of a standard power series form

$$y = y_0 + y_1 q + y_2 q^2 + y_3 q^3 + \cdots$$

27

Substituting this into ([3.6.2](#)) we get

$$(1-q)[y_0'' + y_1''q + y_2''q^2 + y_3''q^3 + \cdots - y_0''] = q\bar{h}[y_0'' + y_1''q + y_2''q^2 + y_3''q^3 + \cdots$$
$$+ (y_0 + y_1q + y_2q^2 + y_3q^3 + \cdots)^2].$$

By the first derivative, we have

$$(1-q)(y_1'' + 2y_2''q + 3y_3''q^2 + \cdots) + (-1)(y_1''q + y_2''q^2 + y_3''q^3 + \cdots)$$
$$= \hbar[y_0'' + y_1''q + y_2''q^2 + y_3''q^3 + \cdots$$
$$+ (y_0 + y_1q + y_2q^2 + y_3q^3 + \cdots)^2] + \hbar q[y_1'' + 2y_2''q + 3y_3''q^2 + \cdots$$
$$+ 2(y_0 + y_1q + y_2q^2 + y_3q^3 + \cdots)(y_1 + 2y_2q + 3y_3q^2 + \cdots)].$$

Taking $q = 0$, we obtain

$$y_1'' = \hbar y_0'' + \hbar y_0^2$$
$$= \hbar(0) + \hbar(1)^2$$
$$= \hbar$$
$$y_1 = \int_0^x \int_0^x \hbar \, dx \, dx$$
$$= \hbar \frac{x^2}{2}.$$

By the second derivative, we have

$$(1-q)(2y_2'' + 6y_3''q + \cdots) + (-1)(y_1'' + 2y_2''q + 3y_3''q^2 + \cdots) + (-1)(y_1'' + 2y_2''q + 3y_3''q^2 + \cdots)$$
$$= \hbar[y_1'' + 2y_2''q + 3y_3''q^2 + \cdots + 2(y_0 + y_1q + y_2q^2 + y_3q^3 + \cdots)(y_1 + 2y_2q + 3y_3q^2 + \cdots)]$$
$$+ \hbar[y_1'' + 2y_2''q + 3y_3''q^2 + \cdots + 2(y_0 + y_1q + y_2q^2 + y_3q^3 + \cdots)(y_1 + 2y_2q + 3y_3q^2 + \cdots)]$$
$$+ \hbar q[2y_2'' + 6y_3''q + \cdots + 2(y_0 + y_1q + y_2q^2 + y_3q^3 + \cdots)(2y_2 + 6y_3q + \cdots)$$
$$+ 2(y_1 + 2y_2q + 3y_3q^2 + \cdots)(y_1 + 2y_2q + 3y_3q^2 + \cdots)].$$

Taking $q = 0$, we obtain

$$2y_2'' - y_1'' - y_1'' = 2\hbar y_1'' + 4\hbar y_0 y_1$$

$$y_2'' = y_1'' + \hbar y_1'' + 2\hbar y_0 y_1$$

$$= \hbar + \hbar(\hbar) + 2\hbar(1)(\hbar\frac{x^2}{2})$$

$$y_2 = \int_0^x \int_0^x \hbar \, dx dx + \hbar \int_0^x \int_0^x \hbar \, dx dx + 2\int_0^x \int_0^x \hbar^2 \frac{x^2}{2} dx dx$$

$$= \hbar\frac{x^2}{2} + \hbar\frac{x^2}{2} + \hbar^2\frac{x^4}{12}.$$

By the third derivative, we have

$$(1-q)(6y_3'' + \cdots) + (-1)(2y_2'' + 6y_3''q + \cdots) + (-1)(2y_2'' + 6y_3''q + \cdots) + (-1)(2y_2'' + 6y_3''q + \cdots)$$

$$= \hbar[2y_2'' + 6y_3''q + \cdots + 2(y_0 + y_1q + y_2q^2 + y_3q^3 + \cdots)(2y_2 + 6y_3q + \cdots)$$

$$+ 2(y_1 + 2y_2q + 3y_3q^2 + \cdots)(y_1 + 2y_2q + 3y_3q^2 + \cdots)]$$

$$+ \hbar[2y_2'' + 6y_3''q + \cdots + 2(y_0 + y_1q + y_2q^2 + y_3q^3 + \cdots)(2y_2 + 6y_3q + \cdots)$$

$$+ 2(y_1 + 2y_2q + 3y_3q^2 + \cdots)(y_1 + 2y_2q + 3y_3q^2 + \cdots)]$$

$$+ \hbar[2y_2'' + 6y_3''q + \cdots + 2(y_0 + y_1q + y_2q^2 + y_3q^3 + \cdots)(2y_2 + 6y_3q + \cdots)$$

$$+ 2(y_1 + 2y_2q + 3y_3q^2 + \cdots)(y_1 + 2y_2q + 3y_3q^2 + \cdots)]$$

$$+ \hbar q[6y_3'' + \cdots$$

Taking $q = 0$, gives

$$6y_3'' - 2y_2'' - 2y_2'' - y_2'' = 6\hbar y_2'' + 12\hbar y_0 y_2 + 6\hbar y_1^2$$

$$y_3'' = y_2'' + \hbar y_2'' + 2\hbar y_0 y_2 + \hbar y_1^2$$

$$= \left(\hbar\frac{x^2}{2} + \hbar\frac{x^2}{2} + \hbar^2\frac{x^4}{12}\right) + \hbar\left(\hbar\frac{x^2}{2} + \hbar\frac{x^2}{2} + \hbar^2\frac{x^4}{12}\right) + 2\hbar(1)\left(\hbar\frac{x^2}{2} + \hbar\frac{x^2}{2} + \hbar^2\frac{x^4}{12}\right)$$

$$+ \hbar\left(\hbar\frac{x^2}{2}\right)^2.$$

29

$$y_3 = \int_0^x \int_0^x \left(\hbar \frac{x^2}{2} + \hbar \frac{x^2}{2} + \hbar^2 \frac{x^4}{12} \right) dxdx + \hbar \int_0^x \int_0^x \left(\hbar \frac{x^2}{2} + \hbar \frac{x^2}{2} + \hbar^2 \frac{x^4}{12} \right) dxdx +$$

$$2 \int_0^x \int_0^x \hbar(1) \left(\hbar \frac{x^2}{2} + \hbar \frac{x^2}{2} + \hbar^2 \frac{x^4}{12} \right) dxdx + \hbar \int_0^x \int_0^x \left(\hbar \frac{x^2}{2} \right)^2 dxdx$$

$$= \hbar \frac{x^4}{24} + \hbar^2 \frac{x^4}{24} + \hbar^2 \frac{x^6}{360} + \hbar^2 \frac{x^4}{24} + \hbar^3 \frac{x^4}{24} + \hbar^3 \frac{x^6}{360} + \hbar^2 \frac{x^4}{12} + \hbar^3 \frac{x^4}{12} + \hbar^3 \frac{x^6}{180}$$

$$+ \hbar^3 \frac{x^6}{120}, \quad etc.$$

Thus the solution is

$$y(x) = y_0 + y_1 + y_2 + y_3 + \cdots$$

$$= 1 + \hbar \frac{x^2}{2} + \hbar \frac{x^2}{2} + \hbar^2 \frac{x^2}{2} + \hbar^2 \frac{x^4}{12} + \hbar \frac{x^4}{24} + \hbar^2 \frac{x^4}{24} + \hbar^2 \frac{x^6}{360} + \hbar^2 \frac{x^4}{24} + \hbar^3 \frac{x^4}{24} + \hbar^3 \frac{x^6}{360} + \hbar^2 \frac{x^4}{12}$$

$$+ \hbar^3 \frac{x^4}{12} + \hbar^3 \frac{x^6}{180} + \hbar^3 \frac{x^6}{120} + \cdots$$

For $\hbar = -1$, we obtain the solution

$$y(x) = y_0 + y_1 + y_2 + y_3 + \cdots$$

$$= 1 - \frac{x^2}{2} - \frac{x^2}{2} + \frac{x^2}{2} + \frac{x^4}{12} - \frac{x^4}{24} + \frac{x^4}{24} + \frac{x^6}{360} + \frac{x^4}{24} - \frac{x^4}{24} - \frac{x^6}{360} + \frac{x^4}{12} - \frac{x^4}{12} - \frac{x^6}{180} - \frac{x^6}{120} + \cdots$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{12} - \frac{x^6}{72} + \cdots$$

which is the same as that obtained by DTM.

3.7 Solving nonlinear partial differential equations using the homotopy analysis method.

Example 3.7.1. Use HAM to solve the first order nonlinear partial differential equation

$$u_t + u_x + u^2 = 0, \tag{3.7.1}$$

with the initial condition

$$u(x, 0) = \frac{1}{2x}.$$

Solution 3.7.1.1. Let $L(y) = u'$, and the initial guess $u_0(x, y) = \frac{1}{2x}$. By the so-called zeroth-order deformation equation

$$(1 - q)L[U(x, t; q) - u_0(x, t)] = q\bar{h}H(x, t)[N(x, t)],$$

we have

$$(1 - q)(u'(x, y) - u_0'(x, y)) = \hbar q(u_t(x, y) + u_x(x, y) + u^2(x, y)). \qquad (3.7.2)$$

We will assume that the solution of the given differential equation is of a standard power series form

$$u = u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots.$$

Substituting this into (3.7.2) gives

$$(1 - q)(u_0' + u_1'q + u_2'q^2 + u_3'q^3 + \cdots - u_0') = \hbar q[\frac{\partial}{\partial t}\left(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots\right) + \frac{\partial}{\partial x}(u_0 + u_1 q$$

$$+ u_2 q^2 + u_3 q^3 + \cdots) + (u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)^2]$$

From the first derivative,

$$(1 - q)(u_1' + 2u_2'q + 3u_3'q^2 + \cdots) + (-1)(u_1'q + u_2'q^2 + u_3'q^3 + \cdots)$$

$$= \hbar[\frac{\partial}{\partial t}(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots) + \frac{\partial}{\partial x}(u_0 + u_1 q + u_2 q^2$$

$$+ u_3 q^3 + \cdots) + (u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)^2]$$

$$+ \hbar q[\frac{\partial}{\partial t}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)$$

$$+ \frac{\partial}{\partial x}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots) + 2(u_0 + u_1 q + u_2 q^2$$

$$+ u_3 q^3 + \cdots)(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)].$$

Take $q = 0$ to get

$$u_1' = \hbar \frac{\partial u_0}{\partial t} + \hbar \frac{\partial u_0}{\partial x} + \hbar u_0^2$$

$$= \hbar(0) + \hbar \frac{\partial}{\partial x}\left(\frac{1}{2x}\right) + \hbar\left(\frac{1}{2x}\right)^2$$

$$= -\frac{\hbar}{2x^2} - \frac{\hbar}{4x^2}$$

$$= -\frac{\hbar}{4x^2}$$

$$u_1 = -\int_0^t \frac{\hbar}{4x^2} dt$$

$$= -\frac{\hbar}{4x^2} t.$$

From the second derivative,

$$(1-q)(2u_2' + 6u_3'q + \cdots) + (-1)(u_1' + 2u_2'q + 3u_3'q^2 + \cdots) + (-1)(u_1' + 2u_2'q + 3u_3'q^2 + \cdots)$$

$$= \hbar[\frac{\partial}{\partial t}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots) + \frac{\partial}{\partial x}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots) + 2(u_0 + u_1 q + u_2 q^2 + u_3 q^3$$

$$+ \cdots)(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)] + \hbar[\frac{\partial}{\partial t}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)$$

$$+ \frac{\partial}{\partial x}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots) + 2(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)]$$

$$+ \hbar q[\frac{\partial}{\partial t}(2u_2 + 6u_3 q + \cdots) + \frac{\partial}{\partial x}(2u_2 + 6u_3 q + \cdots) + 2(u_0 + u_1 q + u_2 q^2$$

$$+ u_3 q^3 + \cdots)(2u_2 + 6u_3 q + \cdots) + 2(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)].$$

Taking $q = 0$, we get

$$2u_2' = 2u_1' + 2\hbar \frac{\partial u_1}{\partial t} + 2\hbar \frac{\partial u_1}{\partial x} + 4\hbar u_0 u_1$$

$$u_2' = u_1' + \hbar \frac{\partial u_1}{\partial t} + \hbar \frac{\partial u_1}{\partial x} + 2\hbar u_0 u_1$$

$$u_2' = -\frac{\hbar}{4x^2} + \hbar \frac{\partial}{\partial t}\left(-\frac{\hbar}{4x^2} t\right) + \hbar \frac{\partial}{\partial x}\left(-\frac{\hbar}{4x^2} t\right) + 2\hbar\left(\frac{1}{2x}\right)\left(-\frac{\hbar}{4x^2} t\right)$$

$$= -\frac{\hbar}{4x^2} - \frac{\hbar^2}{4x^2} + \frac{\hbar^2}{4x^3} t$$

$$= -\int_0^t \frac{\hbar}{4x^2} dx - \int_0^t \frac{\hbar^2}{4x^2} dx + \int_0^t \frac{\hbar^2}{4x^3} t dx$$

$$u_2 = -\frac{\hbar}{4x^2}t - \frac{\hbar^2}{4x^2}t + \frac{\hbar^2}{8x^3}t^2.$$

From the third derivative,

$$(1-q)(6u_3' + \cdots) + (-1)(2u_2' + 6u_3'q + \cdots) + (-1)(2u_2' + 6u_3'q + \cdots) + (-1)(2u_2' + 6u_3'q + \cdots)$$

$$= \hbar[\frac{\partial}{\partial t}(2u_2 + 6u_3q + \cdots)$$

$$+ \frac{\partial}{\partial x}(2u_2 + 6u_3q + \cdots) + 2(u_0 + u_1q + u_2q^2$$

$$+ u_3q^3 + \cdots)(2u_2 + 6u_3q + \cdots) + 2(u_1 + 2u_2q + 3u_3q^2 + \cdots)(u_1 + 2u_2q + 3u_3q^2 + \cdots)]$$

$$+ \hbar[\frac{\partial}{\partial t}(2u_2 + 6u_3q + \cdots)$$

$$+ \frac{\partial}{\partial x}(2u_2 + 6u_3q + \cdots) + 2(u_0 + u_1q + u_2q^2$$

$$+ u_3q^3 + \cdots)(2u_2 + 6u_3q + \cdots) + 2(u_1 + 2u_2q + 3u_3q^2 + \cdots)(u_1 + 2u_2q + 3u_3q^2 + \cdots)]$$

$$+ \hbar[\frac{\partial}{\partial t}(2u_2 + 6u_3q + \cdots)$$

$$+ \frac{\partial}{\partial x}(2u_2 + 6u_3q + \cdots) + 2(u_0 + u_1q + u_2q^2$$

$$+ u_3q^3 + \cdots)(2u_2 + 6u_3q + \cdots) + 2(u_1 + 2u_2q + 3u_3q^2 + \cdots)(u_1 + 2u_2q + 3u_3q^2 + \cdots)]$$

$$+ \hbar q[\frac{\partial}{\partial t}(6u_3 + \cdots) + \cdots$$

Taking $q = 0$, we get

$$6u_3' = 6u_2' + 6\hbar\frac{\partial u_2}{\partial t} + 6\hbar\frac{\partial u_2}{\partial x} + 12\hbar u_0 u_2 + 6\hbar u_1^2$$

$$u_3' = u_2' + \hbar\frac{\partial u_2}{\partial t} + \hbar\frac{\partial u_2}{\partial x} + 2\hbar u_0 u_2 + \hbar u_1^2$$

$$u_3' = -\frac{\hbar}{4x^2} - \frac{\hbar^2}{4x^2} + \frac{\hbar^2}{4x^3}t + \hbar\frac{\partial}{\partial t}\left(-\frac{\hbar}{4x^2}t - \frac{\hbar^2}{4x^2}t + \frac{\hbar^2}{8x^3}t^2\right) + \hbar\frac{\partial}{\partial x}\left(-\frac{\hbar}{4x^2}t - \frac{\hbar^2}{4x^2}t + \frac{\hbar^2}{8x^3}t^2\right) +$$

$$2\hbar\left(\frac{1}{2x}\right)\left(-\frac{\hbar}{4x^2}t - \frac{\hbar^2}{4x^2}t + \frac{\hbar^2}{8x^3}t^2\right) + \hbar\left(-\frac{\hbar}{4x^2}t\right)^2$$

$$u_3' = -\frac{\hbar}{4x^2} - \frac{\hbar^2}{4x^2} + \frac{\hbar^2}{4x^3}t + \hbar\left(-\frac{\hbar}{4x^2} - \frac{\hbar^2}{4x^2} + \frac{\hbar^2}{4x^3}t\right) + \hbar\left(\frac{\hbar}{2x^3}t + \frac{\hbar^2}{2x^3}t - \frac{3\hbar^2}{8x^4}t^2\right) +$$

$$2\hbar\left(\frac{1}{2x}\right)\left(-\frac{\hbar}{4x^2}t - \frac{\hbar^2}{4x^2}t + \frac{\hbar^2}{8x^3}t^2\right) + \hbar\left(-\frac{\hbar}{4x^2}t\right)^2$$

$$u_3' = -\frac{\hbar}{4x^2} - \frac{\hbar^2}{4x^2} + \frac{\hbar^2}{4x^3}t - \frac{\hbar^2}{4x^2} - \frac{\hbar^3}{4x^2} + \frac{\hbar^3}{4x^3}t + \frac{\hbar^2}{2x^3}t + \frac{\hbar^3}{2x^3}t - \frac{3\hbar^3}{8x^4}t^2 +$$

$$- \frac{\hbar^2}{4x^3}t - \frac{\hbar^3}{4x^3}t + \frac{\hbar^3}{8x^4}t^2 + \frac{\hbar^3}{16x^4}t^2.$$

$$u_3 = -\int_0^t \frac{\hbar}{4x^2}dt - \int_0^t \frac{\hbar^2}{4x^2}dt + \int_0^t \frac{\hbar^2}{4x^3}tdt - \int_0^t \frac{\hbar^2}{4x^2}dt - \int_0^t \frac{\hbar^3}{4x^2}dt + \int_0^t \frac{\hbar^3}{4x^3}tdt + \int_0^t \frac{\hbar^2}{2x^3}tdt$$

$$+ \int_0^t \frac{\hbar^3}{2x^3}tdt - \int_0^t \frac{3\hbar^3}{8x^4}t^2dt - \int_0^t \frac{\hbar^2}{4x^3}tdt - \int_0^t \frac{\hbar^3}{4x^3}tdt + \int_0^t \frac{\hbar^3}{8x^4}t^2dt + \int_0^t \frac{\hbar^3}{16x^4}t^2dt$$

$$u_3 = -\frac{\hbar}{4x^2}t - \frac{\hbar^2}{4x^2}t + \frac{\hbar^2}{8x^3}t^2 - \frac{\hbar^2}{4x^2}t - \frac{\hbar^3}{4x^2}t + \frac{\hbar^3}{8x^3}t^2 + \frac{\hbar^2}{4x^3}t^2 + \frac{\hbar^3}{4x^3}t^2 - \frac{\hbar^3}{8x^4}t^3 -$$

$$\frac{\hbar^2}{8x^3}t^2 - \frac{\hbar^3}{8x^3}t^2 + \frac{\hbar^3}{24x^4}t^3 + \frac{\hbar^3}{48x^4}t^3, \quad etc.$$

Thus the solution is given as

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots$$

$$= \frac{1}{2x} - \frac{\hbar}{4x^2}t - \frac{\hbar}{4x^2}t - \frac{\hbar^2}{4x^2}t + \frac{\hbar^2}{8x^3}t^2 - \frac{\hbar}{4x^2}t - \frac{\hbar^2}{4x^2}t + \frac{\hbar^2}{8x^3}t^2 - \frac{\hbar^2}{4x^2}t - \frac{\hbar^3}{4x^2}t + \frac{\hbar^3}{8x^3}t^2$$

$$+ \frac{\hbar^2}{4x^3}t^2 + \frac{\hbar^3}{4x^3}t^2 - \frac{\hbar^3}{8x^4}t^3 - \frac{\hbar^2}{8x^3}t^2 - \frac{\hbar^3}{8x^3}t^2 + \frac{\hbar^3}{24x^4}t^3 + \frac{\hbar^3}{48x^4}t^3 + \cdots$$

For $\hbar = -1$, we obtain the solution

$$u(x,t) = \frac{1}{2x} + \frac{1}{4x^2}t + \frac{1}{4x^2}t - \frac{1}{4x^2}t + \frac{1}{8x^3}t^2 + \frac{1}{4x^2}t - \frac{1}{4x^2}t + \frac{1}{8x^3}t^2 - \frac{1}{4x^2}t + \frac{1}{4x^2}t - \frac{1}{8x^3}t^2$$

$$+ \frac{1}{4x^3}t^2 - \frac{1}{4x^3}t^2 + \frac{1}{8x^4}t^3 - \frac{1}{8x^3}t^2 + \frac{1}{8x^3}t^2 - \frac{1}{24x^4}t^3 - \frac{1}{48x^4}t^3 + \cdots$$

$$= \frac{1}{2x} + \frac{1}{4x^2}t + \frac{1}{8x^3}t^2 + \frac{1}{16x^4}t^3 + \cdots$$

It again coincides with that obtained by DTM.

Example 3.7.2. Use HAM to solve the *modified Korteweg de-Vries (mKdV) equation*

$$u_t - 6u^2 u_x + u_{xxx} = 0$$

with the initial condition

$$u(x,0) = x.$$

This equation models a variety of nonlinear wave phenomena such as shallow water waves, acoustic waves in a harmonic crystal, and ion-acoustic waves in plasmas. It thus has important applications.

Solution 3.7.2.1. Let $L(y) = u'$, and the initial guess given by $u_0(x, y) = x$.

By the so-called zeroth-order deformation equation

$$(1 - q)L[U(x, t; q) - u_0(x, t)] = q\bar{h}H(x, t)[N(x, t)],$$

we have

$$(1 - q)(u' - u_0') = \hbar q(u_t - 6u^2 u_x + u_{xxx}). \tag{3.7.3}$$

We will assume that the solution of the given differential equation is of a standard power series form

$$u = u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots .$$

Substituting this into (3.7.3) gives

$$(1 - q)(u_0' + u_1' q + u_2' q^2 + u_3' q^3 + \cdots - u_0') = \hbar q(\frac{\partial}{\partial t}(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)$$

$$- 6(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)^2 \frac{\partial}{\partial x}(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)$$

$$+ \frac{\partial^3}{\partial x^3}(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)).$$

From the first derivative,

$$(1 - q)(u_1' + 2u_2' q + 3u_3' q^2 + \cdots) + (-1)(u_1' q + u_2' q^2 + u_3' q^3 + \cdots)$$

$$= \hbar[\frac{\partial}{\partial t}(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots) - 6(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)^2$$

$$\frac{\partial}{\partial x}(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots) + \frac{\partial^3}{\partial x^3}(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)]$$

$$+ \hbar q[\frac{\partial}{\partial t}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots) - 12(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)$$

$$\frac{\partial}{\partial x}(u_0 + u_1 q + u_2 q^2 + u_3 q^3 - 6(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)^2 \frac{\partial}{\partial x}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)$$

$$+ \frac{\partial^3}{\partial x^3}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)].$$

Taking $q = 0$, we get

$$u'_1 = \hbar \left[\frac{\partial u_0}{\partial t} - 6u_0^2 \frac{\partial u_0}{\partial x} + \frac{\partial^3 u_0}{\partial x^3} \right]$$

$$= \hbar[-6(x)^2]$$

$$= -\hbar 6x^2$$

$$u_1 = -\hbar 6x^2 t$$

$$= 6x^2 t.$$

From the second derivative,

$$(1 - q)(2u'_2 + 6u'_3 q + \cdots) + (-1)(u'_1 + 2u'_2 q + 3u'_3 q^2 + \cdots) + (-1)(u'_1 + 2u'_2 q + 3u'_3 q^2 + \cdots)$$

$$= 2\hbar[\frac{\partial}{\partial t}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots) - 12(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)$$

$$\frac{\partial}{\partial x}(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots) - 6(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)^2 \frac{\partial}{\partial x}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)$$

$$+ \frac{\partial^3}{\partial x^3}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)] + \hbar q[\frac{\partial}{\partial t}(2u_2 + 6u_3 q + \cdots)$$

$$- 12(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)^2 \frac{\partial}{\partial x}(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)$$

$$- 12(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)(2u'_2 + 6u'_3 q + \cdots)\frac{\partial}{\partial x}(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)$$

$$- 12(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)\frac{\partial}{\partial x}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)$$

$$- 12(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)\frac{\partial}{\partial x}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)$$

$$- 6(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)^2 \frac{\partial}{\partial x}(2u'_2 + 6u'_3 q + \cdots) + \frac{\partial^3}{\partial x^3}(2u'_2 + 6u'_3 q + \cdots)].$$

Taking $q = 0$, we obtain

$$2u'_2 - 2u'_1 = 2\hbar \left[\frac{\partial u_1}{\partial t} - 12u_0 u_1 \frac{\partial u_0}{\partial x} - 6u_o^2 \frac{\partial u_1}{\partial x} + \frac{\partial^3 u_1}{\partial x^3} \right]$$

$$u_2 = 6x^2 t + \hbar \int_0^t [6x^2 - 12x(6x^2 t) - 6x^2(12xt)]dt$$

$$= 6x^2 t + \hbar \int_0^t [6x^2 - 72x^3 t - 72x^3 t]dt$$

$$= 6x^2 t - 6x^2 t + 36x^3 t^2 + 36x^3 t^2$$

$$= 72x^3 t^2.$$

The third derivative gives

$$(1-q)(6u_3' + \cdots) + (-1)(2u_2' + 6u_3'q + \cdots) + (-1)(2u_2' + 6u_3'q + \cdots) + (-1)(2u_2' + 6u_3'q + \cdots)$$

$$= 3\hbar[\frac{\partial}{\partial t}(2u_2 + 6u_3q + \cdots) - 12(u_1 + 2u_2q + 3u_3q^2 + \cdots)^2\frac{\partial}{\partial x}(u_0 + u_1q + u_2q^2 + u_3q^3 + \cdots)$$

$$- 12(u_0 + u_1q + u_2q^2 + u_3q^3 + \cdots)(2u_2 + 6u_3q + \cdots)\frac{\partial}{\partial x}(u_0 + u_1q + u_2q^2 + u_3q^3 + \cdots)$$

$$- 12(u_0 + u_1q + u_2q^2 + u_3q^3 + \cdots)(u_1 + 2u_2q + 3u_3q^2 + \cdots)\frac{\partial}{\partial x}(u_1 + 2u_2q + 3u_3q^2 + \cdots)$$

$$- 12(u_0 + u_1q + u_2q^2 + u_3q^3 + \cdots)(u_1 + 2u_2q + 3u_3q^2 + \cdots)\frac{\partial}{\partial x}(u_1 + 2u_2q + 3u_3q^2 + \cdots)$$

$$- 6(u_0 + u_1q + u_2q^2 + u_3q^3 + \cdots)^2\frac{\partial}{\partial x}(2u_2 + 6u_3q + \cdots) + \frac{\partial^3}{\partial x^3}(2u_2 + 6u_3q + \cdots)]$$

$$+ \hbar q[6u_3 + \cdots$$

Taking $q = 0$, we obtain

$$6u_3' - 6u_2' = 3\hbar\left[2\frac{\partial u_2}{\partial t} - 12u_1^2\frac{\partial u_0}{\partial x} - 24u_0u_2\frac{\partial u_0}{\partial x} - 12u_0u_1\frac{\partial u_1}{\partial x} - 12u_0u_1\frac{\partial u_1}{\partial x} - 12u_0^2\frac{\partial u_2}{\partial x} + 2\frac{\partial^3 u_2}{\partial x^3}\right]$$

$$u_3 = 72x^3t^2 + \frac{1}{2}\hbar\int_0^t[288x^3 - 12(6x^2t)^2 - 24x(72x^3t^2) - 12x(6x^2t)(12xt) - 12x(6x^2t)(12xt)$$

$$- 12x^2(216x^2t^2) + 864t^2]dt$$

$$= 72x^3t^2 + \frac{1}{2}\hbar\int_0^t[288x^3t - 432x^4t^2 - 1728x^4t^2 - 864x^4t^2 - 864x^4t^2 - 2592x^4t^2 + 864t^2]dt$$

$$= 72x^3t^2 + \frac{1}{2}\hbar\int_0^t[288x^3t - 6480x^4t^2 + 864t^2]dt$$

$$= 72x^3t^2 + \hbar\int_0^t[144x^3t - 3240x^4t^2 + 432t^2]dt$$

$$= 72x^3t^2 + \hbar[72x^3t^2 - 1080x^4t^3 + 144t^3]$$

$$= 72x^3t^2 - 72x^3t^2 + 1080x^4t^3 - 144t^3$$

$$= 1080x^4t^3 - 144t^3, \quad etc.$$

Thus the solution is

$$u(x, y) = u_0 + u_1 + u_2 + u_3 + \cdots$$

$$= x + 6x^2t + 72x^3t^2 + 72x^3t^2 - 72x^3t^2 + 1080x^4t^3 - 144t^3 + \cdots$$

$$= x + 6x^2t + 72x^3t^2 + 1080x^4t^3 - 144t^3 + \cdots$$

Example 3.7.3. Solve the following nonlinear partial differential equation

$$u_{yy} + u_x u_{xx} = 2,$$

with the initial conditions

$$u(x, 0) = 0 \quad and \quad u_y(x, 0) = x.$$

Solution 3.7.3.1. Let $L(y) = u''$, and initial guess $u_0(x, y) = xy$.

By the so-called zeroth-order deformation equation

$$(1 - q)L[U(x, t; q) - u_0(x, t)] = q\bar{h}H(x, t)[N(x, t)],$$

we have

$$(1 - q)(u''(x, y) - u_0''(x, y)) = \hbar q(u_{yy} + u_x u_{xx} - 2). \tag{3.7.4}$$

We will assume that the solution of the given differential equation is of a standard power series form

$$u = u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots .$$

Substituting this into (3.7.4) gives

$$(1 - q)(u_0'' + u_1'' q + u_2'' q^2 + u_3'' q^3 + \cdots - u_0'') = \hbar q(\frac{\partial^2}{\partial y^2}(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots) + \frac{\partial}{\partial x}(u_0 + u_1 q$$

$$+ u_2 q^2 + u_3 q^3 + \cdots)\frac{\partial^2}{\partial x^2}(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots) - 2).$$

From the first derivative,

$$(1-q)(u_1^{''} + 2u_2^{''}q + 3u_3^{''}q^2 + \cdots) + (-1)(u_1^{''}q + u_2^{''}q^2 + u_3^{''}q^3 + \cdots)$$

$$= \hbar[\frac{\partial^2}{\partial y^2}(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots) + \frac{\partial}{\partial x}(u_0 + u_1 q + u_2 q^2$$

$$+ u_3 q^3 + \cdots)\frac{\partial^2}{\partial x^2}(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots) - 2]$$

$$+ \hbar q[\frac{\partial^2}{\partial y^2}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)$$

$$+ \frac{\partial}{\partial x}(u_1 q + 2u_2 q + 3u_3 q^2 + \cdots)\frac{\partial^2}{\partial x^2}(u_0 + u_1 q + u_2 q^2$$

$$+ u_3 q^3 + \cdots)] + \frac{\partial}{\partial x}(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)\frac{\partial^2}{\partial x^2}(u_1$$

$$+ 2u_2 q + 3u_3 q^2 + \cdots)].$$

Take $q = 0$ to get

$$u_1^{''} = \hbar\frac{\partial^2 u_0}{\partial x^2} + \hbar\frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2} - 2\hbar$$

$$= \hbar(0) + \hbar(0)(0) - 2\hbar$$

$$= -2\hbar.$$

$$u_1 = \int_0^y \int_0^y -2\hbar \, dy \, dy$$

$$= -\hbar y^2.$$

From the second derivative,

$$(1-q)(2u_2^{''} + 6u_3^{''}q + \cdots) + (-1)(u_1^{''} + 2u_2^{''}q + 3u_3^{''}q^2 + \cdots) + (-1)(u_1^{''} + 2u_2^{''}q + 3u_3^{''}q^2 + \cdots)$$

$$= \hbar[\frac{\partial^2}{\partial y^2}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)$$

$$+ \frac{\partial}{\partial x}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)\frac{\partial^2}{\partial x^2}(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)$$

$$+ \frac{\partial}{\partial x}(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)\frac{\partial^2}{\partial x^2}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)]$$

$$+ \hbar\left[\frac{\partial^2}{\partial y^2}(u_1 + 2u_2q + 3u_3q^2 + \cdots)\right.$$

$$+ \frac{\partial}{\partial x}(u_1q + 2u_2q + 3u_3q^2 + \cdots)\frac{\partial^2}{\partial x^2}(u_0 + u_1q + u_2q^2 + u_3q^3 + \cdots)\right]$$

$$+ \frac{\partial}{\partial x}(u_0 + u_1q + u_2q^2 + u_3q^3 + \cdots)\frac{\partial^2}{\partial x^2}(u_1 + 2u_2q + 3u_3q^2 + \cdots)$$

$$+ \hbar q\left[\frac{\partial^2}{\partial y^2}(2u_2 + 6u_3q + \cdots)\right.$$

$$+ \frac{\partial}{\partial x}(2u_2 + 6u_3q + \cdots)\frac{\partial^2}{\partial x^2}(u_0 + u_1q + u_2q^2 + u_3q^3 + \cdots)\right]$$

$$+ \frac{\partial}{\partial x}(u_1q + 2u_2q + 3u_3q^2 + \cdots)\frac{\partial^2}{\partial x^2}(u_1 + 2u_2q + 3u_3q^2 + \cdots)$$

$$+ \frac{\partial}{\partial x}(u_1q + 2u_2q + 3u_3q^2 + \cdots\frac{\partial^2}{\partial x^2}(u_1 + 2u_2q + 3u_3q^2 + \cdots)$$

$$+ \frac{\partial}{\partial x}(u_0 + u_1q + u_2q^2 + u_3q^3 + \cdots)\frac{\partial^2}{\partial x^2}(2u_2 + 6u_3q + \cdots).$$

Take $q = 0$ to get

$$2u_2^{''} - u_1^{''} - u_1^{''} = \hbar\frac{\partial^2 u_1}{\partial y^2} + \hbar\frac{\partial u_1}{\partial x}\frac{\partial^2 u_0}{\partial x^2} + \hbar\frac{\partial u_0}{\partial x}\frac{\partial^2 u_1}{\partial x^2} +$$

$$\hbar\frac{\partial^2 u_1}{\partial x^2} + \hbar\frac{\partial u_1}{\partial x}\frac{\partial^2 u_0}{\partial x^2} +$$

$$\hbar\frac{\partial u_0}{\partial x}\frac{\partial^2 u_1}{\partial x^2}.$$

$$u_2 = \int_0^y \int_0^y u_1^{''}\, dx dx + \hbar\int_0^y \int_0^y \frac{\partial^2 u_1}{\partial y^2}\, dy dy + \hbar\int_0^y \int_0^y \hbar\frac{\partial u_1}{\partial x}\frac{\partial^2 u_0}{d}y dy + \hbar\frac{\partial u_0}{\partial x}\frac{\partial^2 u_1}{\partial x^2}$$

$$= -\hbar y^2 + \hbar\int_0^y \int_0^y -2\hbar\, dy dy$$

$$u_2 = -\hbar y^2 - \hbar^2 y^2$$

$$= (-\hbar - \hbar^2)y^2.$$

From the third derivative,

$$(1-q)(6u_3^{''} + \cdots) + (-1)(2u_2^{''} + 6u_3^{''}q + \cdots) + (-1)(2u_2^{''} + 6u_3^{''}q + \cdots) + (-1)(2u_2^{''} + 6u_3^{''}q + \cdots)$$

$$= 3\hbar[\frac{\partial^2}{\partial y^2}(2u_2 + 6u_3 q + \cdots)$$

$$+ \frac{\partial}{\partial x}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)\frac{\partial^2}{\partial x^2}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)$$

$$+ \frac{\partial}{\partial x}(2u_2^{''} + 6u_3^{''}q + \cdots)\frac{\partial^2}{\partial x^2}(u_0 + u_1 q + u_2 q^2 + u_3 q^3 + \cdots)$$

$$+ \frac{\partial}{\partial x}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)\frac{\partial^2}{\partial x^2}(u_1 + 2u_2 q + 3u_3 q^2 + \cdots)]$$

$$+ \hbar q[6u_3 + \cdots$$

Take $q = 0$ to get

$$6u_3^{''} - 2u_2^{''} - 2u_2^{''} - 2u_2^{''} = 6\hbar\frac{\partial^2 u_2}{\partial y^2} + 3\hbar\frac{\partial u_1}{\partial x}\frac{\partial^2 u_1}{\partial x^2} + 6\hbar\frac{\partial u_2}{\partial x}\frac{\partial^2 u_0}{\partial x^2} +$$

$$6\hbar\frac{\partial^2 u_0}{\partial x^2} + 3\hbar\frac{\partial u_1}{\partial x}\frac{\partial^2 u_1}{\partial x^2} +$$

$$u_3^{''} = \int_0^y\int_0^y u_2^{''}dxdx + \hbar\int_0^y\int_0^y\frac{\partial^2 u_2}{\partial y^2}dydy + \hbar\int_0^y\int_0^y \hbar\frac{\partial u_1}{\partial x}\frac{\partial^2 u_1}{d}ydy +$$

$$\hbar\int_0^y\int_0^y\frac{\partial u_2}{\partial x}\frac{\partial^2 u_0}{\partial x^2}dxdx + \hbar\int_0^y\int_0^y\frac{\partial u_0}{\partial x}\frac{\partial^2 u_2}{\partial x^2}dxdx$$

$$u_3 = (-\hbar - \hbar^2)y^2 + \hbar(-\hbar - \hbar^2)y^2.$$

Thus the solution is

$$u(y) = xy - \hbar y^2 + (-\hbar - \hbar^2)y^2 + \hbar(-\hbar - \hbar^2)y^2 + \cdots$$

When $\hbar = -1$, then we have

$$u(y) = xy + y^2 + (0)y^2 - (0)y^2 + \cdots$$

$$= y^2 + xy.$$

which is the exact solution.

In this chapter, it has clearly been demonstrated that both DTM and HAM can be directly applied to the equations without requiring linearisation, discretisation or perturbation. We have seen that in most examples that we have solved both methods obtain series solutions that converge to the exact solution. The methods do not involve large amount of computational work. The other strengths of the these two methods is that they do not require an initial guess of the solution as is the case with the other methods like the HPM. Further, HAM contains the auxiliary parameter $\hbar$ that provides a simple way to adjust and control the convergence region and rate of convergence of the series. Finally with these two methods good approximations are obtained in few terms. However, we have observed that the weakness of both methods is that they are not applicable to some classes of nonlinear differential equations.

Chapter 4

Adomian decomposition method

In this chapter, a description of ADM is given. Adomian polynomials for various forms of non-linearity are calculated. The chapter concludes by applying the ADM to solve some selected examples of nonlinear differential equations.

4.1 Description of the Adomian decomposition method.

Consider the following nonlinear differential equation

$$L(y(x)) + R(y(x)) + N(y(x)) = g(x) \tag{4.1.1}$$

where L is the highest order differential operator , R is the linear remainder of the differential operator, N is the nonlinear operator, and $g(x)$ is an inhomogeneous term. The choice for L and its inverse L^{-1} are determined by the equation to be solved. For example, for cases of ordinary differential equations, we choose a different form for the linear operator [31, 32]. Generally we choose

$$L = \frac{d^n}{dx^n}(.)$$

for nth-order ordinary differential equations and thus its inverse L^{-1} follows as the nth-order definite integration operator from 0 to x. Applying the inverse operator L^{-1} to both sides of (4.1.1) we obtain

$$L^{-1}Ly(x) = L^{-1}g(x) - L^{-1}[R(y(x)) + N(y(x))]$$

$$y(x) - \varphi = L^{-1}g(x) - L^{-1}[R(y(x)) + N(y(x))]$$

$$y(x) = \gamma(x) - L^{-1}[R(y(x)) + N(y(x))] \tag{4.1.2}$$

where φ includes the initial values as

$$\varphi = \sum_{r=0}^{n-1} \alpha_r \frac{x^r}{r!}, \quad n = 1, 2, 3, \cdots$$

and

$$\gamma(x) = \varphi + L^{-1}g(x).$$

The ADM decomposes the solution into a series

$$y(x) = \sum_{i=0}^{\infty} y_i(x), \tag{4.1.3}$$

and then decomposes the nonlinear term $N(y(x))$ into a series

$$N(y(x)) = \sum_{i=0}^{\infty} A_i, \tag{4.1.4}$$

where the A_i's are the Adomian polynomials that are evaluated according to the formula

$$A_i = \frac{1}{i!} \frac{d^i}{d\lambda^i} \left[F\left(\sum_{p=0}^{i} \lambda^p y_p \right) \right]_{\lambda=0}, i = 0, 1, 2, \cdots \tag{4.1.5}$$

where F represents the nonlinear term. Substituting (4.1.3) and (4.1.4) into (4.1.2), we obtain

$$\sum_{i=0}^{\infty} y_i(x) = \gamma(x) - L^{-1}[R\left(\sum_{i=0}^{\infty} y_i(x) \right) + \sum_{i=0}^{\infty} A_i]$$

where R is the lower order derivative which is assumed to be invertible. The recursive relation is found to be

$$y_0(x) = \gamma(x)$$
$$y_{q+1}(x) = -L^{-1}[Ry_q(x) + A_q], \quad q \geq 0.$$

4.2 Calculation of Adomian polynomials

In this section, we calculate the Adomian polynomials for different forms of nonlinearity that appear in nonlinear ordinary differential equations or partial differential equations. The Adomian polynomials were first constructed by [31, 32, 33]. The general formula for calculating these Adomian polynomials is given by formula (4.1.5). There are other ways to calculate the Adomian polynomials.

We now use formula (4.1.5) to calculate the Adomian polynomials for different forms of nonlinearity that appear in differential equations.

4.2.1 Nonlinearity of the form $y\frac{dy}{dx}$.

$$A_0 = \frac{1}{0!} \frac{d^0}{d\lambda^0} [F(y_0)] = y_0 \frac{dy_0}{dx}$$
$$A_1 = \frac{1}{1!} \frac{d}{d\lambda} [F(y_0 + \lambda y_1)]_{\lambda=0}$$
$$= \frac{d}{d\lambda} [(y_0 + \lambda y_1) \frac{d}{dx}(y_0 + \lambda y_1)]_{\lambda=0}$$
$$= (y_0 + \lambda y_1) \frac{d}{dx} y_1 + y_1 \frac{d}{dx}(y_0 + \lambda y_1)_{\lambda=0}$$
$$= y_0 \frac{dy_1}{dx} + y_1 \frac{dy_0}{dx}$$

44

$$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}[F(y_0 + \lambda y_1 + \lambda^2 y_2)]_{\lambda=0}$$

$$= \frac{1}{2!}\frac{d^2}{d\lambda^2}[(y_0 + \lambda y_1 + \lambda^2 y_2)\frac{d}{dx}(y_0 + \lambda y_1 + \lambda^2 y_2)]_{\lambda=0}$$

$$= \frac{1}{2!}\frac{d}{d\lambda}[(y_0 + \lambda y_1 + \lambda^2 y_2)\frac{d}{dx}(y_1 + 2\lambda y_2) + (y_1 + 2\lambda y_2)\frac{d}{dx}(y_0 + \lambda y_1 + \lambda^2 y_2)]_{\lambda=0}$$

$$= \frac{1}{2!}[(y_0 + \lambda y_1 + \lambda^2 y_2)\frac{d}{dx}(2y_2) + (y_1 + \lambda y_2)\frac{d}{dx}(y_1 + 2\lambda y_2) + (2y_2)\frac{d}{dx}(y_0 + \lambda y_1 + \lambda^2 y_2) + (y_1 + \lambda y_2)$$

$$\frac{d}{dx}(y_1 + 2\lambda y_2)]_{\lambda=0}$$

$$= \frac{1}{2!}[2y_0\frac{dy_2}{dx} + y_1\frac{dy_1}{dx} + 2y_2\frac{dy_0}{dx} + y_1\frac{dy_1}{dx}]$$

$$= \frac{1}{2!}[2y_0\frac{dy_2}{dx} + 2y_1\frac{dy_1}{dx} + 2y_2\frac{dy_0}{dx}]$$

$$= y_0\frac{dy_2}{dx} + y_1\frac{dy_1}{dx} + y_2\frac{dy_0}{dx}$$

$$A_3 = \frac{1}{3!}\frac{d^3}{d\lambda^3}[F(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)]_{\lambda=0}$$

$$= \frac{1}{3!}\frac{d^3}{d\lambda^3}[(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)\frac{d}{dx}(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)]_{\lambda=0}$$

$$= \frac{1}{3!}\frac{d^2}{d\lambda^2}[(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)\frac{d}{dx}(y_1 + 2\lambda y_2 + 3\lambda^2 y_3) + (y_1 + 2\lambda y_2 + 3\lambda^2 y_3)$$

$$\frac{d}{dx}(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)]_{\lambda=0}$$

$$= \frac{1}{3!}\frac{d}{d\lambda}[(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)\frac{d}{dx}(2y_2 + 6\lambda y_3) + (y_1 + 2\lambda y_2 + 3\lambda^2 y_3)\frac{d}{dx}(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)$$

$$+ (2y_2 + 6\lambda y_3)\frac{d}{dx}(y_1 + 2\lambda y_2 + 3\lambda^2 y_3) + (y_1 + 2\lambda y_2 + 3\lambda^2 y_3)\frac{d}{dx}(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)]_{\lambda=0}$$

$$= \frac{1}{3!}[(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)\frac{d}{dx}(6y_3) + (y_1 + 2\lambda y_2 + 3\lambda^2 y_3)\frac{d}{dx}(2y_2 + 6\lambda y_3) + (2y_2 + 6\lambda y_3)$$

$$\frac{d}{dx}(y_1 + 2\lambda y_2 + 3\lambda^2 y_3) + (y_1 + 2\lambda y_2 + 3\lambda^2 y_3)\frac{d}{dx}(2y_2 + 6\lambda y_3) + 6y_3\frac{d}{dx}(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)$$

$$+ (2y_2 + 6\lambda y_3)\frac{d}{dx}(y_1 + 2\lambda y_2 + 3\lambda^2 y_3) + (y_1 + 2\lambda y_2 + 3\lambda^2 y_3)\frac{d}{dx}(2y_2 + 6\lambda y_3) +$$

$$(2y_2 + 6\lambda y_3)\frac{d}{dx}(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)]_{\lambda=0}$$

$$= \frac{1}{3!}[6y_0\frac{dy_3}{dx} + 2y_1\frac{dy_2}{dx} + 2y_2\frac{dy_1}{dx} + 2y_1\frac{dy_2}{dx} + 6y_3\frac{dy_0}{dx} + 2y_2\frac{dy_1}{dx} + 2y_1\frac{dy_2}{dx} + 2y_2\frac{dy_1}{dx}]$$

$$= \frac{1}{3!}[6y_0\frac{dy_3}{dx} + 6y_1\frac{dy_2}{dx} + 6y_2\frac{dy_1}{dx} + 6y_3\frac{dy_0}{dx}]$$

$$= y_0\frac{dy_3}{dx} + y_1\frac{dy_2}{dx} + y_2\frac{dy_1}{dx} + y_3\frac{dy_0}{dx}$$

$$\vdots$$

4.2.2 Nonlinearity of the form $y^2 + y^3$.

$$A_0 = \frac{1}{0!}\frac{d^0}{d\lambda^0}[F(y_0)] = y_0^2 + y_0^3$$

$$A_1 = \frac{1}{1!}\frac{d}{d\lambda}[F(y_0 + \lambda y_1)]_{\lambda=0}$$

$$= \frac{d}{d\lambda}[(y_0 + \lambda y_1)^2 + (y_0 + \lambda y_1)^3]_{\lambda=0}$$

$$= [2(y_0 + \lambda y_1)(y_1) + 3(y_0 + \lambda y_1)^2(y_1)]_{\lambda=0}$$

$$= 2y_0 y_1 + 3y_0 y_1^2$$

$$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}[F(y_0 + \lambda y_1 + \lambda^2 y_2)]_{\lambda=0}$$

$$= \frac{1}{2!}\frac{d^2}{d\lambda^2}[(y_0 + \lambda y_1 + \lambda^2 y_2)^2 + (y_0 + \lambda y_1 + \lambda^2 y_2)^3]_{\lambda=0}$$

$$= \frac{1}{2!}\frac{d}{d\lambda}[2(y_0 + \lambda y_1 + \lambda^2 y_2)(y_1 + 2\lambda y_2) + 3(y_0 + \lambda y_1 + \lambda^2 y_2)^2(y_1 + 2\lambda y_2)]_{\lambda=0}$$

$$= \frac{1}{2!}[2(y_0 + \lambda y_1 + \lambda^2 y_2)(2y_2) + 2(y_1 + 2\lambda y_2)^2 + 6(y_0 + \lambda y_1 + \lambda^2 y_2)(y_1 + 2\lambda y_2)^2 +$$

$$3(y_0 + \lambda y_1 + \lambda^2 y_2)^2(2y_2)]_{\lambda=0}$$

$$= 2y_0 y_2 + y_1^2 + 3y_0 y_1^2 + 3y_0^2 y_2$$

$$A_3 = \frac{1}{3!}\frac{d^3}{d\lambda^3}[F(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)]_{\lambda=0}$$

$$= \frac{1}{3!}\frac{d^3}{d\lambda^3}[(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)^2 + (y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)^3]_{\lambda=0}$$

$$= \frac{1}{3!}\frac{d^2}{d\lambda^2}[2(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(y_1 + 2\lambda y_2 + 3\lambda^2 y_3) + 3(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)^2$$

$$(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)]_{\lambda=0}$$

$$= \frac{1}{3!}\frac{d}{d\lambda}[2(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(2y_2 + 6\lambda y_3) + 2(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)^2 +$$

$$6(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)^2 + 3(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)^2(2y_2 + 6\lambda y_3)]_{\lambda=0}$$

$$= \frac{1}{3!}[4(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)(y_2 + 6\lambda y_3) + 2(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)(y_2 + 6\lambda y_3) +$$

$$2(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(6y_3)]_{\lambda=0} + 6(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)^3$$

$$+ 12(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)(2y_2 + 6\lambda y_3) +$$

$$6(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)(2y_2 + 6\lambda y_3) +$$

$$3(y_0 + \lambda + y_1 + \lambda^2 y_2 + \lambda^3 y_3)^2(6y_3)]_{\lambda=0}$$

$$= \frac{1}{3}[8y_1 y_2 + 4y_1 y_2 + 12y_0 y_3 + 6y_1^3 + 24y_0 y_1 y_2 + 12y_0 y_1 y_2 + 18y_0^2 y_3]$$

$$= \frac{1}{3}[12y_1 y_2 + 12y_0 y_3 + 6y_1^3 + 36y_0 y_1 y_2 + 18y_0^2 y_3]$$

$$= 2y_1 y_2 + 2y_0 y_3 + y_1^3 + 6y_0 y_1 y_2 + 3y_0^2 y_3$$

$$\vdots$$

4.2.3 Nonlinearity of the form y^3.

$$A_0 = \frac{1}{0!}\frac{d^0}{d\lambda^0}[F(y_0)] = y_0^3$$

$$A_1 = \frac{1}{1!}\frac{d}{d\lambda}[F(y_0 + \lambda y_1)]_{\lambda=0}$$

$$= \frac{d}{d\lambda}[(y_0 + \lambda y_1)^3]_{\lambda=0}$$

$$= [3(y_0 + \lambda y_1)^2(y_1)]_{\lambda=0}$$

$$= 3y_0^2 y_1$$

$$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}[F(y_0 + \lambda y_1 + \lambda^2 y_2)]_{\lambda=0}$$

$$= \frac{1}{2!}\frac{d^2}{d\lambda^2}[(y_0 + \lambda y_1 + \lambda^2 y_2)^3]_{\lambda=0}$$

$$= \frac{1}{2!}\frac{d}{d\lambda}[3(y_0 + \lambda y_1 + \lambda^2 y_2)^2(y_1 + 2\lambda y_2)]_{\lambda=0}$$

$$= \frac{3}{2!}[2(y_0 + \lambda y_1 + \lambda^2 y_2)(y_1 + 2\lambda y_2)^2 + (y_0 + \lambda y_1 + \lambda^2 y_2)^2(2y_2)]_{\lambda=0}$$

$$= 3[y_0 y_1^2 + y_0^2 y_2]$$

$$= 3y_0 y_1^2 + 3y_0^2 y_2$$

$$A_3 = \frac{1}{3!}\frac{d^3}{d\lambda^3}[F(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)]_{\lambda=0}$$

$$= \frac{1}{3!}\frac{d^3}{d\lambda^3}[(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)^3]_{\lambda=0}$$

$$= \frac{1}{3!}\frac{d^2}{d\lambda^2}[3(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)^2(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)]_{\lambda=0}$$

$$= \frac{1}{2}\frac{d}{d\lambda}[2(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)^2 + (y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)^2(2y_2 + 6\lambda y_3)]_{\lambda=0}$$

$$= \frac{1}{2}[2((y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)(2y_2 + 6\lambda y_3) + (y_1 + 2\lambda y_2 + 3\lambda^2 y_3)^3) +$$

$$2(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)(2y_2 + 6\lambda y_3) + (y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)^2(6y_3)]_{\lambda=0}$$

$$= 4y_0 y_1 y_2 + y_1^3 + 2y_0 y_1 y_2 + 3y_0^2 y_3$$

$$= 3y_0^2 y_3 + 6y_0 y_1 y_2 + y_1^3$$

$$\vdots$$

4.2.4 Nonlinearity of the form y^4.

$$A_0 = \frac{1}{0!}\frac{d^0}{d\lambda^0}[F(y_0)] = y_0^4$$

$$A_1 = \frac{1}{1!}\frac{d}{d\lambda}[F(y_0 + \lambda y_1)]_{\lambda=0}$$

$$= \frac{1}{1!}\frac{d}{d\lambda}[(y_0 + \lambda y_1)^4]_{\lambda=0}$$

$$= [4(y_0 + \lambda y_1)^3 (y_1)]_{\lambda=0}$$
$$= 4y_0^3 y_1$$

$$A_2 = \frac{1}{2!} \frac{d^2}{d\lambda^2} [F(y_0 + \lambda y_1 + \lambda^2 y_2)]_{\lambda=0}$$

$$= \frac{1}{2!} \frac{d^2}{d\lambda^2} [(y_0 + \lambda y_1 + \lambda^2 y_2)^4]_{\lambda=0}$$

$$= \frac{1}{2!} \frac{d}{d\lambda} [4(y_0 + \lambda y_1 + \lambda^2 y_2)^3 (y_1 + 2\lambda y_2)]_{\lambda=0}$$

$$= 2[3(y_0 + \lambda y_1 + \lambda^2 y_2)^2 (y_1 + 2\lambda y_2)^2 + (y_0 + \lambda y_1 + \lambda^2 y_2)^3 (2y_2)]_{\lambda=0}$$

$$= 6y_0^2 y_1 + 4y_0^3 y_2$$

$$A_3 = \frac{1}{3!} \frac{d^3}{d\lambda^3} [F(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)]_{\lambda=0}$$

$$= \frac{1}{3!} \frac{d^3}{d\lambda^3} [(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)^4]_{\lambda=0}$$

$$= \frac{1}{3!} \frac{d^2}{d\lambda^2} [4(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)^3 (y_1 + 2\lambda y_2 + 3\lambda^2 y_3)]_{\lambda=0}$$

$$= \frac{2}{3} \frac{d}{d\lambda} [3(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)^2 (y_1 + 2\lambda y_2 + 3\lambda^2 y_3)^2 + (y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)^3 (2y_2 + 6\lambda y_3)]_{\lambda=0}$$

$$= \frac{2}{3} [6(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)^3 +$$

$$6(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)^2 (y_1 + 2\lambda y_2 + 3\lambda^2 y_3)(2y_2 + 6\lambda y_3) +$$

$$3(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)^2 (y_1 + 2\lambda y_2 + 3\lambda^2 y_3)(2y_2 + 6\lambda y_3) + (y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)^3 (6y_3)]_{\lambda=0}$$

$$= \frac{2}{3} [6y_0 y_1^3 + 12y_0^2 y_1 y_2 + 6y_0^2 y_1 y_2 + 6y_0^3 y_3]$$

$$= \frac{2}{3} [6y_0 y_1^3 + 18y_0^2 y_1 y_2 + 6y_0^3 y_3]$$

$$= 4y_0 y_1^3 + 12y_0^2 y_1 y_2 + 4y_0^3 y_3$$

$$\vdots$$

4.2.5 Trigonometry nonlinearity. i. $\sin y$.

$$A_0 = \frac{1}{0!} \frac{d^0}{d\lambda^0} [F(y_0)] = \sin y_0$$

$$A_1 = \frac{1}{1!} \frac{d}{d\lambda} [F(y_0 + \lambda y_1)]_{\lambda=0}$$

$$= \frac{d}{d\lambda} [\sin(y_0 + \lambda y_1)]_{\lambda=0}$$

$$= [\cos(y_0 + \lambda y_1) y_1]_{\lambda=0}$$

$$= y_1 \cos y_0$$

$$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}[F(y_0 + \lambda y_1 + \lambda^2 y_2)]_{\lambda=0}$$

$$= \frac{1}{2!}\frac{d^2}{d\lambda^2}[\sin(y_0 + \lambda y_1 + \lambda^2 y_2)]_{\lambda=0}$$

$$= \frac{1}{2!}\frac{d}{d\lambda}[\cos(y_0 + \lambda y_1 + \lambda^2 y_2)(y_1 + 2\lambda y_2)]_{\lambda=0}$$

$$= \frac{1}{2!}[-\sin(y_0 + \lambda y_1 + \lambda^2 y_2)(y_1 + 2\lambda y_2)^2 + \cos(y_0 + \lambda y_1 + \lambda^2 y_2)(2y_2)]_{\lambda=0}$$

$$= \frac{1}{2!}[-y_1^2 \sin y_0 + 2y_2 \cos y_0]$$

$$= y_2 \cos y_0 - \frac{1}{2!}y_1^2 \sin y_0$$

$$A_3 = \frac{1}{3!}\frac{d^3}{d\lambda^3}[F(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)]_{\lambda=0}$$

$$= \frac{1}{3!}\frac{d^3}{d\lambda^3}[\sin(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)]_{\lambda=0}$$

$$= \frac{1}{3!}\frac{d^2}{d\lambda^2}[\cos(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)]_{\lambda=0}$$

$$= \frac{1}{3!}\frac{d}{d\lambda}[-\sin(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)^2 + \cos(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)$$
$$(2y_2 + 6\lambda y_3)]_{\lambda=0}$$

$$= \frac{1}{3!}[-\cos(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)^3$$
$$- 2\sin(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)(2y_2 + 6\lambda y_3) + \cos(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)$$
$$(6y_3) - \sin(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(2y_2 + 6\lambda y_3)(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)]_{\lambda=0}$$

$$= \frac{1}{3!}[y_1^3 \cos y_0 - 4y_1 y_2 \sin y_0 + 6y_3 \cos y_0 - 2y_1 y_2 \sin y_0]$$

$$= \frac{1}{3!}[y_1^3 \cos y_0 - 6y_1 y_2 \sin y_0 + 6y_3 \cos y_0]$$

$$= y_3 \cos y_0 - y_1 y_2 \sin y_0 - \frac{1}{3!}y_1^3 \cos y_0$$

$$\vdots$$

ii. $\cos 2y$.

$$A_0 = \frac{1}{0!}\frac{d^0}{d\lambda^0}[F(y_0)] = \cos 2y_0$$

$$A_1 = \frac{1}{1!}\frac{d}{d\lambda}[F(y_0 + \lambda y_1)]_{\lambda=0}$$

$$= \frac{d}{d\lambda}[\cos 2(y_0 + \lambda y_1)]_{\lambda=0}$$

$$= [-2\sin 2(y_0 + \lambda y_1)y_1]_{\lambda=0}$$

$$= -2y_1 \sin 2y_0$$

$$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}[F(y_0 + \lambda y_1 + \lambda^2 y_2)]_{\lambda=0}$$

$$= \frac{1}{2!}\frac{d^2}{d\lambda^2}[\cos 2(y_0 + \lambda y_1 + \lambda^2 y_2)]_{\lambda=0}$$

$$= \frac{1}{2!}\frac{d}{d\lambda}[-2\sin 2(y_0 + \lambda y_1 + \lambda^2 y_2)(y_1 + 2\lambda y_2)]_{\lambda=0}$$

$$= \frac{1}{2!}[-4\cos 2(y_0 + \lambda y_1 + \lambda^2 y_2)(y_1 + 2\lambda y_2)^2 - 2\sin 2(y_0 + \lambda y_1 + \lambda^2 y_2)(2y_2)]_{\lambda=0}$$

$$= \frac{1}{2!}[-4y_1^2\cos 2y_0 - 4y_2\sin 2y_0]$$

$$= -2y_2\sin 2y_0 - 2y_1^2\cos 2y_0$$

$$A_3 = \frac{1}{3!}\frac{d^3}{d\lambda^3}[F(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)]_{\lambda=0}$$

$$= \frac{1}{3!}\frac{d^3}{d\lambda^3}[\cos 2(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)]_{\lambda=0}$$

$$= \frac{1}{3!}\frac{d^2}{d\lambda^2}[-2\sin 2(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)]_{\lambda=0}$$

$$= \frac{1}{3!}\frac{d}{d\lambda}[-4\cos 2(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)^2 - 2\sin 2(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)$$
$$(2y_2 + 6\lambda y_3)]_{\lambda=0}$$

$$= \frac{1}{3!}[8\sin 2(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)^3 -$$
$$8\cos 2(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)(2y_2 + 6\lambda y_3) - 4\cos 2(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)$$
$$(6y_3) - 2\sin 2(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(2y_2 + 6\lambda y_3)(y_1 + 2\lambda y_2 + 3\lambda^2 y_3)]_{\lambda=0}$$

$$= \frac{1}{3!}[8y_1^3\sin 2y_0 - 16y_1y_2\cos 2y_0 - 12y_3\cos 2y_0 - 8y_1y_2\cos 2y_0]$$

$$= \frac{1}{3!}[8y_1^3\sin 2y_0 - 24y_1y_2\cos 2y_0 - 12y_3\sin 2y_0]$$

$$= -2y_3\sin 2y_0 - 4y_1y_2\cos 2y_0 + \frac{4}{3!}y_1^3\sin 2y_0$$

$$\vdots$$

4.2.6 Exponential nonlinearity. e^y.

$$A_0 = \frac{1}{0!}\frac{d^0}{d\lambda^0}[F(y_0)] = e^{y_0}$$

$$A_1 = \frac{1}{1!}\frac{d}{d\lambda}[F(y_0 + \lambda y_1)]_{\lambda=0}$$

$$= \frac{d}{d\lambda}[e^{(y_0 + \lambda y_1)}]_{\lambda=0}$$

$$= \frac{d}{d\lambda}[e^{y_0}e^{\lambda y_1}]_{\lambda=0}$$

$$= [y_1 e^{y_0 + \lambda y_1}]_{\lambda=0}$$

$$= y_1 e^{y_0}$$

$$A_2 = \frac{1}{2!} \frac{d^2}{d\lambda^2} [F(y_0 + \lambda y_1 + \lambda^2 y_2)]_{\lambda=0}$$

$$= \frac{1}{2!} \frac{d^2}{d\lambda^2} [e^{(y_0 + \lambda y_1 + \lambda^2 y_2)}]_{\lambda=0}$$

$$= \frac{1}{2!} \frac{d^2}{d\lambda^2} [e^{y_0} e^{\lambda y_1} e^{\lambda^2 y_2}]_{\lambda=0}$$

$$= \frac{1}{2!} \frac{d}{d\lambda} e^{y_0} [y_1 e^{\lambda y_1} e^{\lambda^2 y_2} + e^{\lambda y_1} 2\lambda y_2 e^{\lambda^2 y_2}]_{\lambda=0}$$

$$= \frac{1}{2!} \frac{d}{d\lambda} e^{y_0} [y_1^2 e^{\lambda y_1} e^{\lambda^2 y_2} + y_1 e^{\lambda y_1} 2\lambda y_2 e^{\lambda^2 y_2} + y_1 e^{\lambda y_1} 2\lambda y_2 e^{\lambda^2 y_2} + e^{\lambda y_1}(2 y_2 e^{\lambda^2 y_2} + 4\lambda^2 y_2^2 e^{\lambda^2 y_2})]_{\lambda=0}$$

$$= \frac{1}{2!} e^{y_0} [y_1^2 + 2 y_2]$$

$$= y_2 e^{y_0} + \frac{1}{2!} y_1^2 e^{y_0}$$

$$\vdots$$

4.3 Solving nonlinear ordinary differential equations using Adomian decomposition method.

We apply ADM to selected examples of ODEs to illustrate the working of the method.

Example 4.3.1. Solve

$$y' + y - y^2 = 0, \quad y(0) = 2.$$

The exact solution for this problem is $y(x) = \frac{-2}{-2+e^x}$.

Solution 4.3.1.1. Writing the given differential equation in an operator form gives

$$Ly + y - y^2 = 0, \tag{4.3.1}$$

where $L = \frac{d}{dx}$. Applying the inverse operator L^{-1} to both sides of (4.3.1) we have

$$\int_0^x Ly dx + \int_0^x y dx - \int_0^x y^2 dx = 0.$$

which is

$$y(x) - y(0) = -\int_0^x y\,dx + \int_0^x y^2\,dx$$

$$y(x) = 2 + \int_0^x y^2\,dx - \int_0^x y\,dx$$

Using the assumptions of the Adomian decomposition method we have

$$\sum_{n=0}^{\infty} y_n(x) = 2 + \int_0^x \sum_{n=0}^{\infty} A_n\,dx - \int_0^x \sum_{n=0}^{\infty} y_n(x)\,dx$$

where the A_n's are the Adomian polynomials. This leads to

$$y_0 = 2$$

$$y_{p+1} = \int_0^x A_p\,dx - \int_0^x y_p(x)\,dx.$$

Calculating the first few components of the Adomian polynomials we obtain

$$y_1 = \int_0^x A_0\,dx - \int_0^x y_0(x)\,dx$$

$$A_0 = y_0^2$$

$$= 4$$

$$y_0 = 2$$

$$y_1 = \int_0^x 4\,dx - \int_0^x 2\,dx$$

$$= 4x - 2x$$

$$= 2x$$

$$y_2 = \int_0^x A_1\,dx - \int_0^x y_1(x)\,dx$$

$$A_1 = 2y_0y_1$$

$$= 2(2)(2x)$$

$$= 8x$$

$$y_2 = \int_0^x 8x\,dx - \int_0^x 2x\,dx$$

$$= 3x^2$$

$$y_3 = \int_0^x A_2\,dx - \int_0^x y_2(x)\,dx$$

$$A_2 = 2y_0y_2 + y_1^2$$

$$= 2(2)(3x^2) + (2x)^2$$

$$= 16x^2$$

$$y_3 = \int_0^x 16x^2\,dx - \int_0^x 3x^2\,dx$$

$$= \frac{13}{3}x^3$$

and so on. Thus the solution is given by

$$y(x) = y_0 + y_1 + y_2 + y_3 + \cdots$$

$$= 2 + 2x + 3x^2 + \frac{13}{3}x^3 + \cdots$$

and the solution agrees with that obtained by HAM and DTM. Table 4.1 gives a comparison of the exact solution and the solution computed by ADM, DTM and HAM. The table shows that the three numerical solutions are a good approximate to the exact solution. In fact, Table 4.1 shows that if the truncation point is increased, the solutions by DTM, HAM and ADM will coincide with the exact solution. Thus the series solutions are in all cases very good approximates.

Table 4.1: Comparison between exact solution and numerical solution.

x	Exact solution	Solution by ADM	Solution by HAM	Solution by DTM
-0.5	1.43527	1.59896	1.59896	1.59896
-0.4	1.50412	1.56267	1.56267	1.56267
-0.3	1.58833	1.60362	1.60362	1.60362
-0.2	1.69309	1.69533	1.69533	1.69533
-0.1	1.82621	1.82629	1.82629	1.82629
0.0	2.00000	2.00000	2.00000	2.00000
0.1	2.23506	2.23496	2.23496	2.23496
0.2	2.56872	2.56467	2.56467	2.56467
0.3	3.07625	2.56467	2.56467	2.56467
0.4	3.93565	3.71733	3.71733	3.71733
0.5	5.69348	4.68229	4.68229	4.68229

Figure 4.1 is a graphical illustration of the comparison between the exact solution and the numerical solutions. From the figure it can be seen that for $-0.3 < x < 0.3$ the numerical results coincides exactly with the exact solution. For the values $x < -0.3$ and $x > 0.3$ there is a significant derivation between the numerical solution and the exact solution. This can be attributed to the fact that the series solution only has a few terms (4 terms). Increasing the terms of the series will improve the numerical solution.
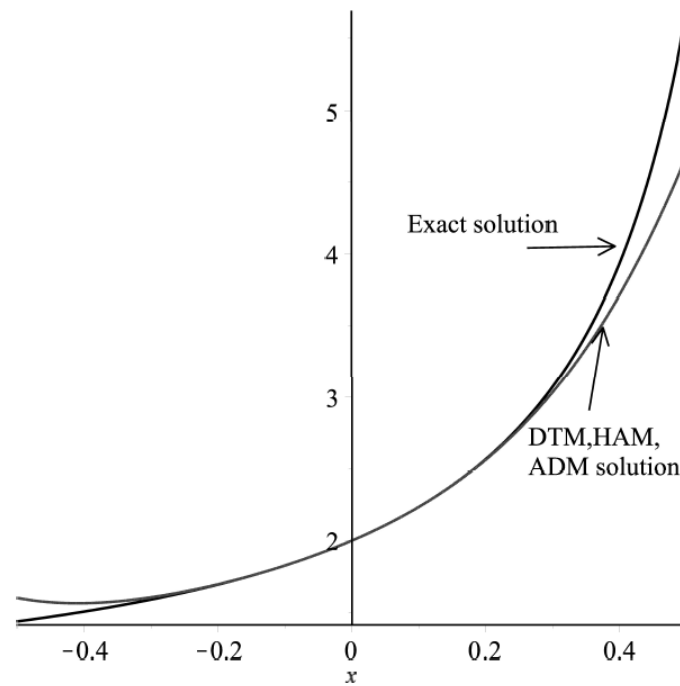
Figure 4.1: Comparison between exact solution and numerical solution.

Example 4.3.2. Solve

$$y'' + y^2 = 0, \qquad y(0) = 1 \quad and \quad y'(0) = 0.$$

Solution 4.3.2.1. Writing the given differential equation in an operator form gives

$$Ly + y^2 = 0, \qquad\qquad (4.3.2)$$

where $L = \frac{d^2}{dx^2}$. Applying the inverse operator L^{-1} to both sides of (4.3.2) we have

$$\int_0^x \int_0^x Ly\,dx\,dx + \int_0^x \int_0^x y^2\,dx\,dx = 0,$$

which is

$$y(x) - xy'(0) - y(0) = -\int_0^x \int_0^x y^2\,dx\,dx,$$

$$y(x) = 1 - \int_0^x \int_0^x y^2\,dx\,dx.$$

Using the assumptions of the Adomian decomposition method we have

$$\sum_{n=0}^{\infty} y_n(x) = 1 - \int_0^x \int_0^x \sum_{n=0}^{\infty} A_n dx dx$$

where the A_n's are the Adomian polynomials. This leads to

$$y_0 = 1$$
$$y_{p+1} = -\int_0^x \int_0^x A_p dx dx$$

Calculating the first few components of the Adomian polynomials we obtain

$$y_1 = -\int_0^x \int_0^x A_0 dx dx$$
$$A_0 = y_0^2$$
$$= 1$$
$$y_1 = -\int_0^x \int_0^x 1 dx dx$$
$$= -\frac{x^2}{2}$$
$$y_2 = -\int_0^x \int_0^x A_1 dx dx$$
$$A_1 = 2y_0 y_1$$
$$= 2(1)\left(-\frac{x^2}{2}\right)$$
$$= -x^2$$

$$y_2 = \int_0^x \int_0^x x^2 dx dx$$
$$= \frac{x^4}{12}$$
$$y_3 = -\int_0^x \int_0^x A_2 dx dx$$
$$A_2 = 2y_0 y_2 + y_1^2$$
$$= 2(1)\left(\frac{x^4}{12}\right) + \left(-\frac{x^2}{2}\right)^2$$
$$= \frac{x^4}{6} + \frac{x^4}{4}$$
$$= \frac{5}{12} x^4$$
$$y_3 = -\frac{5}{12} \int_0^x \int_0^x x^4 dx dx$$
$$= -\frac{x^6}{72}$$

and so on. Thus the solution is given by

$$y(x) = y_0 + y_1 + y_2 + y_3 + \cdots$$
$$= 1 - \frac{x^2}{2} + \frac{x^4}{12} - \frac{x^6}{72} + \cdots$$

and the solution agrees with that obtained by HAM and DTM.

Example 4.3.3. Solve the Lane-Emden equation

$$y'' + \frac{6}{x} y' + 14y = -4 \ln y, \qquad y(0) = 1 \quad and \quad y'(0) = 0.$$

The Lane-Emden equation is used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behaviour of a spherical cloud of gas, isothermal gas spheres, and theory of thermionic currents[34]. The equation is thus of paramount importance in physiscs and the sciences in general.

Solution 4.3.3.1. The difficulty of the given type of Lane-Emden equation is the behaviour of the term $\frac{6}{x}$ when $x = 0$. This kind of Lane–Emden equation has been studied by [35]. To succeed in dealing with the singularity behaviour, we define the differential operator L in terms of the two

derivatives contained in the problem. Following [28], we write our given Lane-Emden equation in an operator form

$$Ly = 4lny - 14y \qquad (4.3.3)$$

where L is the differential operator defined by

$$L = x^{-2}\frac{d}{dx}\left(x^2\frac{d}{dx}\right).$$

Applying the inverse operator L^{-1} to both sides of (4.3.3) we have

$$\int_0^x x^{-6}\int_0^x x^6 Ly dx dx = -\int_0^x x^{-6}\int_0^x x^6 4ylny dx dx - \int_0^x x^{-6}\int_0^x x^6 14y dx dx.$$

This implies that

$$y(x) - xy'(0) - y(0) = -\int_0^x x^{-6}\int_0^x x^6 4ylny dx dx - \int_0^x x^{-6}\int_0^x x^6 14y dx dx.$$

Using the initial conditions gives

$$y(x) - 1 = -\int_0^x x^{-6}\int_0^x x^6 4ylny dx dx - \int_0^x x^{-6}\int_0^x x^6 14y dx dx.$$

Using the assumptions of decomposition method we have

$$\sum_{n=0}^{\infty} y_n(x) = 1 - 4\int_0^x x^{-6}\int_0^x x^6 \left(\sum_{n=0}^{\infty} A_n\right) dx dx - 14\int_0^x x^{-6}\int_0^x x^6 \left(\sum_{n=0}^{\infty} y_n(x)\right) dx dx.$$

This leads to

$$y_0 = 1$$
$$y_{p+1} = -4\int_0^x x^{-6}\int_0^x x^6(A_p) dx dx - 14\int_0^x x^{-6}\int_0^x x^6(y_p(x)) dx dx.$$

Calculating the first few components we obtain

$$y_1 = -4 \int_0^x x^{-6} \int_0^x x^6(A_0)dxdx - 14 \int_0^x x^{-6} \int_0^x x^6(y_0(x))dxdx$$

$$A_0 = y_0 ln y_0$$

$$= 0$$

$$y_1 = -14 \int_0^x x^{-6} \int_0^x x^6(1)dxdx$$

$$= -2 \int_0^x \left(\frac{1}{x^6}\right)(x^7) dx$$

$$= -x^2$$

$$y_2 = -4 \int_0^x x^{-6} \int_0^x x^6(A_1)dxdx - 14 \int_0^x x^{-6} \int_0^x x^6(y_1(x))dxdx$$

$$A_1 = y_1 + y_1 ln y_0$$

$$= -x^2$$

$$y_2 = -4 \int_0^x x^{-6} \int_0^x x^6(-x^2)dxdx - 14 \int_0^x x^{-6} \int_0^x x^6(-x^2)dxdx$$

$$= -4 \int_0^x x^{-6} \left(\frac{x^9}{9}\right) dx - 14 \int_0^x x^{-6} \left(\frac{x^9}{9}\right) dx$$

$$= \frac{4}{9}\left(\frac{x^4}{4}\right) + \frac{14}{9}\left(\frac{4}{4}\right)$$

$$= \frac{x^4}{9} + \frac{7}{18}x^4$$

$$= \frac{x^4}{2}$$

$$y_3 = -4 \int_0^x x^{-6} \int_0^x x^6(A_2)dxdx - 14 \int_0^x x^{-6} \int_0^x x^6(y_2(x))dxdx$$

$$A_2 = y_2 + y_2 ln y_0 + \frac{y_1^2}{2y_0}$$

$$= \frac{x^4}{2} + \frac{x^4}{2}$$

$$= x^4$$

$$y_3 = -4 \int_0^x x^{-6} \int_0^x x^6 (x^4) \, dx \, dx - 14 \int_0^x x^{-6} \int_0^x x^6 \left(\frac{x^4}{2}\right) dx \, dx$$

$$= -4 \int_0^x x^{-6} \left(\frac{x^{11}}{11}\right) dx - 14 \int_0^x x^{-6} \left(\frac{x^{11}}{22}\right) dx$$

$$= -4 \left(\frac{x^6}{66}\right) - 14 \left(\frac{x^6}{132}\right)$$

$$= -\frac{x^6}{6}$$

$$y_4 = -4 \int_0^x x^{-6} \int_0^x x^6 (A_3) \, dx \, dx - 14 \int_0^x x^{-6} \int_0^x x^6 (y_3(x)) \, dx \, dx$$

$$A_3 = y_3 + y_3 \ln y_0 + \frac{y_1 y_2}{y_0} - \frac{y_1^3}{6y_0^2}$$

$$= -\frac{x^6}{6} + (-x^2)\left(\frac{x^4}{2}\right) - \frac{(-x^2)^3}{6}$$

$$= -\frac{x^6}{6} - \frac{x^6}{2} + \frac{x^6}{6}$$

$$= -\frac{x^6}{2}$$

$$y_4 = -4 \int_0^x x^{-6} \int_0^x x^6 \left(-\frac{x^6}{2}\right) dx \, dx - 14 \int_0^x x^{-6} \int_0^x x^6 \left(-\frac{x^6}{6}\right) dx \, dx$$

$$= 2 \int_0^x x^{-6} \left(\frac{x^{13}}{13}\right) dx + \frac{14}{6} \int_0^x x^{-6} \left(\frac{x^{13}}{13}\right) dx$$

$$= 2 \left(\frac{x^8}{104}\right) + \frac{14}{6} \left(\frac{x^6}{104}\right)$$

$$= \frac{x^8}{24}$$

Thus the solution is given as

$$y(x) = y_0 + y_1 + y_2 + y_3 + \cdots$$

$$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}x^4 + \frac{x^8}{24} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n-2}}{(n-1)!} \cdot (-1)^{n+1}$$

It can be easily shown by the ratio test that this solution converges for all the values of n.

4.4 Solving nonlinear partial differential equations using Adomian decomposition method.

In this section we apply ADM to selected PDEs.

Example 4.4.1. Solve

$$u_t + u_x + u^2 = 0, \quad u(x,0) = \frac{1}{2x}.$$

Solution 4.4.1.1. Writing the given equation in an operator form we have

$$L_t u + L_x u + u^2 = 0$$

where

$$L_t = \frac{\partial}{\partial t}, \quad L_x = \frac{\partial}{\partial x}.$$

Making $L_t u$ the subject of the formula we obtain

$$L_t u = -L_x u - u^2. \tag{4.4.1}$$

Applying the inverse operator L_t^{-1} to both sides of (4.4.1) we have

$$\int_0^t L_t u \, dt = -\int_0^t L_x u \, dt - \int_0^t u^2 \, dt.$$

This implies that

$$u(x,t) - u(x,0) = -\int_0^t L_x u \, dt - \int_0^t u^2 \, dt.$$

Using the initial condition we have

$$u(x,t) - \frac{1}{2x} = -\int_0^t L_x u \, dt - \int_0^t u^2 \, dt.$$

where $N(y) = u^2$.

Now, using the assumptions of the Adomian decomposition method we have

$$\sum_{n=0}^{\infty} u_n(x,t) = \frac{1}{2x} - \int_0^t \sum_{n=0}^{\infty} L_x u_n \, dt - \int_0^t \sum_{n=0}^{\infty} A_n \, dt.$$

This leads to

$$u_0(x, t) = \frac{1}{2x},$$

$$u_{p+1} = -\int_0^t L_x u_p dt - \int_0^t A_p dt.$$

Calculating the first few components we obtain

$$u_1 = -\int_0^t L_x u_0 dt - \int_0^t A_0 dt$$

$$A_0 = u_0^2$$

$$= \left(\frac{1}{2x}\right)^2$$

$$= \frac{1}{4x^2}$$

$$u_1 = -\int_0^t L_x \left(\frac{1}{2x}\right) dt - \int_0^t \frac{1}{4x^2} dt$$

$$= \int_0^t \left(\frac{1}{2x^2}\right) dt - \int_0^t \frac{1}{4x^2} dt$$

$$= \frac{1}{2x^2} t - \frac{1}{4x^2}$$

$$= \frac{1}{4x^2} t$$

$$u_2 = -\int_0^t L_x u_1 dt - \int_0^t A_1 dt$$

$$A_1 = 2u_0 u_1$$

$$= 2 \left(\frac{1}{2x}\right) \left(\frac{1}{4x^2} t\right)$$

$$= \frac{1}{4x^3} t$$

$$L_x u_1 = \frac{\partial}{\partial x} \left(\frac{1}{2x}\right)$$

$$= -\frac{1}{2x^3} t$$

$$u_2 = -\int_0^t \left(\frac{1}{2x^3} t\right) dt - \int_0^t \left(\frac{1}{4x^3} t\right) dt$$

$$= \frac{1}{4x^3} t^2 - \frac{1}{16x^3} t^2$$

$$= \frac{1}{8x^3} t^2$$

$$u_3 = -\int_0^t L_x u_2 dt - \int_0^t A_2 dt$$

$$A_2 = 2u_0 u_2 + u_1^2$$

$$= 2\left(\frac{1}{2x}\right)\left(\frac{1}{8x^3}t^2\right) + \left(\frac{1}{4x^2}t\right)^2$$

$$= \frac{1}{8x^4}t^2 + \frac{1}{16x^4}t^2$$

$$= \frac{3}{16x^4}t^2$$

$$L_x u_2 = \frac{\partial}{\partial x}\left(\frac{1}{8x^3}t^2\right)$$

$$= -\frac{3}{8x^4}t^2$$

$$u_3 = \int_0^t \left(\frac{3}{8x^4}t^2\right) dt - \int_0^t \left(\frac{3}{16x^4}t^2\right) dt$$

$$= \frac{1}{8x^4}t^3 - \frac{1}{16x^4}t^3$$

$$= \frac{1}{16x^4}t^3$$

and so on. Thus the solution is given by

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots$$

$$= \frac{1}{2x} + \frac{1}{4x^2}t + \frac{1}{8x^3}t^2 + \frac{1}{16x^4}t^3 + \cdots$$

$$= \frac{1}{2x}\left(1 + \frac{1}{2x}t + \frac{1}{4x^2}t^2 + \frac{1}{8x^3}t^3 + \cdots\right)$$

$$= \frac{1}{2x}\sum_{n=0}^{\infty}\left(\frac{t}{2x}\right)^n$$

Example 4.4.2. Use ADM to solve the *modified Korteweg de-Vries (mKdV) equation*

$$u_t - 6u^2 u_x + u_{xxx} = 0$$

with the initial condition

$$u(x,0) = x.$$

Solution 4.4.2.1. Writing the given *mKdV* equation in an operator form we have

$$L_t u - 6u^2 L_x u + L_{xxx} u = 0,$$

where

$$L_t = \frac{\partial}{\partial t}, \qquad L_x = \frac{\partial}{\partial x}, \qquad and \qquad L_{xxx} = \frac{\partial^3}{\partial x^3}.$$

Making $L_t u$ the subject of the formula we obtain

$$L_t u = 6u^2 L_x u - L_{xxx} u. \qquad (4.4.2)$$

Applying the inverse operator L_t^{-1} to both sides of (4.4.2) we have

$$\int_0^t L_t u\, dt = 6 \int_0^t u^2 L_x u\, dt - \int_0^t L_{xxx} u\, dt.$$

This implies that

$$u(x,t) - u(x,0) = 6 \int_0^t u^2 L_x u\, dt - \int_0^t L_{xxx} u\, dt.$$

Using the initial condition we have

$$u(x,t) - x = 6 \int_0^t u^2 L_x u\, dt - \int_0^t L_{xxx} u\, dt.$$

where $N(y) = u^2 L_x u$.

Now, using the assumptions of the Adomian decomposition method we have

$$\sum_{n=0}^{\infty} u_n(x,t) = x + 6 \int_0^t \sum_{n=0}^{\infty} A_n\, dt - \int_0^t \sum_{n=0}^{\infty} L_{xxx} u_n\, dt.$$

This leads to

$$u_0(x,t) = x$$
$$u_{p+1} = 6 \int_0^t A_p\, dt - \int_0^t L_{xxx} u_p\, dt.$$

Calculating the first few components we obtain

$$u_1 = 6 \int_0^t A_0 dt - \int_0^t L_{xxx} u_0 dt$$

$$A_0 = u_0^2 \frac{\partial u_0}{\partial x}$$

$$= x^2$$

$$L_{xxx} u_0 = 0$$

$$u_1 = 6 \int_0^t x^2 dt$$

$$= 6x^2 t$$

$$u_2 = 6 \int_0^t A_1 dt - \int_0^t L_{xxx} u_1 dt$$

$$A_1 = 2u_0 u_1 \frac{\partial u_0}{\partial x} + u_0^2 \frac{\partial u_1}{\partial x}$$

$$= 2(x)(6xt)(1) + (x)^2 (6t)$$

$$= 12x^3 t + 12x^3 t$$

$$= 24x^3 t$$

$$L_{xxx} u_1 = \frac{\partial^3}{\partial x^3} (6x^2 t)$$

$$= 0$$

$$u_2 = 6 \int_0^t (24x^3 t) dt$$

$$= 6(12x^3 t^2)$$

$$= 72x^3 t^2$$

$$u_3 = 6 \int_0^t A_2 dt - \int_0^t L_{xxx} u_2 dt$$

$$A_2 = 2u_0 u_2 \frac{\partial u_0}{\partial x} + u_1^2 \frac{\partial u_0}{\partial x} + 2u_0 u_1 \frac{\partial u_1}{\partial x} + u_0^2 \frac{\partial u_2}{\partial x}$$

$$= 2(x)(72x^3 t^2) + (6x^2 t)^2 + 2(6x^2 t)(12xt) + x^2 (216x^2 t^2)$$

$$= 144x^4 t^2 + 36x^4 t^2 + 144x^4 t^2 + 216x^4 t^2$$

$$= 540x^4 t^2.$$

So we have

$$L_{xxx}u_2 = L_{xxx}(72x^3t^2)$$

$$= 432t^2$$

$$u_3 = 6\int_0^t (540x^4t^2)dt - \int_0^t 432t^2 dt$$

$$= 6(180x^4t^3) - 144t^3$$

$$= 1080x^4t^3 - 144t^3.$$

Thus the solution is given by

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots$$

$$= x + 6x^2t + 72x^3t^2 + 1080x^4t^3 - 144t^3 + \cdots$$

$$= x + 6x^2t + 72x^3t^2 + (1080x^4 - 144)t^3 + \cdots$$

and it coincides with the one obtained by HAM.

Example 4.4.3. Solve the nonlinear differential equation

$$u_{xx} + uu_y = 2y^2 + 2x^4y^3, \tag{4.4.3}$$

with the initial conditions

$$u(0,y) = 0, u_x(0,y) = 0,$$

where $u = u(x,y)$.

Solution 4.4.3.1. Writing (4.4.3) in an operator form we have

$$L_x u = 2y^2 + 2x^4y^3 - uu_y \tag{4.4.4}$$

where L_x is defined by

$$L_x = \frac{\partial^2}{\partial x^2}.$$

66

The inverse operator L_x^{-1} is defined by

$$L_x^{-1}(\cdot) = \int_0^x \int_0^x (\cdot) dx dx.$$

Applying the inverse operator L_x^{-1} to both sides of (4.4.4) we obtain

$$\int_0^x \int_0^x u_{xx} dx dx = \int_0^x \int_0^x (2y^2 + 2x^4 y^3) dx dx$$
$$- \int_0^x \int_0^x u u_y dx dx.$$

Then we get

$$u(x,y) - u(0,y) - x u_x(0,y) = \int_0^x \int_0^x (2y^2 + 2x^4 y^3) dx dx$$
$$- \int_0^x \int_0^x u u_y dx dx.$$

Using the given initial conditions and substituting for

$$u(x,y) = \sum_{i=0}^{\infty} u_i(x,y)$$

and for the nonlinear term

$$u u_y = \sum_{i=0}^{\infty} A_i$$

where the A_i's are the Adomian polynomials, we get

$$\sum_{i=0}^{\infty} u_i(x,y) = x^2 y^2 + \frac{1}{15} x^6 y^3 - \int_0^x \int_0^x \sum_{i=0}^{\infty} A_i dx dx$$

To use the ADM, we identify the component u_0 by

$$u_0(x,y) = x^2 y^2 + \frac{1}{15} x^6 y^3,$$

and the remaining term

$$- \int_0^x \int_0^x A_i dx dx$$

will be assigned to $u_1(x,y)$ among other terms. Consequently, we obtain the recursive relation

$$u_0(x,y) = x^2 y^2 + \frac{1}{15} x^6 y^3$$
$$u_{q+1}(x,y) = - \int_0^x \int_0^x A_q dx dx.$$

Calculating the first few components we have

$$u_1(x, y) = - \int_0^x \int_0^x A_0 dx dx$$

$$A_0 = u_0 u_{0y}$$

$$= \left(x^2 y^2 + \frac{1}{15} x^6 y^3 \right) \left(2x^2 y + \frac{1}{5} x^6 y^2 \right)$$

$$= 2x^4 y^3 + \frac{1}{5} x^8 y^4 + \frac{2}{15} x^8 y^4 + \frac{1}{75} x^{12} y^5$$

$$= 2x^4 y^3 + \frac{1}{3} x^8 y^4 + \frac{1}{75} x^{12} y^5$$

$$u_1(x, y) = - \int_0^x \int_0^x \left(2x^4 y^3 + \frac{1}{3} x^8 y^4 + \frac{1}{75} x^{12} y^5 \right) dx dx$$

$$= -\frac{1}{15} x^6 y^3 - \frac{1}{270} x^{10} y^4 - \frac{1}{13650} x^{14} y^5$$

$$u_2(x, y) = - \int_0^x \int_0^x A_1 dx dx$$

$$A_1 = u_{0y} u_1 + u_0 u_{1y}$$

$$= \left(2x^2 y + \frac{1}{5} x^6 y^2 \right) \left(-\frac{1}{15} x^6 y^3 - \frac{1}{270} x^{10} y^4 - \frac{1}{13650} x^{14} y^5 \right) + \left(x^2 y^2 + \frac{1}{15} x^6 y^3 \right)$$

$$\left(-\frac{1}{5} x^6 y^2 - \frac{2}{135} x^{10} y^3 - \frac{1}{2730} x^{14} y^4 \right)$$

$$= -\frac{2}{15} x^8 y^4 - \frac{2}{135} x^{12} y^5 - \frac{1}{6825} x^{16} y^6 - \frac{1}{75} x^{12} y^5 - \frac{1}{1350} x^{16} y^6 - \frac{1}{68250} x^{20} y^7 - \frac{1}{5} x^8 y^4 -$$

$$\frac{2}{135} x^{12} y^5 - \frac{1}{2730} x^{16} y^6 - \frac{1}{75} x^{12} y^5 - \frac{2}{2025} x^{16} y^6 - \frac{1}{40950} x^{20} y^7$$

$$= -\frac{1}{3} x^8 y^4 - \frac{11}{225} x^{12} y^5 - \frac{59}{26325} x^{16} y^6 - \frac{4}{102375} x^{20} y^7$$

$$u_2(x, y) = - \int_0^x \int_0^x \left(-\frac{1}{3} x^8 y^4 - \frac{11}{225} x^{12} y^5 - \frac{59}{26325} x^{16} y^6 - \frac{4}{102375} x^{20} y^7 \right) dx dx$$

$$= \frac{1}{270} x^{10} y^4 + \frac{11}{40950} x^{14} y^5 + \frac{50}{805540} x^{16} y^6 - \frac{2}{23648625} x^{20} y^7$$

and so on. The solution is given by

$$u(x, y) = u_0(x, y) + u_1(x, y) + u_2(x, y) + \cdots$$

$$= x^2 y^2 + \frac{1}{15} x^6 y^3 - \frac{1}{15} x^6 y^3 - \frac{1}{270} x^{10} y^4 - \frac{1}{13650} x^{14} y^5 + \frac{1}{270} x^{10} y^4 + \frac{11}{40950} x^{14} y^5$$

$$+ \frac{50}{805540} x^{16} y^6 - \frac{2}{23648625} x^{20} y^7 + \cdots$$

Wazwaz [4] defined the expression "noise terms" as the identical terms with opposite signs that

arise in the components $u_0(x, y)$ and $u_1(x, y)$. By canceling the noise terms between $u_0(x, y)$ and $u_1(x, y)$, even though $u_1(x, y)$ contains further terms, the remaining non-canceled terms of $u_0(x, y)$ may give the exact solution of the PDE. So in our solution we can see that the terms $\frac{1}{15}x^6 y^3$ and $\frac{1}{270}x^{10}y^4$ appear with opposite signs in the components $u_0(x, y)$ and $u_1(x, y)$, so by cancelling these terms from $u_0(x, y)$,

$$u(x, y) = x^2 y^2.$$

Therefore the exact solution is

$$u(x, y) = x^2 y^2.$$

This shows that the noise terms ensure fast convergence.

Chapter 5

Application to fluid flow problems

In this chapter we solve two selected fluid flow and heat transfer problems using ADM. We compare the results with those obtained by the authors of the selected papers. The former used DTM and the later used HAM.

5.1   Unsteady nanofluid flow and heat transfer.

Nanofluids were pioneered by Choi [36]. These fluids are simply suspensions of nanoparticles in a base fluid. The significant advantage of nanofluids over the traditional base fluids is that they have significantly high thermal conductivity that makes them capable of heat transfer enhancement. For this reason, they find use in, among others, biomedical applications, coolants in electronic devices, vehicle cooling and air conditioning equipment in buildings. The advent of nanotechnology has seen many researchers actively engaging in experimental and theoretical research in nanofluid flow and heat transfer.

Buongiorno [37] pioneered a two phase model for convective transport of nanofluid with emphasis on thermophoresis and Brownian motion effects. Usman *et al.* [38] conducted investigation on unsteady two phase nanofluid flow and heat transfer between moving parallel plates in the presence of magnetic field. The authors applied DTM to solve the model equations. We, here, apply ADM to solve the same model equations and perform a comparison.

The modelling equations for the flow are [38]

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0, \tag{5.1.1}$$

$$\rho_f \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \sigma B^2 u, \tag{5.1.2}$$

$$\rho_f \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \tag{5.1.3}$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \frac{4\mu}{(\rho c_p)_f} \left(\frac{\partial u}{\partial x} \right)^2 + \frac{(\rho c_p)_p}{(\rho c_p)_f} [D_B \left\{ \frac{\partial C}{\partial x} \cdot \frac{\partial T}{\partial x} + \frac{\partial C}{\partial y} \cdot \frac{\partial T}{\partial y} \right\}$$
$$\tag{5.1.4}$$

$$+ \left(\frac{D_T}{T_c} \right) \left\{ \left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 \right\}] + \frac{\sigma_{nf} B_0^2}{(\rho C_p)_{nf}} (u^2),$$

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right) + \frac{D_T}{T_c} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right), \qquad (5.1.5)$$

where $u, v, T, C, P, \rho_f, \mu; k, C_p$ and D_B represent the velocity in x direction, velocity in y direction, temperature, concentration, pressure, base fluid's density, dynamic viscosity, thermal conductivity, specific heat and diffusion coefficient of diffusing species respectively.

The boundary conditions are

$$y \longrightarrow h(t), \quad C = 0, \quad v = v_w = \frac{dh}{dt}, \quad T = T_H, \quad C = C_H \qquad (5.1.6)$$

$$y \longrightarrow 0, \quad v = \frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} = \frac{\partial C}{\partial y} = 0. \qquad (5.1.7)$$

The similarity transformations

$$\eta = \frac{y}{\iota(1 - \gamma t)^{\frac{1}{2}}}, \quad u = \frac{\gamma x}{2(1 - \gamma t) F'(\eta)}, \quad v = -\frac{\gamma \iota}{2(1 - \gamma t)^{\frac{1}{2}}} F(\eta) \qquad (5.1.8)$$

$$\theta = \frac{T}{T_H}, \quad \phi = \frac{C}{C_h}$$

transform equations (5.1.1) to (5.1.5) to a system of ODEs:

$$F''''(\eta) - S(\eta F'''(\eta) + 3F''(\eta) + F'(\eta) F''(\eta) - F(\eta) F'''(\eta)) - Ha^2 F''(\eta) = 0, \qquad (5.1.9)$$

$$\theta''(\eta) + PrS(F\theta'(\eta) - \eta\theta'(\eta)) + PrEc(F''(\eta))^2 + N_b \phi'\theta'(\eta) + Nt(\theta'(\eta))^2 \qquad (5.1.10)$$
$$+ PrHaEc(F'(\eta))^2 = 0,$$

$$\phi''(\eta) + S.Sc(F\phi'(\eta) - \eta\phi'(\eta)) + \frac{Nt}{Nb} \theta''(\eta) = 0. \qquad (5.1.11)$$

where $S = \frac{\gamma \iota^2}{2\mu} \rho_f$ is the squeeze number, $Pr = \frac{\mu(\rho c_p)_f}{\rho_f \alpha}$ is the Prandtl number, $Ec = \frac{\rho_f}{(\rho c)_f} \left(\frac{\gamma x}{2(1-\gamma t)} \right)^2$ is the Eckert number, $Sc = \frac{\mu}{\rho_f D}$ is the Schmidt number, $Ha = \iota B \sqrt{\frac{\sigma}{\mu}(1 - \gamma t)}$ is the Hartman number, $Nb = \frac{(\rho c)_p D_B (C_h)}{(\rho c)_f \alpha}$ is the Brownian motion parameter and $Nt = \frac{(\rho c)_p D_T (T_H)}{(\rho c)_f \alpha T_c}$ is the thermophoretic parameter.

The boundary conditions become

$$F(\eta) = 0,$$
$$F''(\eta) = 0,$$
$$\theta'(\eta) = 0,$$
$$\phi'(\eta) = 0, \qquad for \qquad \eta = 0$$

and

$$F(\eta) = 1,$$
$$F'(\eta) = 0,$$
$$\theta(\eta) = 1,$$
$$\phi(\eta) = 1, \qquad for \qquad \eta = 1.$$

5.1.1 Solution by ADM. Writing (5.1.9) in an operator form gives

$$L_\eta F(\eta) = S(\eta F'''(\eta) + 3F''(\eta) + F'(\eta)F''(\eta) - F(\eta)F'''(\eta)) + Ha^2 F''(\eta) \qquad (5.1.12)$$

where the differential operator L_η is defined by

$$L_\eta = \frac{d^4}{d\eta^4}.$$

L_η^{-1} is an integration operator defined by

$$L_\eta^{-1} = \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta (.) d\eta d\eta d\eta d\eta.$$

Applying the inverse operator L_η^{-1} to both sides of (5.1.12) we get

$$\int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta L_\eta F(\eta) d\eta d\eta d\eta d\eta = \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta (S(\eta F'''(\eta) + 3F''(\eta) + F'(\eta)F''(\eta) - F(\eta)F'''(\eta))$$
$$+ Ha^2 F''(\eta)) d\eta d\eta d\eta d\eta,$$

$$F(\eta) - F(0) - F'(0)\eta - F''(0)\frac{\eta^2}{2} - F'''(0)\frac{\eta^3}{6} = \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta (S(\eta F'''(\eta) + 3F''(\eta) + F'(\eta)F''(\eta)$$
$$- F(\eta)F'''(\eta)) + Ha^2 F''(\eta)) d\eta d\eta d\eta d\eta.$$

Using the given boundary conditions we obtain

$$F(\eta) = \alpha_1 \eta + \alpha_2 \frac{\eta^3}{6} + \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta (S(\eta F'''(\eta) + 3F''(\eta) + F'(\eta)F''(\eta) - F(\eta)F'''(\eta))$$
$$+ Ha^2 F''(\eta)) d\eta d\eta d\eta d\eta.$$

ADM introduces the following expression

$$\sum_{n=0}^\infty F_n(\eta) = \alpha_1 \eta + \alpha_2 \frac{\eta^3}{6} + \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta (S(\eta \sum_{n=0}^\infty F_n'''(\eta) + 3\sum_{n=0}^\infty F_n''(\eta) + \sum_{n=0}^\infty C_n - \sum_{n=0}^\infty P_n)$$
$$+ Ha^2 \sum_{n=0}^\infty F_n''(\eta)) d\eta d\eta d\eta d\eta,$$

where C_n and P_n's represent the nonlinear terms calculated by using the Adomian polynomial formula. The recursive relation is found to be

$$F_0(\eta) = \alpha_1\eta + \alpha_2\frac{\eta^3}{6},$$

$$F_{q+1}(\eta) = \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta \left(S(\eta F_q'''(\eta) + 3F_q''(\eta) + C_q - P_q) + Ha^2 F_q''(\eta) \right) d\eta\, d\eta\, d\eta\, d\eta.$$

Determining the first component we have

$$F_1(\eta) = \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta \left(S(\eta F_0'''(\eta) + 3F_0''(\eta) + C_0 - P_0) + Ha^2 F_0''(\eta) \right) d\eta\, d\eta\, d\eta\, d\eta.$$

Since we have F_0, we can find F_0'' and F_0'''.

$$F_0(\eta) = \alpha_1\eta + \alpha_2\frac{\eta^3}{6}$$

$$F_0'(\eta) = \alpha_1 + \alpha_2\frac{\eta^2}{2}$$

$$F_0''(\eta) = \alpha_2\eta$$

$$F_0'''(\eta) = \alpha_2$$

C_0 and P_0 are calculated by the Adomian polynomial formula and they are

$$C_0 = F_0'(\eta)F_0''(\eta) = \left(\alpha_1 + \alpha_2\frac{\eta^2}{2} \right)(\alpha_2\eta)$$

$$= \alpha_1\alpha_2\eta + \frac{\alpha_2^2\eta^3}{2},$$

$$P_0 = F_0(\eta)F_0'''(\eta) = \left(\alpha_1\eta + \alpha_2\frac{\eta^3}{6} \right)(\alpha_2)$$

$$= \alpha_1\alpha_2\eta + \frac{\alpha_2^2\eta^3}{6}.$$

Substituting the above values we obtain

$$F_1(\eta) = \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta \left(S\left(\eta\alpha_2 + 3\alpha_2\eta + \alpha_1\alpha_2\eta + \frac{\alpha_2^2\eta^3}{2} - \alpha_1\alpha_2\eta - \frac{\alpha_2^2\eta^3}{6} \right) + Ha^2\alpha_2\eta \right) d\eta\, d\eta\, d\eta\, d\eta,$$

$$= \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta \left(S\left(4\eta\alpha_2 + \frac{\alpha_2^2\eta^3}{3} \right) + Ha^2\alpha_2\eta \right) d\eta\, d\eta\, d\eta\, d\eta.$$

Therefore we obtain, after integration,

$$F_1(\eta) = \frac{S\alpha_2\eta^5}{30} + \frac{S\alpha_2^2\eta^7}{2520} + \frac{Ha^2\alpha_2\eta^5}{120}$$

$$= \left(\frac{S\alpha_2}{30} + \frac{Ha^2\alpha_2}{120} \right)\eta^5 + \frac{S\alpha_2^2}{2520}\eta^7$$

Similarly, we calculate $F_2(\eta)$, $F_3(\eta)$, $\cdots$. Thus the solution for $F(\eta)$ is given by

$$F(\eta) = F_0(\eta) + F_1(\eta) + \cdots$$
$$= \alpha_1\eta + \alpha_2\frac{\eta^3}{6} + \left(\frac{S\alpha_2}{30} + \frac{Ha^2\alpha_2}{120}\right)\eta^5 + \frac{S\alpha_2^2}{2520}\eta^7 + \cdots$$

$$= \sum_{n=0}^{m} F_n(\eta), \tag{5.1.13}$$

where the series is truncated at $n = m$.

To solve (5.1.10), writing it in an operator form we have

$$L_\eta\theta(\eta) = -PrS(F(\eta)\theta'(\eta) - \eta\theta'(\eta)) - PrEc(F''(\eta))^2 - Nb\phi'\theta'(\eta) - Nt(\theta'(\eta))^2 - PrHaEc(F'(\eta))^2, \tag{5.1.14}$$

where the differential operator L_η is defined by

$$L_\eta = \frac{d^2}{d\eta^2}.$$

Applying the inverse operator L_η^{-1} to both sides of (5.1.14) we obtain

$$\int_0^\eta \int_0^\eta L_\eta\theta(\eta)d\eta d\eta = \int_0^\eta \int_0^\eta (-PrS(F(\eta)\theta'(\eta) - \eta\theta'(\eta)) - PrEc(F''(\eta))^2 - Nb\phi'(\eta)\theta'(\eta) - Nt(\theta'(\eta))^2$$
$$- PrHaEc(F'(\eta))^2)d\eta d\eta.$$

This is

$$\theta(\eta) - \theta(0) - \theta'(0)\eta = \int_0^\eta \int_0^\eta (-PrS(F(\eta)\theta'(\eta) - \eta\theta'(\eta)) - PrEc(F''(\eta))^2 - Nb\phi'(\eta)\theta'(\eta) - Nt(\theta'(\eta))^2$$
$$- PrHaEc(F'(\eta))^2)d\eta d\eta.$$

Using the given boundary conditions we have

$$\theta(\eta) = \alpha_3 + \int_0^\eta \int_0^\eta (-PrS(F(\eta)\theta'(\eta) - \eta\theta'(\eta)) - PrEc(F''(\eta))^2 - Nb\phi'(\eta)\theta'(\eta) - Nt(\theta'(\eta))^2$$
$$- PrHaEc(F'(\eta))^2)d\eta d\eta.$$

Now, applying the definition of ADM we have

$$\sum_{n=0}^{\infty} \theta_n(\eta) = \alpha_3 + \int_0^\eta \int_0^\eta \left(-PrS\left(\sum_{n=0}^{\infty} F_n(\eta)\theta'_n(\eta) - \eta\sum_{n=0}^{\infty}\theta'_n(\eta)\right) - PrEc\left(\sum_{n=0}^{\infty} R_n\right) - Nb\sum_{n=0}^{\infty}\phi'_n(\eta)\theta'_n(\eta)\right.$$
$$\left. - Nt\sum_{n=0}^{\infty} Q_n - PrHaEc\left(\sum_{n=0}^{\infty} G_n\right)\right)d\eta d\eta,$$

where the R_n, Q_n and G_n's are the nonlinear terms calculated by the Adomian polynomial formula. The recursive relation gives

$$\theta_0(\eta) = \alpha_3,$$
$$\theta_{q+1}(\eta) = \int_0^\eta \int_0^\eta (-PrS(F_q(\eta)\theta_q'(\eta) - \eta\theta_q'(\eta)) - PrEc(R_q - Nb\phi_q'(\eta)\theta_q'(\eta) - NtQ_q$$
$$- PrHaEc(G_q))d\eta d\eta.$$

Determining the first component we get the following:

$$\theta_1(\eta) = \int_0^\eta \int_0^\eta \left(-PrS(F_0(\eta)\theta_0'(\eta) - \eta\theta_0'(\eta)) - PrEcR_0 - N_b\phi_0'(\eta)\theta_0'(\eta) - NtQ_0 - PrHaE_cG_0\right) d\eta d\eta,$$
$$F_0(\eta) = \alpha_1\eta + \alpha_2\frac{\eta^3}{6},$$
$$F_0'(\eta) = \alpha_1 + \alpha_2\frac{\eta^2}{2},$$
$$F_0''(\eta) = \alpha_2\eta,$$
$$R_0 = (F_0''(\eta))^2 = \alpha_2^2\eta^2,$$
$$Q_0 = (\theta_0'(\eta))^2 = 0,$$
$$G_0 = (F_0')^2 = (\alpha_1 + \alpha_2\frac{\eta^2}{2})^2 = \alpha_1^2 + \alpha_1\alpha_1\eta^2 + \alpha_2^2\frac{\eta^4}{4}.$$

Therefore we have

$$\theta_1(\eta) = \int_0^\eta \int_0^\eta \left(-PrEc(\alpha_2^2\eta^2) - PrHaEc\left(\alpha_1^2 + \alpha_1\eta^2 + \alpha_2^2\frac{\eta^4}{4}\right)\right) d\eta d\eta,$$
$$= -PrEc\alpha_2^2\frac{\eta^4}{12} - PrHaEc\alpha_1^2\frac{\eta^2}{2} - PrHaEc\alpha_1\alpha_2\frac{\eta^4}{12} - PrHaEc\alpha_2^2\frac{\eta^6}{120}.$$

$\theta_1(\eta)$, $\theta_1(\eta)$, $\cdots$ are obtained similarly.

Thus the solution for $\theta(\eta)$ is given by

$$\theta(\eta) = \theta_0(\eta) + \theta_1(\eta) + \cdots,$$
$$= \alpha_3 - PrHaEc\alpha_1^2\frac{\eta^2}{2} - (PrEc\alpha_2^2 + PrHaEc\alpha_1\alpha_2)\frac{\eta^4}{12} - PrHaEc\alpha_2^2\frac{\eta^6}{120} + \cdots.$$

$$= \sum_{n=0}^m \theta_n(\eta) \tag{5.1.15}$$

Writing (5.1.11) in an operator form we have

$$L_\eta\phi(\eta) = -S.Sc(F(\eta)\phi'(\eta) - \eta\phi'(\eta)) - \frac{Nt}{Nb}\theta''(\eta), \tag{5.1.16}$$

75

where L_η is the second differential operator defined by

$$L_\eta = \frac{d^2}{d\eta^2}.$$

Applying the inverse operator L_η^{-1} to both sides of (5.1.16) we get

$$\int_0^\eta \int_0^\eta L_\eta \phi''(\eta) d\eta d\eta = \int_0^\eta \int_0^\eta \left(-S.Sc(F(\eta)\phi'(\eta) - \eta\phi'(\eta)) - \frac{Nt}{Nb}\theta''(\eta) \right) d\eta d\eta,$$

which is equal to

$$\phi(\eta) = \phi(0) + \phi'(0)\eta + \int_0^\eta \int_0^\eta \left(-S.Sc(F(\eta)\phi'(\eta) - \eta\phi'(\eta)) - \frac{Nt}{Nb}\theta''(\eta) \right) d\eta d\eta.$$

Substituting the given boundary conditions we have

$$\phi(\eta) = \alpha_4 + \int_0^\eta \int_0^\eta \left(-S.Sc(F(\eta)\phi'(\eta) - \eta\phi'(\eta)) - \frac{Nt}{Nb}\theta''(\eta) \right) d\eta d\eta.$$

Using the definition of ADM we have

$$\sum_{n=0}^\infty \phi_n(\eta) = \alpha_4 + \int_0^\eta \int_0^\eta \left(-S.Sc \left(\sum_{n=0}^\infty F_n(\eta)\phi_n'(\eta) - \eta \sum_{n=0}^\infty \phi_n'(\eta) \right) - \frac{Nt}{Nb} \sum_{n=0}^\infty \theta_n''(\eta) \right) d\eta d\eta.$$

The recursive relation is found to be

$$\phi_0(\eta) = \alpha_4,$$
$$\phi_{q+1}(\eta) = \int_0^\eta \int_0^\eta \left(-S.Sc \left(F_q(\eta)\phi_q'(\eta) - \eta\phi_q' \right) - \frac{Nt}{Nb}\theta_q''(\eta) \right) d\eta d\eta.$$

Calculating the first two components we obtain the following

$$\phi_1(\eta) = \int_0^\eta \int_0^\eta \left(-S.Sc \left(F_0(\eta)\phi_0'(\eta) - \eta\phi_0'(\eta) \right) - \frac{Nt}{Nb}\theta_0''(\eta) \right) d\eta d\eta,$$
$$= \int_0^\eta \int_0^\eta \left(-S.Sc(0) - \frac{Nt}{Nb}(0) \right) d\eta d\eta,$$
$$= 0.$$

$$\phi_2(\eta) = \int_0^\eta \int_0^\eta \left(-S.Sc \left(F_1(\eta)\phi_1'(\eta) - \eta\phi_1'(\eta) \right) - \frac{Nt}{Nb}\theta_1''(\eta) \right) d\eta d\eta,$$
$$= -\frac{Nt}{Nb} \int_0^\eta \int_0^\eta \left(-PrEc\alpha_2^3\eta^2 - PrHaEc\alpha_1^2 - PrHaEc\alpha_1\alpha_2\eta^2 - \frac{1}{4}PrHaEc\eta^4 \right) d\eta d\eta,$$
$$= -\frac{Nt}{Nb} \left(-\frac{1}{12}PrEc\alpha_2^2\eta^4 - \frac{1}{2}PrHaEc\alpha_1^2\eta^2 - \frac{1}{12}PrHaEc\alpha_1\alpha_2\eta^4 - \frac{1}{120}PrHaEc\alpha_2^2\eta^6 \right)$$

Thus the solution for $\phi(\eta)$ is given by

$$\phi(\eta) = \phi_0(\eta) + \phi_1(\eta) + \alpha_2 + \cdots$$
$$= \alpha_4 + \frac{Nt}{Nb}\left(\frac{1}{12}PrEc\alpha_2^2\eta^4 + \frac{1}{2}PrHaEc\alpha_1^2\eta^2 + \frac{1}{12}PrHaEc\alpha_1\alpha_2\eta^4 + \frac{1}{120}PrHaEc\alpha_2^2\eta^6\right) + \cdots .$$

$$= \sum_{n=0}^{m}\phi_n(\eta) \qquad\qquad (5.1.17)$$

The accuracy of the series solutions is dependent upon the truncation point. The more terms we have the better the accuracy.

Table 5.1: Comparison of velocity between DTM and ADM solution.

η	F **by DTM** [38]	F **by ADM**
0	0.00000	0.00000
0.1	0.11890	0.11897
0.2	0.23760	0.23773
0.3	0.35578	0.35597
0.4	0.47297	0.47321
0.5	0.58827	0.58855
0.6	0.70007	0.70037
0.7	0.80538	0.80566
0.8	0.89869	0.89889
0.9	0.96982	0.96991
0.10	1.00000	1.00000

Table 5.2: Comparison of temparature between DTM and ADM solution.

η	θ **by DTM** [38]	θ **by ADM**
0	4.76574	5.13796
0.1	4.72338	5.09548
0.2	4.59724	4.96820
0.3	4.39011	4.75650
0.4	4.10665	4.46107
0.5	3.75328	4.08283
0.6	3.33793	3.62298
0.7	2.86889	3.08296
0.8	2.35008	2.46447
0.9	1.76303	1.76943
0.10	1.00000	1.00000

Table 5.3: Comparison of concentration between DTM and ADM solution.

η	ϕ **by DTM** [38]	ϕ **by ADM**
0	-2.74191	-3.13796
0.1	-2.69956	-3.09548
0.2	-2.57346	-2.96820
0.3	-2.36655	-2.75650
0.4	-2.08365	-2.46107
0.5	-1.73142	-2.08283
0.6	-1.31806	-1.62298
0.7	-0.85211	-1.08296
0.8	-0.33766	-0.46447
0.9	0.24370	0.23057
0.10	1.00000	1.00000

5.1.2 Results and discussion. The equations (5.1.13), (5.1.15) and (5.1.17) are coded into MATHEMATICA for easy iteration. Expressions for the unknown constants are also obtained with the aid of the remaining boundary conditions. At the end we obtain the mth partial sums as the approximate solutions of the coupled differential equations. The results are presented in Tables 5.1, 5.2 and 5.3.

The accuracy of the results in Tables 5.1, 5.2 and 5.3 is dependent upon the truncation points. The more the terms we have the better the accuracy. The accuracy of the results of $\theta(\eta)$ is dependent upon $F(\eta)$ solution. The more terms of $F(\eta)$, the better the accuracy of $\theta(\eta)$. Similarly, the accuracy of the results of $\phi(\eta)$ is dependent upon $F(\eta)$ and $\theta(\eta)$ solutions. The more terms of $F(\eta)$ and $\theta(\eta)$, the better the accuracy of $\phi(\eta)$. Table 5.1 shows a good agreement in the results of ADM and DTM. In Tables 5.2 and 5.3, the ADM results for $\theta(\eta)$ and $\phi(\eta)$ do not show good agreement with the DTM results in [38]. The explanation for this big deviation is that in this problem only a few terms of $F(\eta)$ (6 terms) were obtained. The computation for the 7th term demonstrated slow convergence rate of the problem. The code continued running for many days indicating that a higher CPU time was required. This really demonstrated that a computational complexity has been generated in the iterative process. For this reason we could not use more that 6 terms of $F(\eta)$. It is evident that if more terms were used, the results of $\theta(\eta)$ and $\phi(\eta)$ could have been improved. Clearly, this appears to be a limitation of this study.

5.2 MHD Jeffery–Hamel flows in non-parallel walls.

Jefferey-Hamel flows were pioneered by Jeffery and Hamel [39, 40]. These are mostly applied in fluid mechanics, civil, environmental and bio-mechanical engineering. Jeffery-Hamel flows are an exact similarity solution of the Navier-Stokes equations in the special case of two-dimesional flow through a channel with inclined plane walls meeting at a vertex [41]. Since the introduction of Jefferey-Hamel flows, it has been studied by many authors.

Domairry *et al.* [41] studied the Jeffery-Hamel flow and solved its nonlinear ordinary differential equation. Moghimi *et al.* [42] investigated the MHD Jeffery-Hamel flows in non-parallel walls analytically for strongly nonlinear ordinary differential equations. The authors applied HAM to solve the model equation. Here, we apply ADM to solve the same model equation and do the comparison.

The governing equations for the flow are [42]

$$\frac{\rho \partial}{r \partial r}(r u(r, \theta)) = 0, \tag{5.2.1}$$

$$u(r,\theta)\frac{\partial u(r,\theta)}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left[\frac{\partial^2 u(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial \theta^2} - \frac{u(r,\theta)}{r^2}\right] - \frac{\sigma B_0^2}{\rho r^2}u(r,\theta),$$
$$\tag{5.2.2}$$

$$\frac{1}{\rho r}\frac{\partial P}{\partial \theta} - \frac{\partial \nu}{r^2}\frac{\partial u(r,\theta)}{\partial \theta} = 0, \tag{5.2.3}$$

where B_0, r, $u(r, \theta)$, P, ν, ρ are the electromagnetic induction, conductivity of the fluid, velocity along radial direction, fluid pressure, coefficient of kinematic viscosity and the fluid density. From (5.2.1),

$$f\theta = r u(r, \theta).$$

Using dimensionless parameters

$$f(\eta) = \frac{f(\theta)}{f_{max}}, \qquad \eta = \frac{\theta}{\alpha}, \tag{5.2.4}$$

where α is the semi-angle between the two inclined walls, and eliminating P between equations (5.2.2) and (5.2.3), we obtain an ordinary differential equation for the normalized function profile $F(\eta)$:

$$f'''(\eta) + 2\alpha Re f(\eta)f'(\eta) + (4 - H)\alpha^2 f'(\eta) = 0 \tag{5.2.5}$$

with the boundary conditions

$$f(0) = 1, \quad f'(0) = 0, \quad f(1) = 0.$$

5.2.1 Solution by ADM. To solve (5.2.5), we write it in an operator form

$$L_\eta f(\eta) = -2\alpha Re f(\eta) f'(\eta) - (4 - H)\alpha^2 f'(\eta) \qquad (5.2.6)$$

where L_η is defined by

$$L_\eta = \frac{d^3}{d\eta^3}.$$

Applying the inverse operator L_η^{-1} to both sides of (5.2.6) we obtain

$$\int_0^\eta \int_0^\eta \int_0^\eta L_\eta f(\eta) d\eta d\eta d\eta = -\int_0^\eta \int_0^\eta \int_0^\eta [2\alpha Re f(\eta) f'(\eta) + (4 - H)\alpha^2 f'(\eta)] d\eta d\eta d\eta,$$

$$f(\eta) - f(0) - \eta f'(0) - \frac{\eta^2}{2} f''(0) = -\int_0^\eta \int_0^\eta \int_0^\eta [2\alpha Re f(\eta) f'(\eta) + (4 - H)\alpha^2 f'(\eta)] d\eta d\eta d\eta.$$

Using the given boundary conditions we have

$$f(\eta) = 1 + \frac{1}{2}\eta^2 \beta_1 - \int_0^\eta \int_0^\eta \int_0^\eta [2\alpha Re f(\eta) f'(\eta) + (4 - H)\alpha^2 f'(\eta)] d\eta d\eta d\eta.$$

Using the definition of ADM we have

$$\sum_{i=0}^\infty f_i(\eta) = 1 + \frac{1}{2}\eta^2 \beta_1 - \int_0^\eta \int_0^\eta \int_0^\eta \left[2\alpha Re \sum_{i=0}^\infty A_i + (4 - H)\alpha^2 \sum_{i=0}^\infty f_i'(\eta) \right] d\eta d\eta d\eta,$$

where the A's are the Adomian polynomials. The recursive relation is found to be

$$f_0(\eta) = 1 + \frac{1}{2}\eta^2 \beta_1,$$

$$f_{p+1}(\eta) = -\int_0^\eta \int_0^\eta \int_0^\eta \left[2\alpha Re A_p + (4 - H)\alpha^2 f_p'(\eta) \right] d\eta d\eta d\eta.$$

Calculating the first two components of $f(\eta)$ we have

$$f_0(\eta) = 1 + \frac{1}{2}\eta^2 \beta_1$$

$$f_1(\eta) = -\int_0^\eta \int_0^\eta \int_0^\eta \left[2\alpha Re A_0 + (4 - H)\alpha^2 f_0'(\eta) \right] d\eta d\eta d\eta$$

$$f_0' = \eta\beta_1$$

$$A_0 = f_0 f_0'$$

$$= \left(1 + \frac{1}{2}\eta^2\beta_1 \right)(\eta\beta_1)$$

$$= \beta_1\eta + \frac{1}{2}\eta^3\beta_1^2$$

$$f_1(\eta) = -\int_0^\eta \int_0^\eta \int_0^\eta \left[2\alpha Re\left(\beta_1\eta + \frac{1}{2}\eta^3\beta_1^2 \right) + (4 - H)\alpha^2(\beta_1\eta) \right] d\eta d\eta d\eta$$

$$= -\int_0^\eta \int_0^\eta \int_0^\eta \left[2\alpha Re\beta_1\eta + \alpha Re\beta_1^2\eta^3 + 4\alpha^2\beta_1\eta - H\alpha^2\beta_1\beta\eta \right] d\eta d\eta d\eta$$

$$= -\left[\alpha Re\beta_1\frac{\eta^4}{12} + \alpha Re\beta_1^2\frac{\eta^6}{120} + \alpha^2\beta_1\frac{\eta^4}{6} - H\alpha^2\beta_1\frac{\eta^4}{24} \right]$$

$$= -\alpha Re\beta_1\frac{\eta^4}{12} - \alpha Re\beta_1^2\frac{\eta^6}{120} - \alpha^2\beta_1\frac{\eta^4}{6} + H\alpha^2\beta_1\frac{\eta^4}{24}$$

$$= \left(\frac{H}{24}\alpha^2\beta_1 - \alpha^2\beta_1 - \frac{1}{2}\alpha Re\beta_1 \right)\frac{\eta^4}{6} - \alpha Re\beta_1^2\frac{\eta^6}{120}$$

$f_2(\eta)$, $f_3(\eta)$, $\cdots$ are obtained similarly. Thus the solution for $f(\eta)$ is

$$f(\eta) = f_0(\eta) + f_1(\eta) + \cdots$$

$$= 1 + \frac{1}{2}\beta_1\eta^2 + \left(\frac{H}{24}\alpha^2\beta_1 - \alpha^2\beta_1 - \frac{1}{2}\alpha Re\beta_1 \right)\frac{\eta^4}{6} - \alpha Re\beta_1^2\frac{\eta^6}{120} + \cdots$$

$$= \sum_{n=0}^{m} F_n(\eta) \tag{5.2.7}$$

5.2.2 Results and discussion. Equation (5.2.7) is coded into MATHEMATICA for easy iteration. Expression for the unknown constant is also obtained with the aid of the remaining boundary conditions. At the end we obtain the mth partial sum as the approximate solution of the coupled differential equation. The results are presented in Table 5.4. Table 5.4 shows a very good agreement in the results of HAM and ADM. Thus the ADM is as good as the solution obtained in [42] using HAM.

Table 5.4: Comparison of the results for $f(\eta)$ between HAM and ADM solution.

η	Solution by HAM [42]	Solution by ADM
0.05	1.00000	1.00000
0.15	0.997605	0.997605
0.10	0.990427	0.990427
0.15	0.978486	0.978486
0.20	0.961810	0.961810
0.25	0.940437	0.940437
0.30	0.914404	0.914404
0.35	0.883743	0.883743
0.40	0.848474	0.848474
0.45	0.808593	0.808593
0.50	0.764064	0.764064
0.55	0.714806	0.714806
0.60	0.660677	0.660677
0.65	0.601462	0.601462
0.70	0.536852	0.536852
0.75	0.466421	0.466421
0.80	0.389602	0.389602
0.85	0.305652	0.305652
0.90	0.213611	0.213611
0.95	0.112250	0.112250
1.00	0.000000	0.000000

Chapter 6

General discussion, conclusions, recommendations and limitation of the study

6.1   General discussion

In this dissertation, the application of Adomian decomposition method in solving nonlinear ordinary and partial differential equations was investigated. A detailed description of the method was laid down and the method was applied to solve many examples including some important differential equations like the Lane Emden and the mKdV equation. The method was also applied to solve two fluid flow and heat transfer problems. A survey of the other series solution numerical methods, namely the differential transform method and the homotopy analysis method was also carried out in detail and these methods were also applied on the examples and problems on which the Adomian decomposition method was applied. This was done in order to achieve a fair comparison of the methods with the Admian decomposition method.

6.2   Conclusions

A survey of the differential transform method, the perturbation method and the homotopy analysis method was successfully carried out, and a comprehensive description of the Adomian decomposition method was also executed successfully. In most of the examples on which the Adomian decomposition method, the differential transform method and the homotopy analysis method were applied, the three methods produced similar series solutions. In particular, for the homotopy analysis method, when $\hbar = -1$ the solution exactly coincides with that of the Adomian decomposition method. It can thus be concluded that in most examples the Adomian decomposition method is as good as these other series solution methods. In most simple examples, the series solution methods even provide exact solutions. It was also confirmed through this work that the Adomian decomposition method obtains series solutions that are rapidly convergent. In general it was observed that the accuracy of the series solutions is dependent upon the series truncation point. Increasing the truncation point greatly improves the accuracy of the solution. It is however extremely cumbersome to manually calculate the higher order terms. For this reason, in the fluid

flow and heat transfer problems that were solved, the schemes were coded into the symbolic package MATHEMATICA for easy of computation and to achieve higher truncation point where possible. Numerical solutions obtained for the fluid flow and heat transfer problems demonstrated that the Adomian decomposition method and the other two methods provide similar accuracy when the truncation points are increased. However in one of the two fluid flow problems that we solved a computational complexity was encountered in solving the nonlinear problem. The problem demonstrated slow convergence rate and requiring excessively high CPU time. As a result only fewer terms could be obtained and this affected the accuracy of the results in that problem.

6.3   Limitation of the study

The fluid flow problem in section 5.1 demonstrated slow convergence rate and high CPU time indicating that a computational complexity in the iterative process had been generated. Because of this, fewer terms were obtained and the accuracy of the solution was affected. In the nearest future, we hope to conduct further research into finding alternative ways of handling this type of nonlinear problems.

6.4   Recommendations

In this dissertation it has been confirmed that the Adomian decomposition method requires no prior linearisation of the nonlinear terms and discretization of the derivatives. In this way the method has desirable attributes as compared to other numerical methods that require linear approximation and discretization. Linearisation and discretization obviously increase greater deviation from the exact or actual solution. The Adomian decomposition method is direct and not difficult to implement and can easily be coded into symbolic packages for easy of computation, in the process producing easily converging series solutions in most cases. This work thus recommends the Adomian decomposition method, as well as the other series solution methods discussed herein, when faced with a problem that involves nonlinear differential equations. However, in this work we have also observed that in problems with high nonlinear complexity the Adomian decomposition method can have very slow convergence rate and high CPU time. We recommend that in such cases other ways of handling such problems must be sought.

6.5   Future research work

Future research work will involve solving fluid dynamics problems arising from science, engineering and technology using the Adomian decomposition method and the other series solution methods

discussed in this dissertation. Fluid dynamics problems have wide pertinent applications but are difficult to solve owing to the fact that they are inherently nonlinear. It will be an interesting venture, particularly for this candidate, to pursue a PhD through a project involving the application of Adomian decomposition method and/or other series solution methods in solving fluid dynamics problems. Future work will also involve looking at other ways of solving nonlinear problems where the Adomian decomposition method fails to converge.

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Appendix: Mathematica code used for the computations in chapter 5

```
DSolve[{F''''[y] -
    S[yF'''[y] + 3 F''[y] + F'[y] F''[y] - F[y] F'''[y]] -
    Ha^2 F''[y] == 0, F[0] == 0, F''[0] == 0, F[1] == 1,
  F'[1] == 0}, F, y]
DSolve[{-0.5[
     3 (F^\[Prime]\[Prime])[y] +
      Derivative[1][F][y] (F^\[Prime]\[Prime])[
        y] - (1.19003 y - 0.0323768 y^3 - 0.0615159 y^5 +
        7.48755*10^-6 y^7 +
        y^7 (-0.0553789 + 0.0000118158 y^2 + 7.34615*10^-10 y^4) +
        y^9 (-0.0284778 - 0.0000794436 y^2 + 2.20947*10^-6 y^4 -
          3.14769*10^-10 y^6 - 1.3741*10^-14 y^8) +
        y^11 (-0.00953615 - 0.0000318582 y^2 + 7.10326*10^-7 y^4 +
          2.02993*10^-9 y^6 - 1.51721*10^-11 y^8 +
          1.58923*10^-15 y^10 + 5.33819*10^-20 y^12) +
        y^13 (-0.00224007 - 0.0000181534 y^2 + 8.18955*10^-7 y^4 +
          3.30807*10^-9 y^6 - 3.50038*10^-11 y^8 -
          8.21156*10^-15 y^10 + 4.6559*10^-17 y^12 -
```

```
                    3.83933*10^-21 y^14 - 1.04605*10^-25 y^16) +

              y^15 (-0.000388864 - 9.90345*10^-6 y^2 -

                    3.82615*10^-8 y^4 + 2.55165*10^-8 y^6 +

                    6.32675*10^-10 y^8 - 5.31775*10^-12 y^10 -

                    4.0754*10^-14 y^12 + 1.97991*10^-16 y^14 +

                    3.9499*10^-20 y^16 - 1.91866*10^-22 y^18 +

                    3.7105*10^-26 y^20 - 1.41412*10^-30 y^22 -

                    4.02724*10^-35 y^24))

\!\(\*SuperscriptBox[\(F\),

TagBox[

RowBox[{"(", "3", ")"}],

Derivative],

MultilineFunction->None]\)[y] +

\!\(\*SuperscriptBox[\(yF\),

TagBox[

RowBox[{"(", "3", ")"}],

Derivative],

MultilineFunction->None]\)[y]] - 36 (F^\[Prime]\[Prime])[y] +

\!\(\*SuperscriptBox[\(F\),

TagBox[

RowBox[{"(", "4", ")"}],

Derivative],

MultilineFunction->None]\)[y] == 0,

  F[0] == 0, (F^\[Prime]\[Prime])[0] == 0, F[1] == 1,

  Derivative[1][F][1] == 0}, F, y]


Ha = 6

6
```

```
S = 0.5

0.5

Pr = 10

10

Nt = 0.1

0.1

Nb = 0.1

0.1

Sc = 1

Ec = 0.1

1

0.1

Subscript[F, 0][y] = \!\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(A \
\[DifferentialD]y\)\) + \!\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]B \[DifferentialD]y \
\[DifferentialD]y \[DifferentialD]y\)\)\)


1.19003 y - 0.0323768 y^3

Subscript[F, 0]'[y] = A + (y^2 B)/2

1.19003 - 0.0971304 y^2

Subscript[F, 0]''[y] = B y

-0.194261 y

Subscript[F, 0]'''[y] = B

-0.194261

Subscript[C, 0] = Subscript[F, 0]'[y] Subscript[F, 0]''[y]
```

```
-0.194261 y (1.19003 - 0.0971304 y^2)

Subscript[P, 0] = Subscript[F, 0][y] Subscript[F, 0]'''[y]

-0.194261 (1.19003 y - 0.0323768 y^3)

Subscript[F, 1][y] = \!\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\((S \((y\ \(
\*SubscriptBox[\(F\), \(0\)]'''\)[y] + 3 \(
\*SubscriptBox[\(F\), \(0\)]''\)[y] +
\*SubscriptBox[\(C\), \(0\)] -
\*SubscriptBox[\(P\), \(0\)])\) +
\*SuperscriptBox[\(Ha\), \(2\)] \(
\*SubscriptBox[\(F\), \(0\)]''\)[
        y])\) \[DifferentialD]y \[DifferentialD]y \[DifferentialD]y \
\[DifferentialD]y\)\)\)\)

-0.0615159 y^5 + 7.48755*10^-6 y^7


Subscript[F, 1]'[y] = 1/24 Ha^2 y^4 B + 1/6 S y^4 B + (S y^6 B^2)/360

-0.30758 y^4 + 0.0000524129 y^6

Subscript[F, 1]''[y] = 1/6 Ha^2 y^3 B + 2/3 S y^3 B + 1/60 S y^5 B^2

-1.23032 y^3 + 0.000314477 y^5


Subscript[F, 1]'''[y] = 1/2 Ha^2 y^2 B + 2 S y^2 B + 1/12 S y^4 B^2

-3.69096 y^2 + 0.00157239 y^4

Subscript[C, 1] =
 Subscript[F, 1]'[y] Subscript[F, 0]''[y] +
  Subscript[F, 0]'[y] Subscript[F, 1]''[y]
```

```
(1.19003 - 0.0971304 y^2) (-1.23032 y^3 + 0.000314477 y^5) -
 0.194261 y (-0.30758 y^4 + 0.0000524129 y^6)
Subscript[P, 1] =
 Subscript[F, 1][y] Subscript[F, 0]'''[y] +
  Subscript[F, 0][y] Subscript[F, 1]'''[y]
(1.19003 y - 0.0323768 y^3) (-3.69096 y^2 + 0.00157239 y^4) -
 0.194261 (-0.0615159 y^5 + 7.48755*10^-6 y^7)


Subscript[F, 2][y] = \!\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\((S \((y\ \(
\*SubscriptBox[\(F\), \(1\)]'''\)[y] + 3 \(
\*SubscriptBox[\(F\), \(1\)]''\)[y] +
\*SubscriptBox[\(C\), \(1\)] -
\*SubscriptBox[\(P\), \(1\)])\) +
\*SuperscriptBox[\(Ha\), \(2\)] \(
\*SubscriptBox[\(F\), \(1\)]''\)[
        y])\) \[DifferentialD]y \[DifferentialD]y \[DifferentialD]y \
\[DifferentialD]y\)\)\)\)
y^7 (-0.0553789 + 0.0000118158 y^2 + 7.34615*10^-10 y^4)
Subscript[C, 2] =
 D[Subscript[F, 1][y], y] D[D[Subscript[F, 1][y], y], y] +
  D[Subscript[F, 2][y], y] D[D[Subscript[F, 0][y], y], y] +
  2 D[Subscript[F, 0][y], y] D[D[Subscript[F, 2][y], y] y]
```

```
(-1.23032 y^3 + 0.000314477 y^5) (-0.30758 y^4 + 0.0000524129 y^6) -
 0.194261 y (y^7 (0.0000236315 y + 2.93846*10^-9 y^3) +
    7 y^6 (-0.0553789 + 0.0000118158 y^2 + 7.34615*10^-10 y^4)) +
 2 y (1.19003 -
    0.0971304 y^2) (y^7 (0.0000236315 y + 2.93846*10^-9 y^3) +
    7 y^6 (-0.0553789 + 0.0000118158 y^2 + 7.34615*10^-10 y^4))
Subscript[P, 2] =
 Subscript[F, 0][y] D[D[D[Subscript[F, 2][y], y], y], y] +
  2 Subscript[F, 1][y] D[D[D[Subscript[F, 1][y], y], y], y] +
  2 Subscript[F, 2][y] D[D[D[Subscript[F, 0][y], y], y], y]


-0.388522 y^7 (-0.0553789 + 0.0000118158 y^2 + 7.34615*10^-10 y^4) +
 2 (-3.69096 y^2 + 0.00157239 y^4) (-0.0615159 y^5 +
    7.48755*10^-6 y^7) + (1.19003 y -
    0.0323768 y^3) (1.76308*10^-8 y^8 +
    21 y^6 (0.0000236315 + 8.81538*10^-9 y^2) +
    126 y^5 (0.0000236315 y + 2.93846*10^-9 y^3) +
    210 y^4 (-0.0553789 + 0.0000118158 y^2 + 7.34615*10^-10 y^4))


Subscript[F, 3][y] = \!\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\((S \((y\ D[D[D[
\(\*SubscriptBox[\(F\), \(2\)]\)[y], \ y], \ y], \ y] + 3  D[D[
\(\*SubscriptBox[\(F\), \(2\)]\)[y], y], \ y] +
\*SubscriptBox[\(C\), \(2\)] -
\*SubscriptBox[\(P\), \(2\)])\) +
```

```
\*SuperscriptBox[\(Ha\), \(2\)] D[D[

\(\*SubscriptBox[\(F\), \(2\)]\)[y], \ y], \

        y])\) \[DifferentialD]y \[DifferentialD]y \[DifferentialD]y \

\[DifferentialD]y\)\)\)\)
```

y^9 (-0.0284778 - 0.0000794436 y^2 + 2.20947*10^{-6} y^4 -

   3.14769*10^{-10} y^6 - 1.3741*10^{-14} y^8)

```
Subscript[C, 3] =
 D[Subscript[F, 3][y], y] D[D[Subscript[F, 0][y], y], y] +
  D[Subscript[F, 2][y], y] D[D[Subscript[F, 1][y], y], y] +
  D[Subscript[F, 1][y], y] D[D[Subscript[F, 2][y], y], y] +
  D[Subscript[F, 0][y], y] D[D[Subscript[F, 3][y], y], y] y
```


(-0.30758 y^4 +

    0.0000524129 y^6) (y^7 (0.0000236315 + 8.81538*10^{-9} y^2) +

    14 y^6 (0.0000236315 y + 2.93846*10^{-9} y^3) +

    42 y^5 (-0.0553789 + 0.0000118158 y^2 +

       7.34615*10^{-10} y^4)) + (-1.23032 y^3 +

    0.000314477 y^5) (y^7 (0.0000236315 y + 2.93846*10^{-9} y^3) +

    7 y^6 (-0.0553789 + 0.0000118158 y^2 + 7.34615*10^{-10} y^4)) -

 0.194261 y (y^9 (-0.000158887 y + 8.83788*10^{-6} y^3 -

       1.88861*10^{-9} y^5 - 1.09928*10^{-13} y^7) +

    9 y^8 (-0.0284778 - 0.0000794436 y^2 + 2.20947*10^{-6} y^4 -

       3.14769*10^{-10} y^6 - 1.3741*10^{-14} y^8)) +

 y (1.19003 -

    0.0971304 y^2) (y^9 (-0.000158887 y + 8.83788*10^{-6} y^3 -

       1.88861*10^{-9} y^5 - 1.09928*10^{-13} y^7) +

    9 y^8 (-0.0284778 - 0.0000794436 y^2 + 2.20947*10^{-6} y^4 -

       3.14769*10^{-10} y^6 - 1.3741*10^{-14} y^8))

```
Subscript[P, 3] =
 Subscript[F, 3][y] D[D[D[Subscript[F, 0][y], y], y], y] +
  Subscript[F, 2][y] D[D[D[Subscript[F, 1][y], y], y], y] +
  Subscript[F, 1][y] D[D[D[Subscript[F, 2][y], y], y], y] +
  Subscript[F, 0][y] D[D[D[Subscript[F, 3][y], y], y], y]


y^7 (-0.0553789 + 0.0000118158 y^2 +
    7.34615*10^-10 y^4) (-3.69096 y^2 + 0.00157239 y^4) -
 0.194261 y^9 (-0.0284778 - 0.0000794436 y^2 + 2.20947*10^-6 y^4 -
    3.14769*10^-10 y^6 - 1.3741*10^-14 y^8) + (-0.0615159 y^5 +
    7.48755*10^-6 y^7) (1.76308*10^-8 y^8 +
    21 y^6 (0.0000236315 + 8.81538*10^-9 y^2) +
    126 y^5 (0.0000236315 y + 2.93846*10^-9 y^3) +
    210 y^4 (-0.0553789 + 0.0000118158 y^2 +
      7.34615*10^-10 y^4)) + (1.19003 y -
    0.0323768 y^3) (y^9 (0.0000530273 y - 3.77722*10^-8 y^3 -
      4.61699*10^-12 y^5) +
    27 y^8 (-0.000158887 + 0.0000265136 y^2 - 9.44306*10^-9 y^4 -
      7.69498*10^-13 y^6) +
    216 y^7 (-0.000158887 y + 8.83788*10^-6 y^3 - 1.88861*10^-9 y^5 -
      1.09928*10^-13 y^7) +
    504 y^6 (-0.0284778 - 0.0000794436 y^2 + 2.20947*10^-6 y^4 -
      3.14769*10^-10 y^6 - 1.3741*10^-14 y^8))


Subscript[F, 4][y] = \!\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
```

```
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\((S \((y\ D[D[D[
\(\*SubscriptBox[\(F\), \(3\)]\)[y], \ y], \ y], \ y] + 3  D[D[
\(\*SubscriptBox[\(F\), \(3\)]\)[y], y], \ y] +
\*SubscriptBox[\(C\), \(3\)] -
\*SubscriptBox[\(P\), \(3\)])\) +
\*SuperscriptBox[\(Ha\), \(2\)] D[D[
\(\*SubscriptBox[\(F\), \(3\)]\)[y], \ y], \
        y])\) \[DifferentialD]y \[DifferentialD]y \[DifferentialD]y \
\[DifferentialD]y\)\)\)\)
```

y^{11} (-0.00953615 - 0.0000318582 y^2 + 7.10326*10^{-7} y^4 +

2.02993*10^{-9} y^6 - 1.51721*10^{-11} y^8 + 1.58923*10^{-15} y^{10} +

5.33819*10^{-20} y^{12})

```
Subscript[C, 4] =
 D[Subscript[F, 4][y], y] D[D[Subscript[F, 0][y], y], y] +
  D[Subscript[F, 3][y], y] D[D[Subscript[F, 1][y], y], y] +
  D[Subscript[F, 2][y], y] D[D[Subscript[F, 2][y], y], y] +
  D[Subscript[F, 1][y], y] D[D[Subscript[F, 3][y], y] y] +
  D[Subscript[F, 0][y], y] D[D[Subscript[F, 4][y], y] y]
```

```
Subscript[P, 4] =
 Subscript[F, 4][y] D[D[D[Subscript[F, 0][y], y], y], y] +
  Subscript[F, 3][y] D[D[D[Subscript[F, 1][y], y], y], y] +
  Subscript[F, 2][y] D[D[D[Subscript[F, 2][y], y], y], y] +
  Subscript[F, 1][y] D[D[D[Subscript[F, 3][y], y], y], y] +
  Subscript[F, 0][y] D[D[D[Subscript[F, 4][y], y], y], y]


(y^7 (0.0000236315 + 8.81538*10^-9 y^2) +
    14 y^6 (0.0000236315 y + 2.93846*10^-9 y^3) +
    42 y^5 (-0.0553789 + 0.0000118158 y^2 +
        7.34615*10^-10 y^4)) (y^7 (0.0000236315 y +
        2.93846*10^-9 y^3) +
    7 y^6 (-0.0553789 + 0.0000118158 y^2 +
        7.34615*10^-10 y^4)) + (-1.23032 y^3 +
    0.000314477 y^5) (y^9 (-0.000158887 y + 8.83788*10^-6 y^3 -
        1.88861*10^-9 y^5 - 1.09928*10^-13 y^7) +
    9 y^8 (-0.0284778 - 0.0000794436 y^2 + 2.20947*10^-6 y^4 -
        3.14769*10^-10 y^6 - 1.3741*10^-14 y^8)) +
 y (-0.30758 y^4 +
    0.0000524129 y^6) (y^9 (-0.000158887 y + 8.83788*10^-6 y^3 -
        1.88861*10^-9 y^5 - 1.09928*10^-13 y^7) +
    9 y^8 (-0.0284778 - 0.0000794436 y^2 + 2.20947*10^-6 y^4 -
        3.14769*10^-10 y^6 - 1.3741*10^-14 y^8)) -
```

0.194261 y (y^11 (-0.0000637165 y + 2.84131*10^-6 y^3 +

    1.21796*10^-8 y^5 - 1.21377*10^-10 y^7 + 1.58923*10^-14 y^9 +

    6.40583*10^-19 y^11) +

  11 y^10 (-0.00953615 - 0.0000318582 y^2 + 7.10326*10^-7 y^4 +

    2.02993*10^-9 y^6 - 1.51721*10^-11 y^8 + 1.58923*10^-15 y^10 +

    5.33819*10^-20 y^12)) +

y (1.19003 -

  0.0971304 y^2) (y^11 (-0.0000637165 y + 2.84131*10^-6 y^3 +

    1.21796*10^-8 y^5 - 1.21377*10^-10 y^7 + 1.58923*10^-14 y^9 +

    6.40583*10^-19 y^11) +

  11 y^10 (-0.00953615 - 0.0000318582 y^2 + 7.10326*10^-7 y^4 +

    2.02993*10^-9 y^6 - 1.51721*10^-11 y^8 + 1.58923*10^-15 y^10 +

    5.33819*10^-20 y^12))

y^9 (-3.69096 y^2 + 0.00157239 y^4) (-0.0284778 - 0.0000794436 y^2 +

  2.20947*10^-6 y^4 - 3.14769*10^-10 y^6 - 1.3741*10^-14 y^8) -

 0.194261 y^11 (-0.00953615 - 0.0000318582 y^2 + 7.10326*10^-7 y^4 +

  2.02993*10^-9 y^6 - 1.51721*10^-11 y^8 + 1.58923*10^-15 y^10 +

  5.33819*10^-20 y^12) +

y^7 (-0.0553789 + 0.0000118158 y^2 +

  7.34615*10^-10 y^4) (1.76308*10^-8 y^8 +

  21 y^6 (0.0000236315 + 8.81538*10^-9 y^2) +

  126 y^5 (0.0000236315 y + 2.93846*10^-9 y^3) +

  210 y^4 (-0.0553789 + 0.0000118158 y^2 +

    7.34615*10^-10 y^4)) + (-0.0615159 y^5 +

  7.48755*10^-6 y^7) (y^9 (0.0000530273 y - 3.77722*10^-8 y^3 -

    4.61699*10^-12 y^5) +

  27 y^8 (-0.000158887 + 0.0000265136 y^2 - 9.44306*10^-9 y^4 -

```
            7.69498*10^-13 y^6) +

        216 y^7 (-0.000158887 y + 8.83788*10^-6 y^3 - 1.88861*10^-9 y^5 -

            1.09928*10^-13 y^7) +

        504 y^6 (-0.0284778 - 0.0000794436 y^2 + 2.20947*10^-6 y^4 -

            3.14769*10^-10 y^6 - 1.3741*10^-14 y^8)) + (1.19003 y -

        0.0323768 y^3) (y^11 (0.0000170478 y + 2.43592*10^-7 y^3 -

            5.09783*10^-9 y^5 + 1.14424*10^-12 y^7 + 7.04641*10^-17 y^9) +

        33 y^10 (-0.0000637165 + 8.52392*10^-6 y^2 + 6.08979*10^-8 y^4 -

            8.49638*10^-10 y^6 + 1.43031*10^-13 y^8 +

            7.04641*10^-18 y^10) +

        330 y^9 (-0.0000637165 y + 2.84131*10^-6 y^3 +

            1.21796*10^-8 y^5 - 1.21377*10^-10 y^7 + 1.58923*10^-14 y^9 +

            6.40583*10^-19 y^11) +

        990 y^8 (-0.00953615 - 0.0000318582 y^2 + 7.10326*10^-7 y^4 +

            2.02993*10^-9 y^6 - 1.51721*10^-11 y^8 + 1.58923*10^-15 y^10 +

            5.33819*10^-20 y^12))


Subscript[F, 5][y] = \!\(

\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(

\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(

\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(

\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\((S \((y\ D[D[D[

\(\*SubscriptBox[\(F\), \(4\)]\)[y], \ y], \ y], \ y] + 3  D[D[

\(\*SubscriptBox[\(F\), \(4\)]\)[y], y], \ y] +

\*SubscriptBox[\(C\), \(4\)] -

\*SubscriptBox[\(P\), \(4\)])\) +

\*SuperscriptBox[\(Ha\), \(2\)] D[D[

\(\*SubscriptBox[\(F\), \(4\)]\)[y], \ y], \
```

```
        y])\) \[DifferentialD]y \[DifferentialD]y \[DifferentialD]y \
\[DifferentialD]y\)\)\)\)
```

```
y^13 (-0.00224007 - 0.0000181534 y^2 + 8.18955*10^-7 y^4 +
    3.30807*10^-9 y^6 - 3.50038*10^-11 y^8 - 8.21156*10^-15 y^10 +
    4.6559*10^-17 y^12 - 3.83933*10^-21 y^14 - 1.04605*10^-25 y^16)
```

```
Subscript[C, 5] =
 D[Subscript[F, 5][y], y] D[D[Subscript[F, 0][y], y], y] +
  D[Subscript[F, 4][y], y] D[D[Subscript[F, 1][y], y], y] +
  D[Subscript[F, 3][y], y] D[D[Subscript[F, 3][y], y], y] +
  D[Subscript[F, 2][y], y] D[D[Subscript[F, 3][y], y] y] +
  D[Subscript[F, 1][y], y] D[D[Subscript[F, 5][y], y] y] +
  D[Subscript[F, 0][y], y] D[D[Subscript[F, 5][y], y] y]
```

```
Subscript[P, 5] =
 Subscript[F, 5][y] D[D[D[Subscript[F, 0][y], y], y], y] +
  Subscript[F, 4][y] D[D[D[Subscript[F, 1][y], y], y], y] +
  Subscript[F, 3][y] D[D[D[Subscript[F, 3][y], y], y], y] +
  Subscript[F, 2][y] D[D[D[Subscript[F, 3][y], y], y], y] +
  Subscript[F, 1][y] D[D[D[Subscript[F, 5][y], y], y], y] +
  Subscript[F, 0][y] D[D[D[Subscript[F, 5][y], y], y], y]
```

```
y (y^7 (0.0000236315 y + 2.93846*10^-9 y^3) +
    7 y^6 (-0.0553789 + 0.0000118158 y^2 +
        7.34615*10^-10 y^4)) (y^9 (-0.000158887 y +
        8.83788*10^-6 y^3 - 1.88861*10^-9 y^5 - 1.09928*10^-13 y^7) +
```

$$9\ y\char`^8\ (-0.0284778 - 0.0000794436\ y\char`^2 + 2.20947*10\char`^-6\ y\char`^4 -$$

$$3.14769*10\char`^-10\ y\char`^6 -$$

$$1.3741*10\char`^-14\ y\char`^8)) + (y\char`^9\ (-0.000158887 + 0.0000265136\ y\char`^2 -$$

$$9.44306*10\char`^-9\ y\char`^4 - 7.69498*10\char`^-13\ y\char`^6) +$$

$$18\ y\char`^8\ (-0.000158887\ y + 8.83788*10\char`^-6\ y\char`^3 - 1.88861*10\char`^-9\ y\char`^5 -$$

$$1.09928*10\char`^-13\ y\char`^7) +$$

$$72\ y\char`^7\ (-0.0284778 - 0.0000794436\ y\char`^2 + 2.20947*10\char`^-6\ y\char`^4 -$$

$$3.14769*10\char`^-10\ y\char`^6 -$$

$$1.3741*10\char`^-14\ y\char`^8))\ (y\char`^9\ (-0.000158887\ y + 8.83788*10\char`^-6\ y\char`^3 -$$

$$1.88861*10\char`^-9\ y\char`^5 - 1.09928*10\char`^-13\ y\char`^7) +$$

$$9\ y\char`^8\ (-0.0284778 - 0.0000794436\ y\char`^2 + 2.20947*10\char`^-6\ y\char`^4 -$$

$$3.14769*10\char`^-10\ y\char`^6 - 1.3741*10\char`^-14\ y\char`^8)) + (-1.23032\ y\char`^3 +$$

$$0.000314477\ y\char`^5)\ (y\char`^11\ (-0.0000637165\ y + 2.84131*10\char`^-6\ y\char`^3 +$$

$$1.21796*10\char`^-8\ y\char`^5 - 1.21377*10\char`^-10\ y\char`^7 + 1.58923*10\char`^-14\ y\char`^9 +$$

$$6.40583*10\char`^-19\ y\char`^11) +$$

$$11\ y\char`^10\ (-0.00953615 - 0.0000318582\ y\char`^2 + 7.10326*10\char`^-7\ y\char`^4 +$$

$$2.02993*10\char`^-9\ y\char`^6 - 1.51721*10\char`^-11\ y\char`^8 + 1.58923*10\char`^-15\ y\char`^10 +$$

$$5.33819*10\char`^-20\ y\char`^12)) -$$

$$0.194261\ y\ (y\char`^13\ (-0.0000363069\ y + 3.27582*10\char`^-6\ y\char`^3 +$$

$$1.98484*10\char`^-8\ y\char`^5 - 2.8003*10\char`^-10\ y\char`^7 - 8.21156*10\char`^-14\ y\char`^9 +$$

$$5.58707*10\char`^-16\ y\char`^11 - 5.37507*10\char`^-20\ y\char`^13 -$$

$$1.67369*10\char`^-24\ y\char`^15) +$$

$$13\ y\char`^12\ (-0.00224007 - 0.0000181534\ y\char`^2 + 8.18955*10\char`^-7\ y\char`^4 +$$

$$3.30807*10\char`^-9\ y\char`^6 - 3.50038*10\char`^-11\ y\char`^8 - 8.21156*10\char`^-15\ y\char`^10 +$$

$$4.6559*10\char`^-17\ y\char`^12 - 3.83933*10\char`^-21\ y\char`^14 -$$

$$1.04605*10\char`^-25\ y\char`^16)) +$$

$$y\ (1.19003 -$$

$$0.0971304\ y\char`^2)\ (y\char`^13\ (-0.0000363069\ y + 3.27582*10\char`^-6\ y\char`^3 +$$

$$1.98484*10^\wedge-8 \ y^\wedge5 - 2.8003*10^\wedge-10 \ y^\wedge7 - 8.21156*10^\wedge-14 \ y^\wedge9 +$$

$$5.58707*10^\wedge-16 \ y^\wedge11 - 5.37507*10^\wedge-20 \ y^\wedge13 -$$

$$1.67369*10^\wedge-24 \ y^\wedge15) +$$

$$13 \ y^\wedge12 \ (-0.00224007 - 0.0000181534 \ y^\wedge2 + 8.18955*10^\wedge-7 \ y^\wedge4 +$$

$$3.30807*10^\wedge-9 \ y^\wedge6 - 3.50038*10^\wedge-11 \ y^\wedge8 - 8.21156*10^\wedge-15 \ y^\wedge10 +$$

$$4.6559*10^\wedge-17 \ y^\wedge12 - 3.83933*10^\wedge-21 \ y^\wedge14 -$$

$$1.04605*10^\wedge-25 \ y^\wedge16)) +$$

$$y \ (-0.30758 \ y^\wedge4 +$$

$$0.0000524129 \ y^\wedge6) \ (y^\wedge13 \ (-0.0000363069 \ y + 3.27582*10^\wedge-6 \ y^\wedge3 +$$

$$1.98484*10^\wedge-8 \ y^\wedge5 - 2.8003*10^\wedge-10 \ y^\wedge7 - 8.21156*10^\wedge-14 \ y^\wedge9 +$$

$$5.58707*10^\wedge-16 \ y^\wedge11 - 5.37507*10^\wedge-20 \ y^\wedge13 -$$

$$1.67369*10^\wedge-24 \ y^\wedge15) +$$

$$13 \ y^\wedge12 \ (-0.00224007 - 0.0000181534 \ y^\wedge2 + 8.18955*10^\wedge-7 \ y^\wedge4 +$$

$$3.30807*10^\wedge-9 \ y^\wedge6 - 3.50038*10^\wedge-11 \ y^\wedge8 - 8.21156*10^\wedge-15 \ y^\wedge10 +$$

$$4.6559*10^\wedge-17 \ y^\wedge12 - 3.83933*10^\wedge-21 \ y^\wedge14 -$$

$$1.04605*10^\wedge-25 \ y^\wedge16))$$

$$y^\wedge11 \ (-3.69096 \ y^\wedge2 + 0.00157239 \ y^\wedge4) \ (-0.00953615 -$$

$$0.0000318582 \ y^\wedge2 + 7.10326*10^\wedge-7 \ y^\wedge4 + 2.02993*10^\wedge-9 \ y^\wedge6 -$$

$$1.51721*10^\wedge-11 \ y^\wedge8 + 1.58923*10^\wedge-15 \ y^\wedge10 + 5.33819*10^\wedge-20 \ y^\wedge12) -$$

$$0.194261 \ y^\wedge13 \ (-0.00224007 - 0.0000181534 \ y^\wedge2 + 8.18955*10^\wedge-7 \ y^\wedge4 +$$

$$3.30807*10^\wedge-9 \ y^\wedge6 - 3.50038*10^\wedge-11 \ y^\wedge8 - 8.21156*10^\wedge-15 \ y^\wedge10 +$$

$$4.6559*10^\wedge-17 \ y^\wedge12 - 3.83933*10^\wedge-21 \ y^\wedge14 - 1.04605*10^\wedge-25 \ y^\wedge16) +$$

$$y^\wedge7 \ (-0.0553789 + 0.0000118158 \ y^\wedge2 +$$

$$7.34615*10^\wedge-10 \ y^\wedge4) \ (y^\wedge9 \ (0.0000530273 \ y - 3.77722*10^\wedge-8 \ y^\wedge3 -$$

$$4.61699*10^\wedge-12 \ y^\wedge5) +$$

$$27 \ y^\wedge8 \ (-0.000158887 + 0.0000265136 \ y^\wedge2 - 9.44306*10^\wedge-9 \ y^\wedge4 -$$

$$7.69498*10^\wedge-13 \ y^\wedge6) +$$

216 y^7 (-0.000158887 y + 8.83788*10^-6 y^3 - 1.88861*10^-9 y^5 -

   1.09928*10^-13 y^7) +

504 y^6 (-0.0284778 - 0.0000794436 y^2 + 2.20947*10^-6 y^4 -

   3.14769*10^-10 y^6 - 1.3741*10^-14 y^8)) +

y^9 (-0.0284778 - 0.0000794436 y^2 + 2.20947*10^-6 y^4 -

   3.14769*10^-10 y^6 -

   1.3741*10^-14 y^8) (y^9 (0.0000530273 y - 3.77722*10^-8 y^3 -

   4.61699*10^-12 y^5) +

27 y^8 (-0.000158887 + 0.0000265136 y^2 - 9.44306*10^-9 y^4 -

   7.69498*10^-13 y^6) +

216 y^7 (-0.000158887 y + 8.83788*10^-6 y^3 - 1.88861*10^-9 y^5 -

   1.09928*10^-13 y^7) +

504 y^6 (-0.0284778 - 0.0000794436 y^2 + 2.20947*10^-6 y^4 -

   3.14769*10^-10 y^6 - 1.3741*10^-14 y^8)) + (1.19003 y -

0.0323768 y^3) (y^13 (0.0000196549 y + 3.96968*10^-7 y^3 -

   1.17613*10^-8 y^5 - 5.91233*10^-12 y^7 + 6.14578*10^-14 y^9 -

   8.38511*10^-18 y^11 - 3.51474*10^-22 y^13) +

39 y^12 (-0.0000363069 + 9.82746*10^-6 y^2 + 9.9242*10^-8 y^4 -

   1.96021*10^-9 y^6 - 7.39041*10^-13 y^8 + 6.14578*10^-15 y^10 -

   6.98759*10^-19 y^12 - 2.51053*10^-23 y^14) +

468 y^11 (-0.0000363069 y + 3.27582*10^-6 y^3 +

   1.98484*10^-8 y^5 - 2.8003*10^-10 y^7 - 8.21156*10^-14 y^9 +

   5.58707*10^-16 y^11 - 5.37507*10^-20 y^13 -

   1.67369*10^-24 y^15) +

1716 y^10 (-0.00224007 - 0.0000181534 y^2 + 8.18955*10^-7 y^4 +

   3.30807*10^-9 y^6 - 3.50038*10^-11 y^8 - 8.21156*10^-15 y^10 +

   4.6559*10^-17 y^12 - 3.83933*10^-21 y^14 -

   1.04605*10^-25 y^16)) + (-0.0615159 y^5 +

```
    7.48755*10^-6 y^7) (y^13 (0.0000196549 y + 3.96968*10^-7 y^3 -

        1.17613*10^-8 y^5 - 5.91233*10^-12 y^7 + 6.14578*10^-14 y^9 -

        8.38511*10^-18 y^11 - 3.51474*10^-22 y^13) +

    39 y^12 (-0.0000363069 + 9.82746*10^-6 y^2 + 9.9242*10^-8 y^4 -

        1.96021*10^-9 y^6 - 7.39041*10^-13 y^8 + 6.14578*10^-15 y^10 -

        6.98759*10^-19 y^12 - 2.51053*10^-23 y^14) +

    468 y^11 (-0.0000363069 y + 3.27582*10^-6 y^3 +

        1.98484*10^-8 y^5 - 2.8003*10^-10 y^7 - 8.21156*10^-14 y^9 +

        5.58707*10^-16 y^11 - 5.37507*10^-20 y^13 -

        1.67369*10^-24 y^15) +

    1716 y^10 (-0.00224007 - 0.0000181534 y^2 + 8.18955*10^-7 y^4 +

        3.30807*10^-9 y^6 - 3.50038*10^-11 y^8 - 8.21156*10^-15 y^10 +

        4.6559*10^-17 y^12 - 3.83933*10^-21 y^14 -

        1.04605*10^-25 y^16))


Subscript[F, 6][y] = \!\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\((S \((y\ D[D[D[
\(\*SubscriptBox[\(F\), \(5\)]\)[y], \ y], \ y], \ y] + 3  D[D[
\(\*SubscriptBox[\(F\), \(5\)]\)[y], y], \ y] +
\*SubscriptBox[\(C\), \(5\)] -
\*SubscriptBox[\(P\), \(5\)])\) +
\*SuperscriptBox[\(Ha\), \(2\)] D[D[
\(\*SubscriptBox[\(F\), \(5\)]\)[y], \ y], \
        y])\) \[DifferentialD]y \[DifferentialD]y \[DifferentialD]y \
\[DifferentialD]y\)\)\)\)
```

```
y^15 (-0.000388864 - 9.90345*10^-6 y^2 - 3.82615*10^-8 y^4 +
    2.55165*10^-8 y^6 + 6.32675*10^-10 y^8 - 5.31775*10^-12 y^10 -
    4.0754*10^-14 y^12 + 1.97991*10^-16 y^14 + 3.9499*10^-20 y^16 -
    1.91866*10^-22 y^18 + 3.7105*10^-26 y^20 - 1.41412*10^-30 y^22 -
    4.02724*10^-35 y^24)
F[y] = \!\(
\*UnderoverscriptBox[\(\[Sum]\), \(n = 0\), \(6\)]\(
\(\*SubscriptBox[\(F\), \(n\)]\)[y]\)\)


1.19003 y - 0.0323768 y^3 - 0.0615159 y^5 + 7.48755*10^-6 y^7 +
 y^7 (-0.0553789 + 0.0000118158 y^2 + 7.34615*10^-10 y^4) +
 y^9 (-0.0284778 - 0.0000794436 y^2 + 2.20947*10^-6 y^4 -
    3.14769*10^-10 y^6 - 1.3741*10^-14 y^8) +
 y^11 (-0.00953615 - 0.0000318582 y^2 + 7.10326*10^-7 y^4 +
    2.02993*10^-9 y^6 - 1.51721*10^-11 y^8 + 1.58923*10^-15 y^10 +
    5.33819*10^-20 y^12) +
 y^13 (-0.00224007 - 0.0000181534 y^2 + 8.18955*10^-7 y^4 +
    3.30807*10^-9 y^6 - 3.50038*10^-11 y^8 - 8.21156*10^-15 y^10 +
    4.6559*10^-17 y^12 - 3.83933*10^-21 y^14 - 1.04605*10^-25 y^16) +
 y^15 (-0.000388864 - 9.90345*10^-6 y^2 - 3.82615*10^-8 y^4 +
    2.55165*10^-8 y^6 + 6.32675*10^-10 y^8 - 5.31775*10^-12 y^10 -
    4.0754*10^-14 y^12 + 1.97991*10^-16 y^14 + 3.9499*10^-20 y^16 -
    1.91866*10^-22 y^18 + 3.7105*10^-26 y^20 - 1.41412*10^-30 y^22 -
    4.02724*10^-35 y^24)


Q1[y] = D[F[y], y] /. y -> 1
2.22045*10^-16
Q2[y] = F[y] /. y -> 1
```

107

1.

```
Solve[{Q1[y] == 0 && Q2[y] == 1}, {A, B}]

Solve[{False}, {1.19003, -0.194261}]


Q3[x] = F[y] /.
  y -> {0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1}


{0., 0.11897, 0.237727, 0.355973, 0.473212, 0.588553, 0.700367, \
0.805655, 0.898892, 0.969909, 1.}
```

```
DSolve[{ D[ D[ D[U[y], y], y], y] + 2 q E D[ D[U[y], y], y] +
    U[y] D[U[y], y] + (4 - L) q^2  D[U[y], y] == 0, U[0] == 1,
  D[ U[0], y] == 0, U[1] == 1}, U, y]
```

```
DSolve[{(4 - L) q^2 Derivative[1][U][y] + U[y] Derivative[1][U][y] +
    2 E q (U^\[Prime]\[Prime])[y] +
\!\(\*SuperscriptBox[\(U\),
TagBox[
RowBox[{"(", "3", ")"}],
Derivative],
MultilineFunction->None]\)[y] == 0, U[0] == 1, True, U[1] == 1}, U, y]
```

```
Subscript[U, 0][y] = 1 + ( y^2 R)/2
```

```
1 + (R y^2)/2
```

```
Subscript[Z, 0] = Subscript[U, 0][y] D[Subscript[U, 0][y], y]
```

```
R y (1 + (R y^2)/2)
```

```
q =  1/36 \pi
```

```
\pi/36
```

```
Re = 50
```

```
50
```

```
L = 1000
```

```
1000
```

```
D [ D[ D [U_{0}(y), y], y], y]
```

```
0
```

```
2 q 50 D[U_{0}(y), y], y]
```

```
(25 \pi R)/9
```

```
Subscript[U, 1][y] = \!\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(-\((2 \((
\*FractionBox[\(\[Pi]\), \(36\)])\)\)\ 50\
\*SubscriptBox[\(Z\), \(0\)] + \((4 - L)\)
\*SuperscriptBox[\(q\), \(2\)]\ \ D[\
\(\*SubscriptBox[\(U\), \(0\)]\)[y], \
        y])\)\) \[DifferentialD]y \[DifferentialD]y \
\[DifferentialD]y\)\)\)
```

```
(\[Pi] R y^4 (83 \[Pi] - 30 (10 + R y^2)))/2592
```

```
Subscript[Z, 1] =
 Subscript[U, 1][y] D[Subscript[U, 0][y], y] +
  Subscript[U, 0][y] D[Subscript[U, 1][y], y]


(\[Pi] R^2 y^5 (83 \[Pi] - 30 (10 + R y^2)))/2592 + (1 + (R y^2)/
    2) (-(5/216) \[Pi] R^2 y^5 +
    1/648 \[Pi] R y^3 (83 \[Pi] - 30 (10 + R y^2)))


Subscript[U, 2][y] = \!\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(-\((2 \((
\*FractionBox[\(\[Pi]\), \(36\)])\)\)\ 50\
\*SubscriptBox[\(Z\), \(1\)] + \((4 - L)\)
\*SuperscriptBox[\(q\), \(2\)]]\ \ D[\
\(\*SubscriptBox[\(U\), \(1\)]\)[y], \
          y])\)\)\) \[DifferentialD]y \[DifferentialD]y \
\[DifferentialD]y\)\)\)


(\[Pi]^2 R y^6 (48223 \[Pi]^2 - 6225 \[Pi] (56 + 9 R y^2) +
   1500 (420 + 135 R y^2 + 7 R^2 y^4)))/58786560


Subscript[Z, 2] =
 Subscript[U, 2][y] D[Subscript[U, 0][y], y] +
  Subscript[U, 1][y] D[Subscript[U, 1][y], y] +
  Subscript[U, 0][y] D[Subscript[U, 2][y], y]


(\[Pi]^2 R^2 y^7 (48223 \[Pi]^2 - 6225 \[Pi] (56 + 9 R y^2) +
```

```
   1500 (420 + 135 R y^2 +

      7 R^2 y^4)))/58786560 + (\[Pi] R y^4 (83 \[Pi] -

   30 (10 + R y^2)) (-(5/216) \[Pi] R^2 y^5 +

   1/648 \[Pi] R y^3 (83 \[Pi] - 30 (10 + R y^2))))/2592 + (1 + (

   R y^2)/2) ((\[Pi]^2 R y^6 (-112050 \[Pi] R y +

      1500 (270 R y + 28 R^2 y^3)))/

   58786560 + (\[Pi]^2 R y^5 (48223 \[Pi]^2 -

      6225 \[Pi] (56 + 9 R y^2) +

      1500 (420 + 135 R y^2 + 7 R^2 y^4)))/9797760)
Subscript[U, 3][y] = \!\(

\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(

\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(

\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(-\((2 \((

\*FractionBox[\(\[Pi]\), \(36\)])\)\ 50\

\*SubscriptBox[\(Z\), \(2\)] + \((4 - L)\)

\*SuperscriptBox[\(q\), \(2\)]\ \ D[\

\(\*SubscriptBox[\(U\), \(2\)]\)[y], \

        y])\)\)\) \[DifferentialD]y \[DifferentialD]y \

\[DifferentialD]y\)\)\)


1/7263197061120 \[Pi]^3 R y^8 (81765541 \[Pi]^3 -

    88661430 \[Pi]^2 (10 + 3 R y^2) +

    1079000 \[Pi] (2970 + 1782 R y^2 + 127 R^2 y^4) -

    75000 (51480 + 46332 R y^2 + 6604 R^2 y^4 + 231 R^3 y^6))


Subscript[Z, 3] =
 Subscript[U, 3][y] D[Subscript[U, 0][y], y] +

  Subscript[U, 2][y] D[Subscript[U, 1][y], y] +
```

```
    Subscript[U, 0][y] D[Subscript[U, 3][y], y] +
    Subscript[U, 1][y] D[Subscript[U, 2][y], y]


1/7263197061120 \[Pi]^3 R^2 y^9 (81765541 \[Pi]^3 -
      88661430 \[Pi]^2 (10 + 3 R y^2) +
      1079000 \[Pi] (2970 + 1782 R y^2 + 127 R^2 y^4) -
      75000 (51480 + 46332 R y^2 + 6604 R^2 y^4 +
        231 R^3 y^6)) + (\[Pi]^2 R y^6 (48223 \[Pi]^2 -
      6225 \[Pi] (56 + 9 R y^2) +
      1500 (420 + 135 R y^2 + 7 R^2 y^4)) (-(5/216) \[Pi] R^2 y^5 +
      1/648 \[Pi] R y^3 (83 \[Pi] -
        30 (10 + R y^2))))/58786560 + (\[Pi] R y^4 (83 \[Pi] -
      30 (10 + R y^2)) ((\[Pi]^2 R y^6 (-112050 \[Pi] R y +
        1500 (270 R y + 28 R^2 y^3))))/
      58786560 + (\[Pi]^2 R y^5 (48223 \[Pi]^2 -
        6225 \[Pi] (56 + 9 R y^2) +
        1500 (420 + 135 R y^2 + 7 R^2 y^4)))/9797760))/2592 + (1 + (
      R y^2)/2) ((\[Pi]^3 R y^8 (-531968580 \[Pi]^2 R y +
        1079000 \[Pi] (3564 R y + 508 R^2 y^3) -
        75000 (92664 R y + 26416 R^2 y^3 + 1386 R^3 y^5)))/
      7263197061120 +
      1/907899632640 \[Pi]^3 R y^7 (81765541 \[Pi]^3 -
        88661430 \[Pi]^2 (10 + 3 R y^2) +
        1079000 \[Pi] (2970 + 1782 R y^2 + 127 R^2 y^4) -
        75000 (51480 + 46332 R y^2 + 6604 R^2 y^4 + 231 R^3 y^6)))
Subscript[U, 4][y] = \!\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
\*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(
```

*SubsuperscriptBox[\(\[Integral]\), \(0\), \(y\)]\(-\((2 \(((

*FractionBox[\(\[Pi]\), \(36\)])\)\)\ 50\

*SubscriptBox[\(Z\), \(3\)] + \((4 - L)\)

*SuperscriptBox[\(q\), \(2\)]\ \ D[\

\(*SubscriptBox[\(U\), \(3\)]\)[y], \

        y])\)\)\) \[DifferentialD]y \[DifferentialD]y \

\[DifferentialD]y\)\)\)

1/8401194776656281600 \[Pi]^4 R y^10 (807598248457 \[Pi]^4 -

    1061465386800 \[Pi]^3 (11 + 7 R y^2) +

    10540170000 \[Pi]^2 (6006 + 7644 R y^2 + 773 R^2 y^4) -

    476212500 \[Pi] (320320 + 611520 R y^2 + 123680 R^2 y^4 +

        5481 R^3 y^6) +

    11250000 (12252240 + 31187520 R y^2 + 9461520 R^2 y^4 +

        838593 R^3 y^6 + 22099 R^4 y^8))

U[y] = \!\(

*UnderoverscriptBox[\(\[Sum]\), \(n = 0\), \(4\)]\(

\(*SubscriptBox[\(U\), \(n\)]\)[y]\)\)

1 + (R y^2)/2 + (\[Pi] R y^4 (83 \[Pi] -

    30 (10 + R y^2)))/2592 + (\[Pi]^2 R y^6 (48223 \[Pi]^2 -

    6225 \[Pi] (56 + 9 R y^2) +

    1500 (420 + 135 R y^2 + 7 R^2 y^4)))/58786560 +

 1/7263197061120 \[Pi]^3 R y^8 (81765541 \[Pi]^3 -

    88661430 \[Pi]^2 (10 + 3 R y^2) +

    1079000 \[Pi] (2970 + 1782 R y^2 + 127 R^2 y^4) -

    75000 (51480 + 46332 R y^2 + 6604 R^2 y^4 + 231 R^3 y^6)) +

 1/8401194776656281600 \[Pi]^4 R y^10 (807598248457 \[Pi]^4 -

    1061465386800 \[Pi]^3 (11 + 7 R y^2) +

    10540170000 \[Pi]^2 (6006 + 7644 R y^2 + 773 R^2 y^4) -

```
          476212500 \[Pi] (320320 + 611520 R y^2 + 123680 R^2 y^4 +

             5481 R^3 y^6) +

          11250000 (12252240 + 31187520 R y^2 + 9461520 R^2 y^4 +

             838593 R^3 y^6 + 22099 R^4 y^8))

QG[y] = U[y] /. y -> 1

1 + R/2 + (\[Pi] R (83 \[Pi] -

     30 (10 + R)))/2592 + (\[Pi]^2 R (48223 \[Pi]^2 -

     6225 \[Pi] (56 + 9 R) +

     1500 (420 + 135 R +

        7 R^2)))/58786560 + (\[Pi]^3 R (81765541 \[Pi]^3 -

     88661430 \[Pi]^2 (10 + 3 R) +

     1079000 \[Pi] (2970 + 1782 R + 127 R^2) -

     75000 (51480 + 46332 R + 6604 R^2 + 231 R^3)))/7263197061120 +

  1/8401194776656281600 \[Pi]^4 R (807598248457 \[Pi]^4 -

      1061465386800 \[Pi]^3 (11 + 7 R) +

      10540170000 \[Pi]^2 (6006 + 7644 R + 773 R^2) -

      476212500 \[Pi] (320320 + 611520 R + 123680 R^2 + 5481 R^3) +

      11250000 (12252240 + 31187520 R + 9461520 R^2 + 838593 R^3 +

         22099 R^4))

  Solve[{QG[y] == 0}, {R}]

  {{R -> Root[

     8401194776656281600 + (4200597388328140800 -

         972360506557440000 \[Pi] + 359053120384358400 \[Pi]^2 -

         54284411877696000 \[Pi]^3 + 10736124200081280 \[Pi]^4 -

         1178069416524000 \[Pi]^5 + 157880826983880 \[Pi]^6 -

         11676119254800 \[Pi]^7 +

         807598248457 \[Pi]^8) #1 + (-97236050655744000 \[Pi] +

         28939300790400000 \[Pi]^2 - 12025887217344000 \[Pi]^3 +
```

2574898457040000 \[Pi]^4 - 598872176557200 \[Pi]^5 +

80569059480000 \[Pi]^6 -

7430257707600 \[Pi]^7) #1^2 + (1500556337280000 \[Pi]^2 -

572903604000000 \[Pi]^3 + 264945430440000 \[Pi]^4 -

58897962000000 \[Pi]^5 +

8147551410000 \[Pi]^6) #1^3 + (-20039481000000 \[Pi]^3 +

9434171250000 \[Pi]^4 - 2610120712500 \[Pi]^5) #1^4 +

248613750000 \[Pi]^4 #1^5 &, 1]}, {R ->

Root[8401194776656281600 + (4200597388328140800 -

972360506557440000 \[Pi] + 3590531203843584000 \[Pi]^2 -

54284411877696000 \[Pi]^3 + 10736124200081280 \[Pi]^4 -

1178069416524000 \[Pi]^5 + 157880826983880 \[Pi]^6 -

11676119254800 \[Pi]^7 +

807598248457 \[Pi]^8) #1 + (-97236050655744000 \[Pi] +

28939300790400000 \[Pi]^2 - 12025887217344000 \[Pi]^3 +

2574898457040000 \[Pi]^4 - 598872176557200 \[Pi]^5 +

80569059480000 \[Pi]^6 -

7430257707600 \[Pi]^7) #1^2 + (1500556337280000 \[Pi]^2 -

572903604000000 \[Pi]^3 + 264945430440000 \[Pi]^4 -

58897962000000 \[Pi]^5 +

8147551410000 \[Pi]^6) #1^3 + (-20039481000000 \[Pi]^3 +

9434171250000 \[Pi]^4 - 2610120712500 \[Pi]^5) #1^4 +

248613750000 \[Pi]^4 #1^5 &, 2]}, {R ->

Root[8401194776656281600 + (4200597388328140800 -

972360506557440000 \[Pi] + 3590531203843584000 \[Pi]^2 -

54284411877696000 \[Pi]^3 + 10736124200081280 \[Pi]^4 -

1178069416524000 \[Pi]^5 + 157880826983880 \[Pi]^6 -

11676119254800 \[Pi]^7 +

```
            807598248457 \[Pi]^8) #1 + (-97236050655744000 \[Pi] +

            28939300790400000 \[Pi]^2 - 12025887217344000 \[Pi]^3 +

            2574898457040000 \[Pi]^4 - 598872176557200 \[Pi]^5 +

            80569059480000 \[Pi]^6 -

            7430257707600 \[Pi]^7) #1^2 + (1500556337280000 \[Pi]^2 -

            572903604000000 \[Pi]^3 + 264945430440000 \[Pi]^4 -

            58897962000000 \[Pi]^5 +

            8147551410000 \[Pi]^6) #1^3 + (-20039481000000 \[Pi]^3 +

            9434171250000 \[Pi]^4 - 2610120712500 \[Pi]^5) #1^4 +

        248613750000 \[Pi]^4 #1^5 &, 3]}, {R ->

    Root[8401194776656281600 + (4200597388328140800 -

            972360506557440000 \[Pi] + 359053120384358400 \[Pi]^2 -

            54284411877696000 \[Pi]^3 + 10736124200081280 \[Pi]^4 -

            1178069416524000 \[Pi]^5 + 157880826983880 \[Pi]^6 -

            11676119254800 \[Pi]^7 +

            807598248457 \[Pi]^8) #1 + (-97236050655744000 \[Pi] +

            28939300790400000 \[Pi]^2 - 12025887217344000 \[Pi]^3 +

            2574898457040000 \[Pi]^4 - 598872176557200 \[Pi]^5 +

            80569059480000 \[Pi]^6 -

            7430257707600 \[Pi]^7) #1^2 + (1500556337280000 \[Pi]^2 -

            572903604000000 \[Pi]^3 + 264945430440000 \[Pi]^4 -

            58897962000000 \[Pi]^5 +

            8147551410000 \[Pi]^6) #1^3 + (-20039481000000 \[Pi]^3 +

            9434171250000 \[Pi]^4 - 2610120712500 \[Pi]^5) #1^4 +

        248613750000 \[Pi]^4 #1^5 &, 4]}, {R ->

    Root[8401194776656281600 + (4200597388328140800 -

            972360506557440000 \[Pi] + 359053120384358400 \[Pi]^2 -

            54284411877696000 \[Pi]^3 + 10736124200081280 \[Pi]^4 -
```

```
       1178069416524000 \[Pi]^5 + 157880826983880 \[Pi]^6 -

       11676119254800 \[Pi]^7 +

       807598248457 \[Pi]^8) #1 + (-97236050655744000 \[Pi] +

       28939300790400000 \[Pi]^2 - 12025887217344000 \[Pi]^3 +

       2574898457040000 \[Pi]^4 - 598872176557200 \[Pi]^5 +

       80569059480000 \[Pi]^6 -

       7430257707600 \[Pi]^7) #1^2 + (1500556337280000 \[Pi]^2 -

       572903604000000 \[Pi]^3 + 264945430440000 \[Pi]^4 -

       58897962000000 \[Pi]^5 +

       8147551410000 \[Pi]^6) #1^3 + (-20039481000000 \[Pi]^3 +

       9434171250000 \[Pi]^4 - 2610120712500 \[Pi]^5) #1^4 +

     248613750000 \[Pi]^4 #1^5 &, 5]}}




    {{R -> -1.91635}, {R -> -4.61078 - 21.6807 I}, {R -> -4.61078 +

     21.6807 I}, {R -> 15.9154 - 10.7311 I}, {R ->

    15.9154 + 10.7311 I}}




   R = -1.9163528468097948`

   -1.91635

   Q9[x] = U[y] /.

  y -> {0.0, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45,

     0.50, 0.55, 0.60, 0.65, 0.70, 0.75, 0.80,  0.85, 0.90, 0.95, 1}




    {1., 0.997605, 0.990427, 0.978486, 0.96181, 0.940437, 0.914404, \
0.883743, 0.848474, 0.808593, 0.764064, 0.714806, 0.660677, 0.601462, \
0.536852, 0.466421, 0.389602, 0.305652, 0.213611, 0.11225,
```

118

5.5728*10^-17}