On completion and connectedness properties of Császár frames

By

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I declare that the dissertation hereby submitted to the University of Limpopo, for the degree of Master of Science (Mathematics) has not previously been submitted by me for a degree at this or any other university; that it is my work in design and in execution, and that all material contained herein has been duly acknowledged.

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Dedication

I dedicate this dissertation to my mother, Kicky, my children, Ntlhari and Nhlukiso, the Shikweni and Qaba famalies. May God bless and keep you.

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Contents

Abstract

A Császár frame is a pointfree version of syntopogenous space, itself a concept that is attributed to Ákos Császár [14]. In his two papers, Chung ([12] and [13]) characterised few types of Császár frames and extended Hong's construction [21] to the Cauchy completions in Császár frames. From his results, we anchored objectives of our study on the actions of certain frame homomorphisms on proximal Császár frames, as well as co-reflective subcategories of Cauchy complete Császár frames.

We conclude the dissertation by constructing the compactification of proximal Császár frames by applying the methods of Banaschewski and Mulvey [7]. We introduce a weak notion of connectedness of Császár frames and show, following the approach of Baboolal and Banaschewski [4], that most of the standard results on connectedness are do-able in the setting of Császár frames.

LIST OF SYMBOLS

$\mathbb{F}$**rm** : The category of frames and frame homomorphisms

$\mathbb{C}$**s**$\mathbb{F}$**rm** : The category of Cs*á*sz*á*r frames and frame homomorphisms

$\mathbb{PC}$**s**$\mathbb{F}$**rm** : The category of proximal Cs*á*sz*á*r frames and uniform homomorphisms

$\mathbb{CCC}$**s**$\mathbb{F}$**rm** : The category of Cauchy Complete Cs*á*sz*á*r frames and Cauchy homomorphisms

$\mathbb{CCUC}$**s**$\mathbb{F}$**rm** : The category of Cauchy Complete Uniform Cs*á*sz*á*r frames and frame homomorphisms

Chapter 0

Introduction and summary

In his celebrated article "The point of pointless topology", Johnstone [24] showed that many results from topology were extended to generalized spaces by John Isbell in [23]. Frames are complete lattices which satisfy a distributive law and the morphisms are maps which preserves finite meet and arbitrary joins.

Following on the groundwork laid by Isbell [22] on uniform frames, Banishes and Pultr [8] introduced complete nearness frames, especially *Cauchy complete nearness* frames. In addition to constructing *the* completion of nearness frames, they showed that the completion of nearness frames gives rise to the category of almost uniform spaces. Isbell [22] also showed that the use of uniform structures is required on results on paracompactness, hence uniform frames are important in Topology.

Since the introduction of Pointfree Topology, many topological facts are re-discovered as generalisations from results in this pointfree setting. For example, abstractly defined lattices of open sets are regarded as frames. However, many proofs in Pointfree Topology are often more suggestive and transparent than they would have been in General Topology [6]. It is in this respect that Pointfree Topology is more fundamental than Classical Topology.

When Chung [12] introduced Cauchy complete Cs*á*sz*á*r frames, he used strict extensions of frames to construct their Cauchy completion. It was shown that the Cauchy completion gives rise to a coreflection in the categories of Cs*á*sz*á*r frames and uniform Cs*á*sz*á*r frames. In extending syntopogeneous spaces to other platforms of General Topology, Chung [12] further showed that connection properties are possible in syntopogenous spaces. When connectedness was introduced into the category of frames and frame homomorphism, it was applied in constructions related to Stone-Cech compactification. A frame homomorphism is a map between frames which preserves the bottom (zero) and the (top) as well as the generalised distributive law.

Connectedness and its importance in relation to other topological properties was studied by many people, see for instance Baboolal and Banaschewski [4]. However, in his paper titled "On the local connectedness of frames", Chen [11], showed that connectedness is possible in frames where amongst many results, he showed that any Tychonoff space containing X as a dense subspace is locally connected if and only if X is locally connected and pseudocompact. Baboolal and Banaschewski also showed that the Stone-Cech compactification βX of a Tychonoff space X is locally connected if and only if X is locally connected and pseudocompact has a frame-theoretic counterpart.

Synopsis of the dissertation

Chapter 1

In this chapter we define some topological concepts. We study syntopogenous and Cs*á*sz*á*r orders and their properties as introduced by [14]. The Cs*á*sz*á*r orders are studied according to [12], see also [14]. We also show how to construct Cs*á*sz*á*r order from the domain to the codomain and from the codomain to the domain. We look at their properties and show that the right adjoint of frame homomorphism between Cs*á*sz*á*r frames reflects symmetric and strong Cs*á*sz*á*r orders.

Chapter 2

Our focus is in properties of filters and selected frame homomorphisms on proximal Cs*á*sz*á*r frames. Amongst the important results we prove is the fact that uniform frame homomorphism is a Cauchy homomorphism, essentially meaning that the category $\mathbb{UC}\mathbf{s}\mathbb{F}\mathbf{rm}$ of uniform Cs*á*sz*á*r frames and uniform homomorphisms is a subcategory of the category $\mathbb{PC}\mathbf{s}\mathbb{F}\mathbf{rm}$ of proximal Cs*á*sz*á*r frames and Cauchy homomorphisms.

Chapter 3

In this chapter, we first outline the role of filters on completion of frames, especially nearness frames paving the way for the constructions in the remaining three sections: critical to our later constructions is the concept of strict extension which Apfel has(correctly by the way) called "Hong's construction" (see Apfel [2]). We then go on to look into some properties of completions of Cs*á*sz*á*r frames and construct their Cauchy completions according to [12]: we give detailed proofs of the following results:

(i) The category of Cauchy Complete Cs*á*sz*á*r frames and Cauchy homomorphisms is coreflective in the category of Cs*á*sz*á*r frames and Cauchy homomorphisms

(ii) The category of Cauchy complete uniform Cs*á*sz*á*r frames is coreflective in the category of uniform Cs*á*sz*á*r frames and uniform frame homomorphisms.

(iii) The category of Cauchy proximal Cs*á*sz*á*r frames is coreflective in the category of proximal Cs*á*sz*á*r frames and continuous frame homomorphisms.

Chapter 4

We focus on the compactification of proximal frames connectedness of Cs*á*sz*á*r frames. This follows since connectedness was not looked into when Cs*á*sz*á*r introduced syntopogenuos spaces in [14]. We refer to Pervin and Sieber in [32] and translate their approach into the setting of Cs*á*sz*á*r frames. We follows Baboolal's paper [3]

Whereas the first three chapters are devoted to basic facts related to Chung's Cs*á*sz*á*r frames and the expansion of sketchy proofs and revisiting coreflective subcategories of the category $\mathbb{C}$s$\mathbb{F}$**rm** of Cs*á*sz*á*r frames and frame homomorphisms, as far as we know the results on compactification of proximal Cs*á*sz*á*r frames, and $\mathcal{L}$-connectedness in Chapter 4 are new.

Chapter 1

Syntopogenous and Cs*á*sz*á*r Orders

This chapter commences with standard pointfree concepts and basic facts that will be required in the study. We study syntopogenous orders and their properties as introduced by Cs*á*sz*á*r [14], and from them we look at constructions related to Cs*á*sz*á*r orders whose origin is attributed to Chung [12]: specifically, we will show how to construct Cs*á*sz*á*r order from the domain to the codomain and from the codomain to the domain, as well as show that the right adjoint of frame homomorphism between Cs*á*sz*á*r frames reflects symmetric and strong Cs*á*sz*á*r orders.

1.1 Preliminary Concepts

We recall that a frame is a complete lattice L in which the following generalized distributive law is satisfied:

$$a \wedge \bigvee S \;=\; \bigvee \{a \wedge s \mid s \in S\},$$

for all $a \in L$ and all $S \subseteq L$. A *frame homomorphism* $h : M \to L$ is a map between frames which preserves finite meets (including the **top element** e) and arbitrary joins (including the **bottom element** 0). Frame homomorphisms are closed under compositions, so we have the category $\mathbb{F}$**rm** of frames and frame homomorphisms between them. Unless specified otherwise, our knowledge of frames and locales stems from Johnstone in [24], Picado and Pultr in [30] and Pultr in [31].

An element u in a frame L is said to be *rather below* an element $v \in L$ (written $u \prec v$) if there is an element $w \in L$ such that $u \wedge w = 0$ and $v \vee w = e$. This definition is equivalent to saying $u \prec v$ if and only if $u^* \vee v = e$, where u^* is called the *pseudocomplement* of u and is defined by

$$u^* = \bigvee \{t \in L \mid u \wedge t = 0\}.$$

A frame homomorphism $h : M \to L$ is said to be *dense* if $h(x) = 0_L$ implies $x = 0_M$. Dually, h is called *co-dense* if $h(u) = e_L$ implies that $u = e_M$.

Remark 1.1.1. We will need the following elementary properties of pseudocomplements in any frame M: for any $u, v \in M$, we have

a): $e^* = 0$ **and** $0^* = e$**:** This follows from the fact that if $u \wedge e = 0$ then $u = 0$ and (from the definition of *) the largest u for which $u \wedge 0 = 0$ is e.

b): $u \leq u^{**}$**:** We know that $u^* \wedge u = 0$ and $u^* \wedge u^{**} = 0$, so $u \leq u^{**}$ follows from u^{**} being the largest for which the meet with u^* is 0.

c): **If** $u \leq v$ **then** $v^* \leq u^*$**:** This is immediate from the calculation $u \wedge v \leq u \wedge v^* = 0$ implies that $u \wedge v^* = 0$; so, together with $u \wedge u^* = 0$, we conclude that $v^* \leq u^*$.

d): $u = u^{***}$**:** Since $u \leq u^{**}$, we have $u^{***} \leq u^*$. But $u^* \leq (u^*)^{**}$, so we must have $u^* = u^{***}$.

e): $(u \vee v)^* = u^* \wedge v^*$**:** That $(u \vee v)^* \leq u^*$ and $(u \vee v)^* \leq v^*$ so that $(u \vee v)^* \leq u^* \wedge v^*$ follows from c). For the reverse inclusion, we note that

$$(u^* \wedge v^*) \wedge (u \vee v) = (u^* \wedge v^* \wedge u) \vee (u^* \wedge v^* \wedge v) = 0$$

so, together with the definition of $(u \wedge v)^*$, this ensures that $u^* \wedge v^* \leq (u \wedge v)^*$. □

Given a frame homomorphism $h : M \to L$, the *Galois* right adjoint $h_* : L \to M$ is defined by

$$h(x) \leq y \text{ if and only if } x \leq h_*(y)$$

where

$$h_*(y) = \bigvee \{x \in M \mid h(x) \leq y\}.$$

Lemma 1.1.2. *Let $h : M \to L$ be a frame homomorphism, then:*

i) $x \leq h_*(h(x))$ *for all $x \in M$.*

ii) *If h is onto then $h(x) \wedge y^* = 0$ implies $y^* \leq h(x^*)$.*

Proof:

i) If we set $y = h(x)$ in the definition and use the fact that $h(x) \leq h(x)$, then $x \leq h_*(h(x))$.

ii) Note that if h is onto, it holds that $h(x^*) = h(x)^*$ (see Pultr [31]). For, by definition of pseudocomplement, we have that

$$\begin{aligned} h(x^*) &= h[\bigvee\{z \in M \mid z \wedge x = 0\}] \\ &= \bigvee\{h(z) \in L \mid h(z \wedge x) = 0\} \\ &= \bigvee\{h(z) \in L \mid h(z) \wedge h(x) = 0\} \\ &= \bigvee\{y \in L \mid y \wedge h(x) = 0\} \\ &= h(x)^*. \end{aligned}$$

Now we find that (by definition of y) $h(x) \wedge y^* = 0$ implies $y^* \leq h(x)^* = h(x^*)$. □

Lemma 1.1.3. For a dense frame homomorphism $h : M \to L$ it holds that

$$h_*(h(x)) \leq x^{**}.$$

Proof:

Note that $x \leq x^{**}$, then $x^{**} \wedge x = x$. This implies that

$$\begin{aligned} h(x^{**}) \wedge h(x) &= h(x) \\ \Rightarrow h(x^{**}) \wedge (id)h(x) &= (id)(h(x)) \\ \Rightarrow h(x^{**}) \wedge (hh_*)h(x) &= (hh_*)(h(x)) \\ \Rightarrow h(x^{**}) \wedge h(h_*h(x)) &= h(h_*h(x)) \\ \Rightarrow x^{**} \wedge h_*(h(x)) &= h_*(h(x)) \end{aligned}$$

Therefore $h_*h(x) \leq x^{**}$. □

Lemma 1.1.4. *Let $h : M \to L$ be a frame homomorphism between frames.*

a) If $x \prec y$ in M then $h(x) \prec h(y)$ in L.

b) If $x \prec y$ in M then $x \prec y^{**}$ in M: If $x \prec y$ then from the properties of pseudocomplement we have $x^* \vee y = e$; so, $x \leq x^{**}$ implies that

$$x^* \vee x^{**} \geq x^* \vee x = e,$$

which means that

$$x^* \vee x^{**} = e \text{ and so } x \prec x^{**}. \square$$

We say a frame L is a *regular* frame if for all $v \in L$ we have

$$v = \bigvee\{u \mid u^* \vee v = e\} = \bigvee\{u \mid u \prec v\}.$$

We say a frame homomorphism $h : M \to L$ is a monomorphism if for any two frame homomorphisms $g, f : K \to M$ with $hg = hf$ then $g = f$. One well-known result that we will need in the sequel is the following (whose proof is adapted from Murugan [27] who has assembled a number of important results on dense homomorphisms):

Proposition 1.1.5. *A dense homomorphism on regular frames is a monomorphism.*

Proof. Take a dense homomorphism $h : M \to L$ between regular frames, and two frame homomorphisms $f,\ g : K \to M$ such that $h \circ f = h \circ g$. Then for $x \in K$, we have $h \circ f(x) = h \circ g(x)$. The fact that K is regular implies that for all $y \in K$, $y = \bigvee\{x \in K \mid x \prec y\}$. But $x \prec y$ implies that there exists $z \in K$ such that $x \wedge z = 0$ and $y \vee z = e$. We will then have that

$$(h \circ f)(x \wedge z) = (h \circ f)(0), \quad \text{and so } h(f(x)) \wedge h(f(z)) = 0.$$

Since $h \circ f = h \circ g$, it follows that

$$h(f(x)) \wedge h(g(z)) = 0 \text{ and } h(f(x) \wedge g(z)) = 0.$$

Since h is dense, we must have $f(x) \wedge g(z) = 0$.

On the other hand, $y \vee z = e$ implies $g(y \vee z) = g(e)$; thus $g(y) \vee g(z) = e$. Then

$$\begin{aligned} f(x) &= f(x) \wedge e \\ &= f(x) \wedge (g(y) \vee g(z)) \\ &= (f(x) \wedge g(y)) \vee (f(x) \wedge g(z)) \\ &= (f(x) \wedge g(y)) \vee 0 \\ &= f(x) \wedge g(y) \\ &\leq g(y). \end{aligned}$$

This implies that

$$f(y) = \bigvee_{x \prec y} f(x) \leq g(y).$$

Then $f(y) \leq g(y)$. By symmetry we have $g(y) \leq f(y)$, hence $g(y) = f(y)$ □

Moving on to compactness in frames, we have the following (which is not difficult to translate from a classical setting):

Let L be a frame, then a subset C of L is a *cover* of L if $\bigvee C = e$.

Definition 1.1.6. A frame L is said to be *compact* if for every subset U of L with $\bigvee U = e$ there exists a finite subset T of U with $\bigvee T = e$.

1.2 Properties of Syntopogenous orders

We study syntopogenous orders in accordance to Császár [14]. Motivating examples of syntopogenous orders are given and we include some properties of syntopogenous from [17].

Motivating example 1.2.1. Given a topological space (X, τ), define a binary relation $\leq$ on $\mathcal{P}(X) \times \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X, by

$$U \leq V \text{ if and only if } U \subseteq V^{\circ}, \text{ where } V^{\circ} \text{ is the interior of } V$$

Then $\leq$ satisfies the following five properties:

i) $\emptyset \leq \emptyset$ and $X \leq X$.

ii) $U \leq V \Rightarrow U \subseteq V$.

iii) $(U \leq V,\ H \leq K) \Rightarrow U \cap H \leq V \cap K$.

iv) $(U \leq V,\ H \leq K) \Rightarrow U \cup H \leq V \cup K$.

v) $U \subseteq V \leq W \subseteq H \Rightarrow U \leq H$.

Proof:

i) This is trivial since $\emptyset = \emptyset^\circ$ and $X = X^\circ$

ii) Suppose that $U \leq V$, then by definition of $\leq$, it follows that $U \subseteq V^\circ$. But we know that $V^\circ \subseteq V$, thus we have $U \subseteq V$.

iii) Suppose that $U \leq V$ and $H \leq K$. Then it follows that

$$U \subseteq V^\circ \text{ and } H \subseteq K^\circ,$$

so that

$$U \cap H \subseteq V^\circ \cap K^\circ = (V \cap K)^\circ.$$

But $(V \cap K)^\circ \subseteq (V \cap K)$, hence we have that $U \cap H \leq V \cap K$.

iv) Suppose that $U \leq V$ and $H \leq K$. Then $U \subseteq V^\circ$ and $H \subseteq K^\circ$. Since $V \subseteq V \cup K$ and $K \subseteq V \cup K$, we find that

$$V^\circ \subseteq (V \cup K)^\circ \text{ and } K^\circ \subseteq (V \cup K)^\circ.$$

This implies that

$$U \cup H \subseteq V^\circ \cup K^\circ \subseteq (V \cup K)^\circ.$$

Therefore $U \cup H \leq V \cup K$.

v) Suppose that $U \subseteq V \leq W \subseteq H$. Then $V \subseteq W^\circ$. We know that $W^\circ \subseteq W$, and then

$$U \subseteq V \subseteq W^\circ \subseteq H.$$

But H° is the largest open subset in H, hence it holds that $W^\circ \subseteq H^\circ$. Thus it follows that $U \subseteq H^\circ$ and so $U \leq H$ results. □

Motivating example 1.2.2. Given a topological space (X, τ) a binary relation $\leq$ on $\mathcal{P}(X)$ defined by

$$W \leq V \text{ if and only if } \vartheta(W) \subseteq V,$$

(where $\vartheta(W) \subseteq V$ means $\vartheta(W)$ is a uniform neighbourhood of W, that is for all $x \in W$, then $x \in \vartheta(W)$). Then $\leq$ satisfies the following five properties:

i) $\emptyset \leq \emptyset$ and $X \leq X$.

ii) $U \leq V \Rightarrow U \subseteq V$.

iii) $(U \leq V,\ H \leq K) \Rightarrow U \cap H \leq V \cap K$.

iv) $(U \leq V,\ H \leq K) \Rightarrow U \cup H \leq V \cup K$.

v) $U \subseteq V \leq W \subseteq H \Rightarrow U \leq H$.

Proof:

i) Since $\vartheta(\emptyset) \subseteq \emptyset$ and $\vartheta(X) \subseteq X$ (trivially), we must have $\emptyset \leq \emptyset$ and $X \leq X$.

ii) Suppose that $U \leq V$. By definition, it follows that $\vartheta(U) \subseteq V$. Since $\vartheta(U)$ is a neighbourhood of U, it implies that for all $x \in U$, $x \in \vartheta(U)$. Then $x \in V$, which then implies that $U \subseteq V$.

iii) Suppose that $U \leq V$ and $H \leq K$. It then follows that $\vartheta(U) \subseteq V$ and $\vartheta(H) \subseteq K$. We have that

$$\vartheta(U) \cap \vartheta(H) \subseteq V; \quad \vartheta(U) \cap \vartheta(H) \subseteq K,$$

so that

$$\vartheta(U) \cap \vartheta(H) \subseteq V \cap K.$$

Therefore $U \cap H \leq V \cap K$.

iv) Suppose that $U \leq V$ and $H \leq K$. Then it follows that $\vartheta(U) \subseteq V$ and $\vartheta(H) \subseteq K$. We have the following relations

$$\vartheta(U) \subseteq V \cup K, \vartheta(H) \subseteq V \cup K.$$

This implies that

$$\vartheta(U) \cup \vartheta(H) \subseteq V \cup K.$$

Therefore $U \cup H \leq V \cup K$.

v) Suppose that $U \subseteq V \leq W \subseteq H$. This implies that $\vartheta(V) \subseteq W$, but $\vartheta(U) \subseteq \vartheta(V)$, hence we have that $\vartheta(U) \subseteq H$. Therefore $U \leq H$.

We follow Császár [14] in defining a syntopogenous space. See also Flax [17].

Definition 1.2.3. A Let X be a non-empty set and $\leq$ a relation on X. Then $\leq$ is said to be a *semi-topogenous ordern* on X if the following conditions are satisfied:

(i) $\emptyset \leq \emptyset$ and $X \leq X$

(ii) $A \leq B$ implies $A \subseteq B$

(ii) $A \subseteq C \leq D \subseteq B$ implies $A \leq B$

Definition 1.2.3. B A relation $\leq$ on X is a *topogenous order* if it satisfies all the three conditions in Definition 1.2.3 A and the following contition:

$$A \leq B \text{ and } C \leq D \text{ implies } (A \cap C) \leq (B \cap D) \text{ and } (A \cup B) \leq (B \cup D)$$

Definition 1.2.3 C Let a family of topogenous orders on X be denoted by $\mathcal{S}$. Then $\mathcal{S}$ is said to be a *syntopogenous structure* on X if and only if the following conditions are satisfied:

(i) If $\leq_1, \leq_2 \in \mathcal{S}$ then there exists $\leq_3 \in \mathcal{S}$ finer than both $\leq_1$ and $\leq_2$.

(ii) If $\leq \in \mathcal{S}$, then there exists $\leq_1 \in \mathcal{S}$ such that $A \leq B$ implies that there exists D with $A \leq_1 D \leq_1 B$.

Observation 1.2.4. Given a syntopogenous space (X, γ), say $\gamma = \{\leq_i \mid i \in I\}$, we can topologize X as follows. Define the topology τ_γ associated with γ by

$$U \in \tau_\gamma \text{ if and only if for each } x \in U \text{ there exists } i \in I \text{ such that } \{x\} \leq_i U.$$

Proof:

i) That $\emptyset,\ X \in \tau_\gamma$ is trivial.

ii) If $U, V \in \tau_\gamma$ and suppose that $x \in U \cap V$, then there exists $i \in I$ such that $\{x\} \leq_i U$ and $\{x\} \leq_i V$. By SNY5, we get $\{x\} \leq_i U \cap V$, hence $U \cap V \in \tau_\gamma$.

iii) Suppose that $x \in \bigcup U$. Then $x \in U_x$ for some $U_x \in \tau_\gamma$ and then $\{x\} \leq_i U_x \subseteq \bigcup_{U \in \tau_\gamma} U$. by SNY3 it follows that $\{x\} \leq_i \bigcup_{U \in \tau_\gamma} U$ and the $\bigcup_{U \in \tau_\gamma} U \in \tau_\gamma$. □

Proposition 1.2.5. *For any topological space (X, τ), the relation $\leq$ defined on $\mathcal{P}(X)$ by $U \leq V$ if and only if there are $H, K \in \tau$ such that $U \subseteq H \subseteq K \subseteq V$ is syntopogenous, making $(X, \leq)$ a syntopogenous space.*

Proof:

i) We know that $\emptyset \leq \emptyset$ and $X \leq X$.

ii) Suppose that $U \leq V$, so there are $H, K \in \tau$ such that $U \subseteq H \subseteq K \subseteq V$ from which $U \subseteq V$ is clear.

iii) Suppose that $U \subseteq V \leq W \subseteq H$. Then by definition it follows that $U \leq H$.

iv) Suppose $U \in \mathcal{A}$, where $U \leq V$ and $\mathcal{A} \subseteq \mathcal{P}(X)$. We then have that $\bigcup U \leq V$. Therefore $\bigcup \mathcal{A} \leq V$.

v) Suppose that $U \leq V$ and $U \leq W$. Then there are $H, K, M, N \in \tau$ such that

$$U \subseteq H \subseteq K \subseteq V \text{ and } U \subseteq M \subseteq N \subseteq W,$$

which imply

$$U \subseteq H \cap M \subseteq K \cap N \subseteq V \cap W.$$

Therefore $U \leq V \cap W$. □

1.3 Properties of Cs*á*sz*á*r orders

We study Cs*á*sz*á*r orders and their properties and also show how to construct a Cs*á*sz*á*r order from a given Cs*á*sz*á*r order. Whilst Chung [12] clearly translated a Cs*á*sz*á*r order from the founding monograph of general topology [14], we also used Flax [17].

Definition 1.3.1. A *Császár order* on a frame M is a binary relation $\lhd^M$ on M satisfying the following properties:

$CO_1) : 0 \lhd^M 0$ and $e \lhd^M e$;

$CO_2) : x \lhd^M y \Rightarrow x \leq y$; and

$CO_3) : x \leq a \lhd^M b \leq y \Rightarrow x \lhd^M y$.

The pair $(M, \lhd^M)$ is a *Császár frame.*

Lemma 1.3.2. [12] *Given a frame L and a family $\mathcal{L} = \{\lhd_i^L \mid i \in I\}$ of Császár orders on L, then the set $\lhd_{\mathcal{L}}$ is a Császár order on L, making $(L, \mathcal{L})$ a Császár frame where $\lhd_{\mathcal{L}} = \bigcup\{\lhd_i^L \mid i \in I\}$.*

Proof:

$CO_1)$: Since each $\lhd_i^L$ is a Császár order on L, we have that $0 \lhd_i^L 0$ for any $i \in I$. Therefore $0 \lhd_{\mathcal{L}} 0$.

Similarly, since $\lhd_i^L$ is a Császár order on L, it follows that $e \lhd_i^L e$ for $i \in I$ and hence $e \lhd_{\mathcal{L}} e$.

$CO_2)$: Suppose that $x \lhd_{\mathcal{L}} y$. This implies that $x \lhd_i^L y$ for some $i \in I$. But $\lhd_i^L$ is a Császár order on L for each $i \in I$, so $x \leq y$.

$CO_3)$: Suppose that $x \leq a \lhd_{\mathcal{L}} b \leq y$. This implies that $x \leq a \lhd_i^L b \leq y$ for some $i \in I$. It then follows that $x \lhd_i^L y$ since $\lhd_i^L$ is a Császár order. Thus we have $x \lhd_{\mathcal{L}} y$. Therefore $\lhd_{\mathcal{L}}$ is a Császár order on L. □

We strengthen our approach to Császár frames by the following

Definition 1.3.3. Let L be a frame.

i) Let $\mathcal{L}$ be a family of Császár orders on L. We say $\mathcal{L}$ is *admissible* if, for all $a \in L$, it holds that

$$a = \bigvee\{b \in L \mid b \lhd_{\mathcal{L}} a\}$$

ii) A family of Császár orders $\mathcal{L}$ on a frame L is said to be a *Császár structure* on L if it satisfies the following:

CS_1: $\mathcal{L}$ is up-directed, that is, for $a, b \in L$ there is $c \in L$ such that $a \lhd_{\mathcal{L}} c$ and $b \lhd_{\mathcal{L}} c$;

CS_2: $\lhd_{\mathcal{L}}$ is a meet-sublattice of $L \times L$, that is, $a \lhd_{\mathcal{L}} b$ and $a \lhd_{\mathcal{L}} c$ imply $a \lhd_{\mathcal{L}} b \wedge c$;

CS_3: $\mathcal{L}$ is admissible.

Remark 1.3.4. In his dissertation, Flax [17] showed that if $\leq$ is a topogenous order on a set X, then the relation $c(\leq)$ defined by $A\ c(\leq)\ B$ if and only if $X - B \leq X - A$. In what follows, we translate this result to Császár frames.

Motivating Example 1.3.5. Given a Császár order $\lhd^M$ on a frame M, the relation $\lhd_c^M$ defined by

$$u \lhd_c^M v \text{ if and only if } v^* \lhd^M u^*$$

is also a Császár order on M.

Proof:

$CO_1)$: We have that $0^* \lhd^M 0^*$ since $\lhd^M$ is a Császár order, hence $0 \lhd_c^M 0$. Similarly, $e^* \lhd^M e^*$ implying that $e \lhd_c^M e$.

$CO_2)$: Suppose $u \lhd_c^M v$. It then follows that $v^* \lhd^M u^*$. $\lhd^M$ is a Császár order, hence we have $v^* \leq u^*$ which then implies that $u \leq v$ as it was to be shown.

$CO_3)$: Suppose $u \leq x \lhd_c^M y \leq v$. We want show that $u \lhd_c^M v$. From the definition of $\lhd_c^M$, we have $y^* \lhd^M x^*$, hence we will have $v^* \leq y^* \lhd^M x^* \leq u^*$. Since $\lhd^M$ is a Császár order it follows that $v^* \lhd^M u^*$ and this shows that $u \lhd_c^M v$. □

The above definition justifies the following concept (due to Chung [13]).

Definition 1.3.6. Let L be a frame.

(i) A *symmetric Császár order* on L is a Császár order satisfying the following property:

$$\text{If } a \lhd^L b \text{ then } b^* \lhd^L a^*$$

ii) A Császár frame $(L, \mathcal{L})$ is said to be *symmetric* if every member of $\mathcal{L}$ is symmetric.

iii) A Császár frame $(L, \mathcal{L})$ is said to be *strong* if for each $\lhd^L \in \mathcal{L}$, there is $\lhd^L_\circ \in \mathcal{L}$ such that $a \lhd^L b$ implies $a \lhd^L_\circ c \lhd^L_\circ b$ for some $c \in L$.

Definition 1.3.7. Let $(M, \mathcal{M})$ and $(L, \mathcal{L})$ be Császár frames.

(i) A frame homomorphism $h : M \to L$ is said to be *continuous* if for each $\lhd^M \in \mathcal{M}$ there is a $\lhd^L \in \mathcal{L}$ with $h(\lhd^M) \subseteq \lhd^L$, where $h(\lhd^M)$ is defined as follows

$$x\, h(\lhd^M)\, y \; if \text{ and only } if \text{ there exist } a,\ b \in M \text{ such that } x \leq h(a),\ a \lhd^M b \text{ and } h(b) \leq y.$$

(ii) A frame homomorphism $h : M \to L$ is said to be a *surjection* if it is dense, onto and

$$\mathcal{M} = \{h_*(\lhd^L) \mid \lhd^L \in \mathcal{L}\},$$

that is, $\mathcal{M}$ is generated by $h_*(\lhd^L)$, where $h_*(\lhd^L)$ is defined as follows

$$x\ h_*(\lhd^L)\ y \; if \text{ and only } if \text{ there exist } a, b \in L \text{ such that } h(x) \leq a \lhd b;\ h_*(b) \leq y.$$

In what follows, we prove a characterisation for generating a Császár order through the right adjoint. Moreover, the right adjoint of a dense homomorphism reflects symmetric orders (see Chung [12]).

Theorem 1.3.8. *Given an onto frame homomorphism $h : M \to (L, \lhd^L)$ between Császár frames M and L. Then the following hold:*

i) *$(M, h_*(\lhd^L))$ is a Császár frame.*

ii) *If h is dense and $\lhd^L$ is symmetric, then $h_*(\lhd^L)$ is symmetric on M.*

iii) *If $\lhd^L_1$ and $\lhd^L_2$ are Császár orders with $\lhd^L_1 \subseteq \lhd^L_2$, then $h_*(\lhd^L_1) \subseteq h_*(\lhd^L_2)$.*

iv) *$x \lhd^L y$ if and only if $h_*(x)\ h_*(\lhd^L)\ h_*(y)$.*

Proof:

i) We will establish the three axioms of a Császár order on M:

$CO_1)$: Since $h(0_M) = 0_L \lhd^L 0_L$ and $h_*(0_L) = 0_M$. We therefore have $0_M\ h_*(\lhd^L)\ 0_M$. Similarly, since $h(e_M) = e_L \lhd^L e_L$ and $h_*(e_L) = e_M$, then $e_M\ h_*(\lhd^L)\ e_M$.

$CO_2)$: Suppose that $x\ h_*(\lhd^L)\ y$, take $a, b \in L$ such that

$$h(x) \leq a \lhd^L b,\ h_*(b) \leq y.$$

But $\lhd^L$ is a Császár order hence we have $a \leq b$. We then have

$$x \leq h_*h(x) \leq h_*(a) \leq h_*(b) \leq y.$$

Therefore $x \leq y$.

$CO_3)$: Suppose that

$$x \leq a\ h_*(\lhd^L)\ b \leq y.$$

Take $u, v \in L$ such that

$$h(a) \leq u \lhd^L v \text{ and } h_*(v) \leq y.$$

From $x \leq a$ (so that $h(x) \leq h(a)$) and $b \leq y$ we have that

$$h(x) \leq u \lhd^L v,\ h_*(v) \leq y.$$

Therefore $x\ h_*(\lhd^L)\ y$. Thus $h_*(\lhd^L)$ is a Császár order on M.

ii) Suppose that $x\ h_*(\lhd^L)\ y$; take $a, b \in L$ such that

$$h(x) \leq a \lhd^L b \text{ and } h_*(b) \leq y.$$

Since $\lhd^L$ is symmetric, it then follows that $b^* \lhd^L a^*$ and hence we have $y^* \leq (h_*(b))^*$ and $a^* \leq (h(x))^*$, then $h_*(a^*) \leq h_*(h(x)^*)$. This implies that

$$h(y)^* \leq b^* \lhd^L a^* \text{ and } h_*(a)^* \leq x^*.$$

Therefore $y^*\ h_*(\lhd^L)\ x^*$.

iii) Suppose that $\lhd_1^L \subseteq \lhd_2^L$ and let $x\ h_*(\lhd_1^L)\ y$. Take $a, b \in L$ such that

$$h(x) \leq a \ \lhd_1^L \ b, \ h_*(b) \leq y.$$

Since $\lhd_1^L \subseteq \lhd_2^L$, we must have that $a \lhd_2^L b$, so that

$$h(x) \leq a \ \lhd_2^L \ b, \ h_*(b) \leq y,$$

from which it follows that

$$x \ h_*(\lhd_2^L) \ y,$$

that is, $h_*(\lhd_1^L) \subseteq h_*(\lhd_2^L)$.

iv) Suppose $x \ \lhd^L \ y$. Since h is onto, we have that

$$hh_*(x) \ hh_*(\lhd^L) \ hh_*(y)$$

so that

$$h(h_*(x)) \leq x \ \lhd^L \ y \text{ and } h_*(y) \ \leq \ h_*(y).$$

Therefore $h_*(x) \ h_*(\lhd^L) \ h_*(y)$. Conversely, suppose $h_*(x) \ h_*(\lhd^L) \ h_*(y)$. Take $u, v \in L$ such that

$$h(h_*(x) \ \leq \ u \ \lhd^L \ v \text{ and } h_*(v) \ \leq \ h_*(y).$$

But $x \leq hh_*(x)$ and $h_*(v) \ \leq h_*(y) \ \Rightarrow \ v \leq y$, hence it follows that $x \leq \ u \lhd^L v \leq y$. Since $\lhd_L$ is a Császár order we therefore have that $x \lhd^L y$. □

Dense onto frame homomorphisms reflect symmetric and strong orders in Császár frames (see [13]) in the sense that

Lemma 1.3.9. *Let $h : M \to L$ be an onto frame homomorphism and $\lhd^L$ a meet-sublattice of $L \times L$. Then $h_*(\lhd^L)$ is a meet-sublattice of $M \times M$.*

Proof:

Given any $u, v, w \in M$, suppose that $u \ h_*(\lhd^L) \ v$ and $u \ h_*(\lhd^L) \ w$. We claim that $u \ h_*(\lhd^L) \ (v \wedge w)$. By Theorem 1.3.8, there exist $a, b, c, d \in L$ such that

$$h(u) \leq \ a \lhd^L b, \quad h_*(b) \leq v \text{ and } h(a) \leq c \ \lhd^L d, \ h_*(d) \leq w.$$

Since $\lhd^L$ is a meet-sublattice, it follows from the properties of $\lhd^L$ that

$$h(u) \leq (a \wedge c) \ \lhd^L \ (b \wedge d) \text{ and } h_*(b) \wedge h_*(d) = h_*(b \wedge d) \leq v \wedge w.$$

It follows from CO_3 of Definition 1.3.1 that $u \ h_*(\lhd^L) \ (v \wedge w)$. □

Given a Császár frame $(L, \mathcal{L})$, following the notation introduced in Section 3.1 above, we set $\mathcal{L}^* = \{(c_L)_*(\lhd^L) \mid \lhd^L \in \mathcal{L}\}$ for $c_L : t_X L \to L$. To get to a Császár frame completion, we start with the following result. We recall that *admissibility* in a frame L means that if L is a frame and $\mathcal{L}$ a family of Császár orders on L, then for any $a \in L$ we have

$$a = \bigvee \{x \in L \mid x \lhd_{\mathcal{L}} a\}.$$

In the next result, we show how to induce a Császár order on the codomain, and also that an onto frame homomorphism preserves symmetric Császár orders, and that frame homomorphisms are monotone on Császár orders.

Theorem 1.3.10. *Given a frame homomorphism $h : M \to L$ from a Császár frame $(M, \lhd^M)$ to a frame L, define $h(\lhd^M)$ on L. Then the following hold:*

i) *$(L, h(\lhd^M))$ is a Császár frame.*

ii) *If h is onto and $(M, \lhd^M)$ is symmetric, then $(L, h(\lhd^M))$ is symmetric.*

iii) *For Császár orders $\lhd_1^M$ and $\lhd_2^M$ on M, if $\lhd_1^M \subseteq \lhd_2^M$, then $h(\lhd_1^M) \subseteq h(\lhd_2^M)$.*

iv) *If $x \lhd^M y$, then $h(x) \ h(\lhd^M) \ h(y)$.*

v) *If h is onto and $\lhd^L$ is a Császár order on L, then $\lhd^L = hh_*(\lhd^L)$.*

Proof:

i) We shall verify the three axioms of Császár order for $h(\lhd^M)$ on L.

$CO_1) :$ $0_L = h(0_M)$, $0_M \lhd^M 0_M$ and $h(0_M) = 0_L$, therefore $0_L \ h(\lhd^M) \ 0_L$. Similarly, $e_L = h(e_M)$, $e_M \ \lhd^M \ e_M$ and $h(e_M) = e_M$, therefore $e_L \ h(\lhd^M) \ e_l$.

$CO_2) :$ Suppose that $x\ h(\lhd^M)\ y$. Take $u,\ v \in M$ such that

$$x \ \leq \ h(a),\ u \ \lhd^M \ v \ and\ h(v) \ \leq y.$$

Since $u \ \lhd^M \ v$ implies that $u \leq v$ in M, it then follows that $x \leq h(u) \leq h(v) \leq y$. Therefore $x \leq y$ as desired.

$CO_3) :$ Suppose that $x \ \leq \ u\ h(\lhd^M)\ v \ \leq \ y$ in L. Take $s,\ t \in M$ such that

$$u \ \leq \ h(s),\ s \ \lhd^M \ t \ and\ h(t) \ \leq \ v.$$

Then we have

$$x \ \leq \ u \leq \ h(s),\ s \ \lhd^M \ t \ and\ h(t) \ \leq \ v \leq \ y.$$

Therefore $x\ h(\lhd^M)\ y$. This shows that $h(\lhd_M)$ is a Cs*á*sz*á*r order.

ii) Suppose that $x\ h(\lhd^M)\ y$ and take $u,\ v \in M$ such that

$$x \ \leq \ h(u),\ u \ \lhd^M \ v \text{ and } h(v) \ \leq \ y$$

Then we have that

$$y^* \ \leq \ h(v)^* \ = \ h(v^*),\ v^* \ \lhd^M \ u^* \text{ and} h(u)^* \ = \ h(u^*) \ \leq \ x^*.$$

Therefore $y^*\ h(\lhd^M)\ x^*$.

iii) Suppose that $\lhd_1^M \ \subseteq \ \lhd_2^M$ in M and let $x\ h(\lhd_1^M)\ y$. Take $u,\ v \in M$ such that

$$x \ \leq \ h(u),\ u \ \lhd_1^M \ v \text{ and } h(v) \ \leq \ y.$$

It then follows that

$$x \ \leq \ h(u),\ u \ \lhd_2^M \ v \text{ and} h(v) \ \leq \ y.$$

Therefore $x\ h(\lhd_2^M)\ y$.

iv) Suppose that $x \lhd^M y$. This implies that $x \leq y$ and hence $h(x) \leq h(y)$. We have that

$$h(x) \ \leq \ h(y),\ x \ \lhd^M \ y \text{ and } h(y) \ \leq \ h(y).$$

Therefore $h(x)\ h(\lhd^M)\ h(y)$.

v) This will follows from the fact that for an onto frame homomorphism $h : M \to L$, it holds that $h \circ h_* = id_L$ (see [12]) because then:

$$\begin{aligned} x \ \lhd^L \ y \quad &\Leftrightarrow \quad x \ id_L \ (\lhd^L) \ y \\ &\Leftrightarrow \quad x \ (h \circ h_*) \ (\lhd^L) \ y. \end{aligned}$$

□

Proposition 1.3.11. *For a symmetric and strong Császár frame $(L, \mathcal{L})$ and a dense onto homomorphism $h : M \to L$, the Császár frame $(M, h_*(\mathcal{L}))$ is symmetric and strong, where*

$$h_*(\mathcal{L}) \ = \ \{h_*(\lhd^L) \mid \lhd^L \in \mathcal{L}\}.$$

Proof:

Since h dense and $\lhd^L$ is symmetric on L, from Theorem 1.3.8(ii) it follows that $h_*(\lhd^L)$ is also symmetric on M. To see that $h_*(\lhd^L)$ is strong, suppose that $x \ h_*(\lhd^L) \ y$ for $x, y \in M$ and $\lhd^L \in \mathcal{L}$. From Theorem 1.3.10 (iv) it follows that

$$h(x) \ h(h_*(\lhd^L)) \ h(y).$$

By Theorem 1.3.10 (v) we have that $h(x) \ \lhd^L \ h(y)$. Since $\lhd^L$ is strong, there exists $h(z) \in L$ such that

$$h(x) \ \lhd^L \ h(z) \ \lhd^L \ h(y).$$

According to Theorem 1.3.8(iv) it follows that

$$h_*(h(x)) \ h_*(\lhd^L) \ h_*(h(z)) \ h_*(\lhd^L) \ h_*(h(y))$$

and thus

$$x \ h_*(\lhd^L) \ z \ h_*(\lhd^L) y.$$

This proves that $h_*(\lhd^L)$ is strong. □

Definition 1.3.12 ([12]). *If $\lhd^M$ is a Császár order on M, we say an element $w \in M$ is $\lhd^M$-small if whenever $u \ \lhd^M \ v$ then either $w \le u^*$ or $w \le v$.*

Remark 1.3.13

i) If $s,\ t \in M$ are $\lhd^M$-small, then $s \wedge t$ and $s \vee t$ are $\lhd^M$-small: For, if $u \lhd^M\ v$ then $s\ \leq\ u^*$ or $s\ \leq v$ so that together with $t\ \leq\ u^*$ or $t\ \leq\ v$ we find that $s \wedge t\ \leq\ u^*$ or $s \wedge t\ \leq\ v$. A similar argument establishes the $\lhd^M$-smallness of $s \vee t$.

ii) Another notion of *smallness* that we know of is called *u-smallness* in the paper [29] of Picado and Pultr. In their case, "smallness" was used in relation to covers in quasi-uniformities.

Given a poset $(L, \leq)$, we call a subset D of L a *downset* if, whenever $v \in D$ and $u \in L$ with $u \leq v$, then $u \in D$. Equivalently, a subset D of a frame (or a semi-lattice) M is a *downset* if for each $u \in D$, it holds that

$$\downarrow u = \{v \in L \mid v \leq u\} \subseteq D.$$

Notation 1.3.14. If $\lhd^M$ is a Császár order on M, we will denote by $\mathcal{B}_{\lhd^M}$ the $\lhd^M$*-small set*

$$\mathcal{B}_{\lhd^M}\ =\ \{u \in M \mid u\ is\ \lhd^M -small\}.$$

In consequence, $\mathcal{B}_{\lhd^M_\circ}$ denotes the set of elements that are $\lhd^M_\circ$-small (with respect to the Császár order $\lhd^M_\circ$ on M).

In the following result [12], we show that a $\lhd$-small set is a downset.

Lemma 1.3.15. *For Császár orders $\lhd^M$ and $\lhd^M_\circ$ on M, we have:*

i) *$\mathcal{B}_{\lhd^M}$ is a downset.*

ii) *If $\lhd^M\ \subseteq\ \lhd^M_\circ$, then $\mathcal{B}_{\lhd^M_\circ}\ \subseteq\ \mathcal{B}_{\lhd^M}$.*

Proof:

i) Suppose that $v \in \mathcal{B}_{\lhd^M}$ and $u \in M$ such that $u \leq v$. To see that $u \in \mathcal{B}_{\lhd^M}$, we assume that for some w, $w \lhd z$ and note that $v \leq w^*$ or $v \leq z$ which then give rise to $u \leq w^*$ or $u\ \leq\ z$ so that u is $\lhd^M$-small.

ii) Suppose that $w \in \mathcal{B}_{\lhd_\circ^M}$, that is w is "small" with respect to the Császár order $\lhd_\circ^M$ and let $u \lhd^M v$. By hypothesis we find that $u \lhd_\circ^M v$ and thus $w \leq u^*$ or $w \leq v$, which proves that w is *small* with respect to the Császár order $\lhd^M$ as was to be shown. □

We now show that the right adjoint of a frame homomorphism "preserves" $\lhd$-smallness. See also [12].

Theorem 1.3.16. *For a frame homomorphism $h : M \to (L, \lhd^L)$, it holds that:*

i) *If h is a dense homomorphism, then $h_*(\mathcal{B}_{\lhd^L}) \subseteq \mathcal{B}_{h_*(\lhd^L)}$.*

ii) *If h is an onto homomorphism, then $\mathcal{B}_{h_*(\lhd^L)} \leq h_*(\mathcal{B}_{\lhd^L})$.*

iii) *If h a dense onto homomorphism, then $h(\mathcal{B}_{h_*(\lhd^L)}) = \mathcal{B}_{\lhd^L}$.*

Proof:

i) Suppose that $a\ h_*(\lhd^L)\ b$, for $a,\ b \in M$. By definition of $h_*(\lhd^L)$, there are $c,\ d \in L$ such that

$$h(a) \leq c \lhd^L d \text{ and } h_*(d) \leq b.$$

Let $s \in \mathcal{B}_{\lhd^L}$. Then s is $\lhd^L$-small and then we have that $s \leq c^*$ or $s \leq d$. If $h_*(s) \wedge a \neq 0$, then

$$hh_*(s) \wedge h(a) \neq 0 \text{ and } h(a) \leq c.$$

But $hh_*(s) \leq s$, so we must have $s \wedge c \neq 0$. Now $s \leq c^*$ and $s \leq d$ imply that $h_*(s) \leq h_*(c^*) = (h_*(c))^*$ and $h_*(s) \leq h_*(d)$, respectively. Therefore $h_*(s) \leq h_*(c))^*$ or $h_*(s) \leq h_*(d)$. This implies that $h_*(s)$ is $h_*(\lhd^L)$-small and thus $h_*(s) \in \mathcal{B}_{h_*(\lhd^L)}$. Then $s \in h(\mathcal{B}_{h_*(\lhd^L)})$ and it then follows that

$$\mathcal{B}_{\lhd^L} \subseteq h(\mathcal{B}_{h_*(\lhd^L)}).$$

Therefore our desired result, namely $h_*(\mathcal{B}_{\lhd^L}) \subseteq \mathcal{B}_{h_*(\lhd^L)}$, follows.

ii) Suppose that $a \lhd^L b$. We then have $h_*(a)\ h_*(\lhd^L)\ h_*(b)$. Let $s \in \mathcal{B}_{h_*(\lhd^L)}$, so that then

$$s \leq h_*(a^*) \text{ or } s \leq h_*(b).$$

If $h(s) \wedge a \neq 0$ then

$$h_*h(s) \wedge h_*(a) = s \wedge h_*(a) \neq 0.$$

But then $s \leq h_*(a)^*$ and $h(s) \leq b$ imply that $h(s) \leq a^*$ or $h(s) \leq b$. Respectively, from which it follows that $h(s)$ is $\lhd_L$-small and hence $h(s) \in \mathcal{B}_{\lhd^L}$. Since h is onto, it follows that $s \in h_*(\mathcal{B}_{\lhd^L})$ which then implies that

$$\mathcal{B}_{h_*(\lhd^L)} \subseteq h_*(\mathcal{B}_{\lhd^L}).$$

Therefore $\mathcal{B}_{h_*(\lhd^L)} \leq h_*(\mathcal{B}_{\lhd^L})$.

iii) From (i) and (ii) have that $h_*(\mathcal{B}_{\lhd_L}) = \mathcal{B}_{h_*(\lhd^L)}$. Since h is onto then $hh_* = id$, therefore it holds that $hh_*(\mathcal{B}_{\lhd^L}) = \mathcal{B}_{\lhd^L} = h(\mathcal{B}_{h_*(\lhd^L)})$. □

Concluding Remark. In the next chapter, we will look at the role of Cs*á*sz*á*r orders on proximal Cs*á*sz*á*r frames, and $\lhd$-smallness will feature in the discussion on Cauchy filters and uniform homomorphisms.

Chapter 2

Proximal Cs$\acute{a}$sz$\acute{a}$r Frames

In this chapter, we introduce and study properties of proximal Cs$\acute{a}$sz$\acute{a}$r frames, regular (Cauchy) filters, uniform and Cauchy homomorphisms on these frames and related filters.

2.1 Introduction

The concept of a proximal frame comes out of that of a proximity structure. One of the earliest treatment of proximity spaces that has shaped our approach is that of Dowker in [15] in the definition below. We will show that Cs$\acute{a}$sz$\acute{a}$r orders on proximal frames are symmetric and strong.

Definition 2.1.1. A *proximity structure* on a set X is a relation $\leq \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$ which satisfies the following:

P_1 If $A \leq B$, then $A \subseteq B$;

P_2 If $A \subseteq B \leq C \subseteq D$, then $A \leq D$;

P_3 Given that $i \in \{1;\ 2;\ 3;\ ...;\ n\}$:

i) If $A_i \ \leq\ B$, then $\bigcup_i A_i \ \leq\ B$;

ii) If $A \ \leq\ B_i$, then $A \ \leq\ \bigcap_i\ B_i$;

P_4 If $A \leq C$, there exists a B such that $A \leq B \leq C$.

Motivating example 2.1.2. Recall that a frame is *completely regular* if for u in a frame L, we have

$$u = \bigvee\{v \mid v \prec\!\prec u\},$$

where $v \prec\!\prec u$ means there is a system

$$\{a_r \mid r \in \mathbb{Q} \cap [0,1]\}$$

satisfying $a_0 = v,\ a_1 = u$ and $a_{n_1} \prec a_{n_2}$ whenever $n_1 < n_2$. Now, the relation $\prec\!\prec$ satisfies the following properties in the frame $(M,\ \leq)$:

i) If $u \prec\!\prec v$, then $u \leq v$;

ii) If $u \leq v \prec\!\prec w \leq t$, then $u \prec\!\prec t$;

iii) If $u \prec\!\prec w$ and $v \prec\!\prec w$, then $u \vee v \prec\!\prec w$;

iv) If $u \prec\!\prec v$ and $u \prec\!\prec w$, then $u \prec\!\prec v \wedge w$;

v) If $u \prec\!\prec v$, then $v^* \prec\!\prec u^*$.

Proof:

i) Suppose $u \prec\!\prec v$. With $u = c_0$ and $v = c_1$ it easily follows from the definition that $u \prec v$. Now $u^* \vee v = e$ implies that

$$u = u \wedge e = u \wedge (u^* \vee v) = (u \wedge u^*) \vee (u \wedge u) = 0 \vee (u \wedge v) = u \wedge v$$

from which $u \leq v$ follows.

ii) Suppose $u \leq v \prec\!\prec w \leq t$. It follows from i) that $v \prec w$ that $u \leq v \prec w \leq t$ so that $u \leq t$. By definition find a system $\{a_r \mid r \in \mathbb{Q} \cap [0,1]\}$ such that

$$u = v = a_0,\ w = t = a_1 \text{ and } a_0 \prec a_1.$$

Therefore $u \prec\!\prec t$.

iii) Suppose $u \prec\!\prec v$ and $v \prec\!\prec w$. Find $\{a_r \mid r \in \mathbb{Q} \cap [0,1]\}$ and $\{c_s \mid s \in \mathbb{Q} \cap [0,1]\}$ such that

$$a_0 = u,\ c_0 = v,\ \text{and } a_1 = w = c_1.$$

Putting $d_0 = a_0 \wedge c_0 = u \wedge v$, it is immediate that $u \vee v \prec\!\prec w$.

iv) Suppose $u \prec\!\prec v$ and $u \prec\!\prec w$. We find $\{a_r \mid r \in \mathbb{Q} \cap [0,1]\}$ and $\{c_s \mid s \in \mathbb{Q} \cap [0,1]\}$ such that

$$a_0 = u = c_0,\ a_1 = v \text{ and } c_1 = w.$$

Then $d_0 = a_0 \vee c_0 = u$ and $d_1 = a_1 \vee c_1 = v \vee w$ ensures that $u \prec\!\prec v \wedge w$.

v) Suppose $u \prec\!\prec v$. We find $\{a_r \mid r \in \mathbb{Q} \cap [0,1]\}$ such that $a_0 = u$ and $a_1 = v$. Then $b_0 = v^*$, $b_1 = u^*$ and $v^* \prec u^*$ imply $v^* \prec\!\prec u^*$ as desired. □

Definition 2.1.3. Let L be a frame, then a binary relation $\lhd$ on L is a *strong inclusion* if it satisfies the following properties:

SI_1 : If $x \leq a \lhd b \leq y$, then $x \lhd y$

SI_2 : $\lhd$ is a sublattice of $L \times L$

SI_3 : If $a \lhd b$, then $a \prec b$

SI_4 : If $a \lhd b$, then $a \lhd c \lhd b$ for some $c \in L$

SI_5 : If $a \lhd b$, then $b^* \lhd a^*$

SI_6 : For each $a \in L$, $a = \bigvee\{x \in L \mid x \lhd a\}$

As far as we are aware, proximal frames were introduced by Frith in [18] and it is easily seen that there is a relationship between the above two definitions and this one.

Definition 2.1.4([18]). A *proximal frame* is a pair $(M, \preccurlyeq)$ where M is a frame and $\preccurlyeq$ is a relation on M satisfying the following:

PF_1 : $0 \preccurlyeq 0$ and $e \preccurlyeq e$;

PF_2 : If $u \preccurlyeq v$, then $u \leq v$;

PF_3 : If $u \leq v \preccurlyeq s \leq w$, then $u \preccurlyeq w$;

PF_4 : (i) If $u_i \preccurlyeq v$ for $i \in \{1; 2; 3; ...; n\}$, then $\bigvee_i u_i \preccurlyeq v$;

(ii) If $u \preccurlyeq v_i$, for $i \in \{1; 2; 3; ...; n\}$, then $u \preccurlyeq \bigwedge_i v_i$;

PF_5: If $u \preccurlyeq w$, there exists a $v \in M$ such that $u \preccurlyeq v \preccurlyeq w$;

PF_6 : If $u \preccurlyeq v$, then $v^* \preccurlyeq u^*$;

PF_7 : For each $v \in M$, it holds that $v = \bigvee\{u \in M \mid u \preccurlyeq v\}$.

The above relation $\preccurlyeq$ is said to be a *proximal* relation.

Example 2.1.5. If $(M, \leq)$ is a frame, then the relation $\preccurlyeq$ on M defined by

$$u \preccurlyeq v \text{ if and only if } u \leq v$$

is proximal, making $(M, \preccurlyeq)$ a proximal frame. □

Definition 2.1.6 ([13]). Let $(M, \mathcal{M})$ be a Császár frame

i) The frame M is said to be a *regular* Császár frame if every $\lhd^M \in \mathcal{M}$ is coarser than $\prec$, i.e. $\lhd^M \subseteq \prec$ if $u \lhd^M v$ implies that $u \prec v$.

ii) The frame M is said to be a *proximal* Császár frame if $\lhd^M$ is strong, symmetric and regular for all $\lhd^M \in \mathcal{M}$.

iii) The frame M is said to be *uniform* if it is proximal and every member of $\mathcal{M}$ is a meet-complete sublattice of $M \times M$.

The following result is immediate from the definition of a proximal Császár frame.

Lemma 2.1.7. *If $(L, \preccurlyeq)$ is a proximal frame, then L is a regular frame.* □

For the rest of this chapter, unless stated otherwise, we will assume all frames to be proximal Császár frames.

Theorem 2.1.8 ([13]). *For a proximal Császár frame $(L, \mathcal{L})$ we have that for all $\lhd^L \in \mathcal{L}$:*

i) *$\lhd^L$ is a symmetric order.*

ii) *$\lhd^L$ is strong order.*

iii) *$\lhd^L$ is a sub-lattice of $L \times L$.*

Proof:

i) Suppose that $a \lhd_{\mathcal{L}} b$ for $a, b \in L$. This implies that $a \lhd^L b$ for any $\lhd^L \in \mathcal{L}$. L is a proximal frame, then if follows that each $\lhd^L$ is symmetric, hence we have $b^* \lhd^L a^*$. Thus $b^* \lhd_{\mathcal{L}} a^*$.

ii) Suppose that $a \lhd_{\mathcal{L}} b$ for $a, b \in L$. Then $a \lhd^L b$ for any $\lhd^L \in \mathcal{L}$. The fact that L is a proximal frame ensures that $\lhd^L$ is strong, hence there exists $c \in L$ such that $a \lhd^L c \lhd^L b$. Therefore we have $a \lhd_{\mathcal{L}} c \lhd_{\mathcal{L}} b$.

iii) Suppose that $x \lhd_{\mathcal{L}} y$ and $x \lhd_{\mathcal{L}} z$ for $x, y, z \in L$. Then $x \lhd^L y$ and $x \lhd^L y$ for any $\lhd^L \in \mathcal{L}$. Since L is up-directed, there is $\lhd^L_\circ \in \mathcal{L}$ such that $x \lhd^L_\circ y$ and $x \lhd^L_\circ z$. But $\lhd^L_\circ$ is a meet-sublattice hence we will have $x \lhd^L_\circ y \wedge z$ and therefore $x \lhd_{\mathcal{L}} y \wedge z$.

On the other hand, if $y \lhd_{\mathcal{L}} x$ and $z \lhd_{\mathcal{L}} x$. Since $\lhd_{\mathcal{L}}$ is strong, we have that $y \lhd_{\mathcal{L}} a \lhd_{\mathcal{L}} x$ and $z \lhd_{\mathcal{L}} b \lhd_{\mathcal{L}} x$ for $a, b \in L$. The fact that $\lhd_{\mathcal{L}} \subseteq \prec$ implies that there are $c, d \in L$ such that

$$y \lhd_{\mathcal{L}} a \prec c \lhd_{\mathcal{L}} x \text{ and } z \lhd_{\mathcal{L}} b \prec d \lhd_{\mathcal{L}} x.$$

It then follows that $(y \vee z) \lhd_{\mathcal{L}} (a \vee b) \prec c \vee d \leq x$. Therefore we have

$$\begin{aligned} (y \vee z) &\lhd_{\mathcal{L}} (a \vee b)^{**} \\ &\prec (c \vee d) \\ &\leq x \end{aligned}$$

Therefore we have $(y \vee z) \lhd_{\mathcal{L}} x$ as it was to be shown. □

2.2 Filters on proximal Cs*á*sz*á*r Frames

In this section, we study regular Cauchy filters in proximal Cs*á*sz*á*r Frames and show, amongst few results, that;

i) regular Cauchy filters are minimal Cauchy

ii) Cauchy filters are preserved by the right adjoints of uniform homomorphisms

iii) surjections preserve regular Cauchy filters

iv) the round filters (as in Bezanishvili [9]) are precisely those filters that are completely prime.

Definition In a frame M, we say that $F \subseteq M$ is a *filter* if the following three conditions are satisfied:

i) $F \neq \emptyset$,

ii) If $u,\ v \in F$ then $u\ \wedge\ v \in F$, and

iii) If $u\ \leq\ v$ and $u \in F$ then $v \in F$.

Recall that in Chapter 1 (Notation 1.3.14), we denoted by $\mathcal{B}_{\lhd^M}$ the set of elements in Császár frame M which are "small" with respect to some $\lhd^M$ in a Császár order $\mathcal{M}$ on M, which is needed in the definition below (see [13]).

Definition 2.2.1. A filter F on a frame $(L, \mathcal{L})$ is said to be:

i) A *regular* filter if for all $a\ \in F$, there exists $b\ \in F$ such that $b\ \lhd^L\ a$ for $\lhd^L\ \in \mathcal{L}$.

i) A *Cauchy* filter if and only if for any $\lhd^L \in \mathcal{L}$, it holds that $F \cap \mathcal{B}_{\lhd^L} \neq \emptyset$.

ii) A *regular* Cauchy filter if whenever F is a Cauchy filter and $v \in F$, then there exists $u \in F$ with $u\ \lhd^L\ v$, for some $\lhd^L\ \in \mathcal{L}$.

In what follows, we prove standard and well-known results of (regular) Cauchy filters (see also [21]).

Proposition 2.2.2. *Let $(L, \mathcal{L})$ be a Császár frame and F a filter on L. Then:*

i) *If F is a Cauchy filter, then for any $a \lhd^L b$, either $b \in F$ or $a^* \in F$.*

ii) *If F is a regular Cauchy filter, then F is a minimal Cauchy filter.*

iii) *A regular Cauchy filter is a regular filter.*

Proof:

i) Let F be a Cauchy filter and $a \lhd^L b$. It follows by CO_2 in definition 1.3.1 that $a \leq b$. F being Cauchy implies that $F \cap \mathcal{B}_{\lhd^L} \neq \emptyset$. Take $u \in F \cap \mathcal{B}_{\lhd^L}$ such that $u \leq a^*$ or $u \leq b$. We then have that $a^* \in F$ or $b \in F$.

ii) Let F be a regular Cauchy filter. The regularity of F ensures that for any $a \in F$, there exists $b \in F$ such that $b \lhd^L a$. If G is another Cauchy filter on M with $G \subseteq F$, then we will have that $b^* \in G$ or $a \in G$. Suppose that $b^* \in G$, then we will have $b^* \in F$ and thus $\emptyset = b^* \wedge b \in F$ which is a contradiction, hence we must have $a \in G$. This implies that $F \subseteq G$

iii) This follows immediately from the fact that for every regular Cauchy filter F, $a \in F$ implies that there is $b \in F$ such that $b \lhd^L a$. □

Our next result shows that surjections preserve regular Cauchy filters.

Proposition 2.2.3. *If a frame homomorphism $h : M \to L$ is a surjection and F is a regular Cauchy filter on M, then $h(F)$ is a regular Cauchy filter on L.*

Proof:

Since h is a surjection, it is dense and onto and therefore $h(F)$ is a filter base. Let $x \in h(F)$ and suppose $x \leq y$. There exists $s \in F$ such that $h(s) = x$ and so $h(s) \leq y$. It then follows that $h_*(h(s)) \leq h_*(y)$ which implies that $s \leq h_*(y)$ since h is onto. Therefore $y \in h(F)$ and hence $h(F)$ is a filter. To see that $h(F)$ is a regular Cauchy filter, suppose $p \lhd^L q$ in L. It then follows from Theorem 1.3.8 that

$$h_*(p)\ h_*(\lhd^L)\ h_*(q) \text{ in } M.$$

F is Cauchy, hence $h_*(p)^* \in F$ or $h_*(q) \in F$. This implies that

$$h(h_*(p)^*) \in h(F) \text{ or } h(h_*(q)) \in h(F).$$

Since h is onto we have $p^* \in h(F)$ or $q \in h(F)$. □

Turning to prime filters, we recall ([20]) that a filter F is *prime* if $a \vee b \in F$ implies $a \in F$ or $b \in F$, and it is *completely prime* if for $S \subset L$, $\bigvee S \in F$ implies $S \cap F \neq \emptyset$.

Definition 2.2.4. We say a filter F in a frame L is *convergent* if for any cover T of L we have $F \wedge T \neq \emptyset$.

Proposition 2.2.5. *A regular filter on a frame M is convergent if and only if it is completely prime.*

Proof:

$\Rightarrow$: If F is a regular filter, we take $\bigvee S \in F$ and pick $u \in F$ satisfying $u \prec \bigvee S$. Then $F \cap (S \cup \{x^*\}) \neq \emptyset$. Since $F \subseteq M$ is proper, it follows that $F \cap S \neq \emptyset$.

$\Leftarrow$: We take a regular filter F that is a completely prime. Since $\bigvee S = e \in F$ ensures that $F \cap S \neq \emptyset$, it follows that F converges. □

Remark 2.2.6. Note that completely prime filters are also important in the sense that there is a bijective correspondence between points of a locale and completely prime filters. Suppose $p : L \to \mathbf{2}$ is *a point*. We set $F = p^{-1}(1)$. To see that $u \wedge v \in F$ for any $u, v \in F$, note that

$$u, v \in F \Leftrightarrow p(u) = 1 = p(v) \Leftrightarrow p(u \wedge v) = 1 \Leftrightarrow u \wedge v \in F.$$

On the other hand, if $\bigvee S \in F$, then

$$\begin{aligned} p(\bigvee S) = 1 \quad &\Rightarrow \quad \bigvee_{s \in S} p(s) = 1 \\ &\Rightarrow \quad p(s) = 1, for\ some\ s \in S \\ &\Rightarrow \quad S \cap F \neq \emptyset, \end{aligned}$$

making F completely prime. In the opposite direction, if F is a completely prime filter, then the mapping $p : F \to \mathbf{2}$ defined by $p(u) = 1$ for $u \in F$ and $p(u) = 0$ for $u \in M - F$ defines a point. □

Completely prime filters also relate to round filters (in the sense of Bezhanishvili ([9]). First, given a compact regular frame L and a subset S of L, we define

$$\downarrow S = \{u \in L \mid u \prec s\ for\ some\ s \in S\}$$

and

$$\uparrow S = \{u \in L \mid s \prec u\ for\ some\ s \in S\}$$

An ideal I in L is said to be *round* if $I =\Downarrow I$ and a filter F is said to be *round* if $F =\Uparrow F$. Here is the relationship (see [9]):

Theorem 2.2.7. *For a compact regular frame L and a filter F of L, these are equivalent.*

i) *F is a completely prime filter.*

ii) *F is a round prime filter.*

iii) *$F = G$, for some prime filter G.*

iv) *F is a meet-prime element of the lattice of round filters of L ordered by set inclusion $\subseteq$.*

Proof

i) $\Rightarrow$ ii) : Suppose that F is a completely prime filter in L and so primeness is trivial. To see that it is also round, we take $u \in F$ and note that compact regularity of L ensures that

$$u = \bigvee \{v \in L \mid v \prec u\}.$$

It then follows from complete primeness of F that $v \prec u$, for some $v \in F$.

ii) $\Rightarrow$ iii) : Suppose F is a round prime filter. Then $F \ = \Uparrow F$ and hence for each $a \in F$ there exists $b \in F$ with $b \prec a$. If G is another prime filter on L, then it is minimal since L is regular compact. Thus $G \subseteq F$. But $b \prec a$ implies $b^* \in G$ or $a \in G$: If $b^* \in G$, then $b^* \wedge b \in G$. But $G \subseteq F$, hence $b^* \wedge b \in F$. We know that $b^* \wedge b = 0$ and that $0 \notin F$ which will contradict the fact that $b^* \wedge b \in F$. Therefore $b^* \notin G$ and this implies that $a \in G$. This shows that $F \subseteq G$ and therefore $G = F$.

iii) $\Rightarrow$ iv) : In a lattice of round filters, meet is given by intersection. Suppose that H and K are round filters such that $H \nsubseteq F$ and $K \nsubseteq F$, thus there are $u \in H - F$ and $v \in K - F$. Since H and K are round filters, there are $w_u \in H$ and $w_v \in K$ such that $w_u \prec u$ and $w_v \prec v$, so that $w_u \notin G$ and $w_v \notin G$. But G is prime, so $w_u \vee w_v \notin G$; it then follows from $w_u \vee w_v \prec u \vee v$ that $u \vee v \notin F$. Since $u \vee v \in H \cap K$, we must have that $H \cap K \subseteq F$.

iv) $\Rightarrow$ i) : We assume $\bigvee S \in F$. By hypothesis F is round, so there is a $u \in F$ satisfying $u \prec \bigvee S$, and so by the familiar property $u^* \vee (\bigvee S) = e$. Since L is compact, let us find $\{u_i \in S \mid 1 \leq i \leq n\}$ such that

$$u^* \vee u_1 \vee \ldots \vee u_n \ = \ e.$$

Since L is regular, we know that each u_i satisfies

$$u_i = \bigvee \{v_{j_i} \mid v_{j_i} \prec u_i\},$$

so compactness of L ensures that

$$u^* \vee v_{j_1} \vee \cdots \vee v_{j_n} = e,$$

for some $v_{j_i} \prec u_i$. Therefore, we have

$$u \prec v_{j_1} \vee \ldots \vee v_{j_n} \text{ so that } v_{j_1} \vee \ldots \vee v_{j_n} \in F.$$

Thus,

$$\Uparrow v_{j_1} \cap \ldots \cap \Uparrow v_{j_n} \ = \Uparrow (v_{j_1} \vee \cdots \vee v_{j_n}) \ \subseteq \ F.$$

By assumption F is meet prime in the lattice of round filters, so $\Uparrow v_{j_i} \subseteq F$ for some v_{j_i}, giving some $u_i \in F$. This completes the proof that F is completely prime. □

2.3 Homomorphisms on proximal Cs*á*sz*á*r frames

In this section, we study properties of frame homomorphisms between proximal Cs*á*sz*á*r frames and show that the category of proximal Cs*á*sz*á*r frames and uniform homomorphisms exist; in particular, we show that:

i) on these frames, dense onto homomorphisms are uniform;

ii) the right adjoint of a uniform homomorphism reflects Cauchy filters;

iii) uniform homomorphisms and surjections between these frames are Cauchy.

Definition 2.3.1 ([10]). A *proximity morphism* between proximal frames is a map $f : (M, \preccurlyeq_M) \to (L, \preccurlyeq_L)$ satisfying the following axioms:

i) $f(0) = 0$ and $f(1) = 1$.

ii) $f(a \wedge b) = f(a) \wedge f(b)$.

iii) If $a_1 \preccurlyeq_M b_1$ and $a_2 \preccurlyeq_M b_2$, then $f(a_1 \vee a_2) \preccurlyeq_L f(b_1) \vee f(b_2)$.

iv) $f(b) = \bigvee\{f(a) : a \preccurlyeq_M b\}$.

The category $\mathbb{UC}\mathbf{s}\mathbb{F}\mathbf{rm}$ of uniform Császár frames and uniform homomorphisms arises as follows. But we need the concept of a uniform homomorphism first. The definition of Császár smallness is related to uniform homomorphisms between uniform Császár frames as follows (from [12]):

Definition 2.3.2. Given uniform Császár frames $(L, \mathcal{L})$ and $(M, \mathcal{M})$, we say a frame homomorphism $h : M \to L$ is *a uniform homomorphism* if whenever $\lhd^M \in \mathcal{M}$, then there exits a $\lhd^L \in \mathcal{L}$ such that $\mathcal{B}_{\lhd^L} \subseteq h(\mathcal{B}_{\lhd^M})$.

Now the following arise:

Lemma 2.3.3. *The composition of uniform frame homomorphism is again a uniform homomorphism.*

Proof:

We start with two uniform frame homomorphisms $f : (N, \mathcal{N}) \to (M, \mathcal{M})$ and $g : (M, \mathcal{M}) \to (L, \mathcal{L})$, and take a $\lhd^N \in \mathcal{N}$. There is a $\lhd^M \in \mathcal{M}$ such that

$$\mathcal{B}_{\lhd^M} \subseteq f(\mathcal{B}_{\lhd^N}).$$

But for this $\lhd^M$ there is a $\lhd^L \in \mathcal{L}$ such that

$$\mathcal{B}_{\lhd^L} \subseteq g(\mathcal{B}_{\lhd^M}).$$

Combining the two relations, we find that

$$\mathcal{B}_{\lhd^L} \subseteq g(f(\mathcal{B}_{\lhd^N})) = (g \circ f)(\mathcal{B}_{\lhd^N}),$$

showing that $g \circ f$ is a uniform homomorphism. □

Theorem 2.3.4. *Any dense onto continuous homomorphism $h : (M, \mathcal{M}) \to (L, \mathcal{L})$ between proximal Császár frames is a uniform homomorphism.*

Proof:

Suppose that $x \lhd^M y$ for $\lhd^M \in \mathcal{M}$. Since M is a proximal Császár frame, there exists a $c \in M$ such that $x \lhd^M c \lhd^M y$. By continuity of h, it follows that for each $\lhd^M \in M$ there is a $\lhd^L \in L$ such that $h(\lhd^M) \subseteq \lhd^L$. Then we have $\mathcal{B}_{\lhd^L} \subseteq \mathcal{B}_{h(\lhd^M)}$. Now suppose $a \in \mathcal{B}_{h(\lhd^M)}$ and $s \lhd^M t$. Since M is a proximal frame, it follows that $s \lhd^M r \lhd^M t$ for $r \in M$. Then

$$h(s)\ h(\lhd^M)\ h(r)\ h(\lhd^M)\ h(t).$$

For $a \in \mathcal{B}_{h(\lhd^M)}$ we will have

$$a \leq (h(s))^* \text{ or } a \leq h(r).$$

Since h is dense, then

$$h_*(a) \wedge s = 0 \text{ or } h_*(a) \leq t.$$

Therefore $h_*(a) \in \mathcal{B}_{\lhd^M}$ and it then follows that $h_*(\mathcal{B}_{h(\lhd^M)}) \subseteq \mathcal{B}_{\lhd^M}$ and thus $\mathcal{B}_{h(\lhd^M)} \subseteq h(\mathcal{B}_{\lhd^M})$. Hence h is uniform. □

Remark 2.3.5. Proximal Császár frames are Császár frames; in particular, uniform frame homomorphisms on proximal Császár frames are accordingly uniform on Császár frames. Thus, the above result is equally true in the setting of Császár frames. Please refer to [16].

The right adjoint of a uniform homomorphism preserves Cauchy filters [12] in the following sense

Proposition 2.3.6. *If $h : (M, \mathcal{M}) \to (L, \mathcal{L})$ is a uniform homomorphism, where M and L are proximal Császár frames with $\lhd^M \in \mathcal{M}$ and $\lhd^L \in \mathcal{L}$, respectively, and G a Cauchy filter on L, then $h_*(G)$ is a Cauchy filter on M.*

Proof:

Suppose G is a Cauchy filter on L. Then $G \cap \mathcal{B}_{\lhd^L} \neq \emptyset$ for all $\lhd^L \in \mathcal{L}$. Since h is uniform, we have $\mathcal{B}_{\lhd^L} \subseteq h(\mathcal{B}_{\lhd^M})$, for all $lhd^M \in \mathcal{M}$, which then give rise to $h_*(\mathcal{B}_{\lhd^L}) \subseteq \mathcal{B}_{\lhd^M}$. To see that $h_*(G)$ is a Cauchy filter, take $a \in G \cap \mathcal{B}_{\lhd^L}$. Then

$$h_*(a) \in h_*(G \cap \mathcal{B}_{\lhd^L}) \subseteq h * (G) \cap h_*(\mathcal{B}_{\lhd^L}).$$

But $h_*(\mathcal{B}_{\lhd^L}) \subseteq \mathcal{B}_{\lhd^M}$, hence $h_*(a) \in \mathcal{B}_{\lhd^M}$ and therefore

$$h_*(a) \in h_*(G) \cap \mathcal{B}_{\lhd^M}.$$

Thus $h_*(G)$ is a Cauchy filter in M. □

Definition 2.3.7. A frame homomorphism $h : M \to L$ between Császár frames is a *Cauchy frame homomorphism* if for any regular Cauchy filter F in L, there exists a regular Cauchy filter G in M such that $G \subseteq h_*(F)$.

Remark 2.3.8. Given two Cauchy homomorphisms $h : M \to L$ and $g : L \to K$ between Császár frames, let F be a regular filter on K. Pick a regular Cauchy filter G on L such that $G \subseteq g_*(F)$. There is also a regular Cauchy filter H on M such that $H \subseteq h_*(G)$. Since $(g \circ h)_* = h_* \circ g_*$, it follows that $H \subseteq (g \circ h)_*(F)$. We therefore have the category of proximal Császár frames and Cauchy homomorphisms, which we will denote by $\mathbb{PC}$**s**$\mathbb{F}$**rm**.

In the following result [13], we show the relationship between uniform and Cauchy homomorphisms. To wit, it follows from Lemma 2.3.3 that $\mathbb{UC}$**s**$\mathbb{F}$**rm** $\subseteq$ $\mathbb{PC}$**s**$\mathbb{F}$**rm**.

Theorem 2.3.9. *Every uniform homomorphism $h : (M, \mathcal{M}) \to (L, \mathcal{L})$ between proximal Császár frames is a Cauchy homomorphism.*

Proof.

Let F and G be regular Cauchy filters in L and M, respectively. Since h is uniform, we have that $h_*(F)$ is a regular Cauchy filter in M. But G is a minimal Cauchy filter and thus $G \subseteq h_*(F)$. Therefore h is a Cauchy homomorphism. □

Observation 2.3.10.

i) Since dense onto continuous homomorphisms, it follows that every dense onto continuous homomorphism between proximal Császár frames is a Cauchy homomorphism.

ii) Every surjection is a Cauchy homomorphism.

Proof:

i) A dense onto continuous homomorphism $h : M \to L$ is a uniform homomorphism and we have have that every uniform homomorphism between proximal Cs*á*sz*á*r frames is a Cauchy homomorphism.

ii) We know that if $\lhd^M \in \mathcal{M}$ and h is a surjection then $h(h_*(\lhd^L)) = \lhd^L$ such that $h(\lhd^M) \subseteq \lhd^L$. Thus h is continuous and hence a Cauchy homomorphism. □

Concluding Remarks: Cauchy homomorphisms and their properties will be used in constructing coreflective subcategories of Cs*á*sz*á*r frames in the last three sections of the next chapter.

Chapter 3

Completion Properties of Cs*á*sz*á*r Frames

In the first section, we look at some concepts and results on nearness frames that is needed in constructing completions of three Cs*á*sz*á*r frames in the following sections. The concept of a strict extension is introduced in accordance with Apfel's "Hong's construction" in [2]. We then go on to look into some properties of completions of Cs*á*sz*á*r frames and construct their Cauchy completions according to Chung in [12]. In this chapter, unless otherwise stated, all frames are Cs*á*sz*á*r frames. To avoid confusion with the relation $\lhd$, we will use superscripts on the relations we encounter here to emphasise the Cs*á*sz*á*r orders we are working with.

3.1 The role of filters in completion of frames

The importance of filters in topology in general is well-known. They have been studied in their own right and have also been used to construct extensions and completions related to mathematical structures. We know cases where mathematical structure of graduate studies, for instance [2] and [26].

Recall [8] that a *nearness structure* on a frame L is a collection $\mathcal{N} \subseteq Cov(L)$ satisfying:

i) $\mathcal{N}$ is non-empty upset, (that is if $a \in \mathcal{N}$ and $a \leq b$ then $b \in \mathcal{N}$), in $(Cov(L), \leq)$ such that for any $C, D \in \mathcal{N}$,

$$C \wedge D \ = \ \{c \wedge d \mid c \in C, d \in D\} \in \mathcal{N}.$$

ii) For any $u \in L$, we have

$$u = \bigvee\{v \in L, v \lhd u\},$$

where $v \lhd u$ means that there is $C \in \mathcal{N}$ with $Cv \leq u$ and the C-star of v is defined by

$$Cv = \bigvee\{w \in L \mid w \wedge v \neq 0\}.$$

The pair $(L, \mathcal{N})$ is then called a *nearness frame*. A frame homomorphism $h : (M, \mathcal{M}) \to (L, \mathcal{N})$ is said to be *uniform* if for any $U \in \mathcal{M}$, it holds that $h(U) \in \mathcal{N}$; it is said to be a *surjection* if it is onto and for any $V \in \mathcal{N}$, then $h_*(V)$ is a cover of M and $\{h_*(V) \mid V \in \mathcal{N}\}$ generates the filter $\mathcal{M}$.

A nearness frame $(L, \mathcal{N})$ is *complete* if any dense surjection $h : M \to L$ is an isomorphism; and it is *Cauchy complete* if every regular Cauchy filter in $(L, \mathcal{N})$ is completely prime.

We will say that a filter F in a frame L is *convergent* if for every cover U of L, it holds that $F \cap U \neq \emptyset$, We observe (see for instance, [20]) that:

i) Every completely prime filter is convergent: For, suppose F is a completely prime. Then $\bigvee U = e \in F$, we have that $C \cap F \neq \emptyset$. Thus F is convergent.

ii) A filter containing a convergent filter is convergent, since if F is a convergent filter and G another filter with $F \subseteq G$, then for a cover U of L, we have that $F \cap U \neq \emptyset$. Now if $G \cap U = 0$ were true, then (since $F \subseteq G$) it would mean that $F \cap U \subseteq G \cap U \neq \emptyset$, which is a contradiction to the convergence of F. Therefore we must have $G \cap U \neq \emptyset$.

iii) A filter containing a completely prime filter is convergent. Let $F \subseteq G$ where F is a completely prime filter. Then F is convergent by (i) and so the convergence of G follows from (ii). □

In nearness frames (see for instance, [21]), completely prime filters are regular Cauchy in the following sense

Proposition 3.1.1:

i) *Every completely prime filter on a nearness frame is a regular Cauchy filter.*

ii) *A nearness frame* $(L, \mathcal{N})$ *is Cauchy complete if and only if every regular Cauchy filter on* $(L, \mathcal{N})$ *is convergent.*

Proof:

i) Take a completely prime filter F in a nearness frame $(L, \mathcal{N})$. It follows then that any subset $U \subseteq L$ satisfying $\bigvee U \in F$, it holds that $U \cap F \neq \emptyset$; that is F is a Cauchy filter. Now, suppose that $v \in F$ and pick $u \in L$ satisfying $v \lhd u$. By CO_2) in Definition 1.3.1 it holds that $v \leq u$. But F is a filter, so it follows that $u \in F$, which proves that F is a regular Cauchy filter.

ii) Suppose $(L, \mathcal{N})$ is a Cauchy complete nearness frame and take F to be a regular Cauchy filer in L. Then, by definition of this frame L, every regular Cauchy filter in L is a completely prime filter, so F is completely prime. By definition again, every completely prime filter is convergent, and so does F. □

For the reverse implication, we assume that every regular Cauchy filter in the nearness frame $(L, \mathcal{N})$ is convergent. Then it immediately follows that $(L, \mathcal{N})$ is Cauchy complete.

Filters play a crucial role in completion of frames, especially in the uniform and nearness Cauchy completions. See for example, Banaschewski and Pultr in [8]. In what follows, we will follow Apfel and refer to construction of strict extensions as “Hong's Construction” because indeed the idea came from him in [20].

Let L be a frame, and let X be the set of filters on L. We set

$$s_X L = \{(a, \sum) \subseteq L \times \mathcal{P}(X) \mid a \in F, F \in \sum\}$$

We then define $c_L : s_X L \to L, (a, \sum) \mapsto a$, that is, it is the restriction of the first projection map $\pi_1 : L \times \mathcal{P}(X) \to L$. Then c_L is obviously a frame homomorphism. Denoting by $(c_L)_*$ the right adjoint of c_L, we find that

$$(c_L)_*(a) = \bigvee \{a, \sum_a\} = (a, \sum_a), \text{where} \sum_a = \{F \in \mathcal{S} \mid a \in F\}.$$

We denote by $t_X L$ the subframe of $s_X L$ that is generated by $(c_L)_*(L)$, and set $c|_{t_X L} = t$, that is t the restriction of s to $t_X L$.

By definition t is strict, that is, $t_*(L)$ generates $t_X L$, and for any $a \in L$ we find that $(t \circ c_{L_*})(a) = t(a, X_a) = a$ so that t is onto; moreover, it is a strict extension.

In the setting of nearness frames, $t_X L$ with the associated induced nearness structure is known to be a complete nearness frame as well, and the map $c_L : cL \to L$ is also a dense surjection. Moreover, it is Cauchy complete and so $c_L : cL \to L$ is the Cauchy completion of L. See details in [21]. We conclude this section by a related result in strong nearness frames which we will need in the construction of a Cauchy Completion of a Cs*á*sz*á*r frame. First, we recall that a nearness frame $(L, \mathcal{N})$ is said to be *strong* if whenever $U \in \mathcal{N}$, the set

$$\check{U} = \{u \in L \mid u \lhd v \text{ for } \text{ some } v \in U\}$$

also belong to $\mathcal{N}$.

Theorem 3.1.2. *If $(L, \mathcal{N})$ is a strong nearness frame and*

$$F^\circ \ = \ \{x \in L \mid y \lhd x \ for \ some \ y \in F\}$$

for any filter F in L, then the following hold:

i) If F is a Cauchy filter, then F° is a regular Cauchy filter.

ii) F is a regular Cauchy filter if and only if it is a minimal Cauchy filter.

iii) If F is a Cauchy filter, then F° is the unique regular Cauchy filter contained in F.

iv) $(L, \mathcal{N})$ is Cauchy complete if and only if every Cauchy filter is convergent.

Proof:

i) Let $(L, \mathcal{N})$ be a strong nearness frame. Then for $U \in \mathcal{N}$, by definition

$$\check{U} = \{u \in L \mid u \lhd v \text{ for } \text{ some } v \in U\}.$$

Now, if F is a Cauchy filter in $(L, \mathcal{N})$, then we must have that $F \cap \check{U} \neq \emptyset$. So, there is some element $u \in F \cap \check{U}$, so that then there exists $v \in U$ such that $u \lhd v$, which proves that $v \in F^{\circ}$. Therefore $F^{\circ} \cap U \neq \emptyset$; thus F° is a Cauchy filter.

Since $F^{\circ} \cap U \neq \emptyset$, take any $u \in F^{\circ} \cap U$. This implies that $u \in F^{\circ}$ and $u \in U$. Then there is $v \in F$ such that $v \lhd u$. This shows that F° is regular.

ii) $\Rightarrow$ We refer to Proposition 2.2.2(ii).

$\Leftarrow$ Conversely, suppose that F is a minimal Cauchy filter. By (i) above, we know that F° is a Cauchy filter, so we need only show that $F^{\circ} \subseteq F$. Take $x \in F^{\circ}$, so that $a \lhd x$ for some $a \in F$. From the fact that $\lhd \subseteq \prec$ (Definition 2.1.7 (i)), we know that $a \prec x$, and so $a \leq x$, which means that $x \in F$ because $a \in F$. So $F^{\circ} \subseteq F$.

iii) Suppose F and G are Cauchy filters with $G \subseteq F^{\circ}$. We know that for any $a \in F^{\circ}$, there exists $b \in F$ with $b \lhd a$. Since G is also a Cauchy filter, we have $b^* \in G$ or $a \in G$. Since $G \subseteq F^{\circ} \subseteq F$, $b^* \notin G$, thus $a \in G$. This implies that F° is minimal Cauchy filter and hence a regular Cauchy filter. Therefore $F^{\circ} \subseteq G$ which implies that $F^{\circ} \subseteq G$.

iv) $(\Rightarrow)$: Suppose $(L, \mathcal{N})$ is Cauchy complete. This implies that every regular Cauchy filter is a completely prime filter. We know that a completely prime filter is convergent.

$(\Leftarrow)$: Suppose every Cauchy filter in $(L, \mathcal{N})$ is convergent. Then it follows immediate that $(L, \mathcal{N})$ is Cauchy complete. □

3.2 Completion Properties of Császár frames

Unless stated otherwise, the results of this section are taken from Chung's treatise in [12].

Proposition 3.2.1.

For any Császár frame $(L, \mathcal{L})$, the pair $(t_X L, \mathcal{L}^*)$ is a Császár frame.

Proof:

By Theorem 1.3.8 (i), we know that $\mathcal{L}^*$ is a Császár order on $t_X L$. We begin by showing that $\mathcal{L}^*$ is admissible, namely that for every $A \subseteq L$, it holds that

$$(\bigvee A, \sum_a) = \bigvee\{(\bigvee B, \sum_b) \mid (\bigvee B, \mathcal{S}_b) \lhd^{t_X L} (\bigvee A, \sum_a)\}$$

Since $c_L(x, \sum_a) = a$, we must have

$$\bigvee\{(c_L)_*(u) \mid u \in A\} = (\bigvee A, \sum_a).$$

We also have that if $v \lhd^{t_X L} u$, then

$$(c_L)_*(v) \ \lhd^{t_X L} \ (c_L)_*(u),$$

which reduces to having to show that

$$(c_L)_*(u) = \bigvee\{(c_L)_*(v) \mid v \lhd^{t_X L} u\}, \text{ for each } u \in L.$$

Note that

$$\bigcup\{\mathcal{S}_v \mid v \lhd^{t_X L} u\} \subseteq \mathcal{S}_u.$$

Now if $F \in \mathcal{S}_u$ then (F being regular) there is a $v \in F$ such that $v \lhd^L u$, so that then

$$F \in \bigcup\{\mathcal{S}_v \mid v \lhd^{t_X L} u\} \subseteq \mathcal{S}_u.$$

By Definition 1.3.3 (CS_2), we must have that $\lhd^{t_X L}$ is a meet-sublattice of $t_X L \times t_X L$.

It remains to show that $(t_X L, \mathcal{L}^*)$ is regular. To this end, we assume that $(u, \mathcal{S}_a) \ \lhd^L$ $(v, \mathcal{S}_b)$. Then, by Definition 1.3.7 (ii), there are $x, \ y \in L$ for which

$$(u, \mathcal{S}_a) \ \leq \ x \ \lhd^L \ y \text{ and} (c_L)_*(y) \ \leq \ (v, \mathcal{S}_b).$$

But $(L, \mathcal{L})$ is regular, and $u \lhd^L y$ (from $u \leq x \lhd^L y$) implies that $u \prec y$, so $u^* \vee y = e$. On the other hand, F is a regular Cauchy filter so that either $u^* \in F$ or $v \in F$; therefore

$$(c_L)_*(u^*) \vee (c_L)_*(v) = (e, X).$$

Now

$$\begin{aligned} (u, \mathcal{S}_u)^* \ &\geq \ (c_L)_*(u^*) \\ \Rightarrow (u, \mathcal{S}_u)^* \vee (v, \mathcal{S}_v) \ &\geq \ (c_L)_*(u^*) \vee (c_L)_*(v) \ = \ e \end{aligned}$$

and so

$$(u, \mathcal{S}_u)^* \vee (v, \mathcal{S}_v) \ = \ e,$$

which shows that

$$(u, \mathcal{S}_u) \prec (v, \mathcal{S}_v).$$ □

Proposition 3.2.2. *Let $(L, \mathcal{L})$ be Császár frame. Then we have the following:*

i) $c_L : (t_X L, \mathcal{L}^*) \to (L, \mathcal{L})$ *is a Cauchy homomorphism.*

ii) *For any regular Cauchy filter* G *on* $(t_X L, \mathcal{L}^*)$, $c_L(G)$ *is a regular Cauchy filter on* $(L, \mathcal{L})$.

Proof:

i) Suppose F is a regular Cauchy filter in a Császár frame $(L, \mathcal{L})$ and set

$$G = \{u \in t_X L \mid v \leq u \text{ for some } v \in (c_L)_*(F)\}.$$

<u>G **is a filter**</u>: From the fact that the right adjoint of c_L, $(c_L)_*$, preserves finite meets, it follows that G is a filter in $(t_X L, \mathcal{L}^*)$.

<u>**We claim that** $G \subseteq (c_L)_*(F)$</u>: For, assume $\in G$. Then there exists a $v \in (c_L)_*(F)$ such that $v \leq u$. Since c_L is onto. we must have that $c_L(v) \in F$. But together with $c_L(v) \leq c_L(u)$, this implies that $c_L(u) \in F$, and then $u \in (c_L)_*(F)$ as asserted.

<u>G **is Cauchy**</u>: We take any $\lhd^{t_X L} \in \mathcal{L}^*$. We must show that $G \cap \mathcal{B}_{\lhd^{t_X L}} \neq \emptyset$, where $\mathcal{B}_{\lhd^{t_X L}}$ is the set of $\lhd^{t_X L}$-small elements. Then we find $\lhd^L \in \mathcal{L}$ such that $\lhd^{t_X L} = c_{L_*}(\lhd^L)$. From the fact that F is a Cauchy filter in L, we have $F \cap \mathcal{B}_{\lhd^L} \neq \emptyset$ so that $c_{L_*}(F) \cap c_{L_*}\mathcal{B}_{\lhd^L} \neq \emptyset$. It follows that $\mathcal{B}_{\lhd^L} \cap G \neq \emptyset$ since $G \subseteq (c_L)_*(F)$ and therefore G is a Cauchy filter on $(t_X L, \mathcal{L}^*)$.

To see that G is regular, let $a \in G$. Then there is $z \in F$ with $c_*(z) \leq a$. F is a regular filter on $(L, \mathcal{L})$, hence there is $w \in F$ such that $w \lhd^L z$. It then follows that $c_*(w)c_*(\lhd^L)c_*(z)$ and hence $c_*(w) \lhd^L a$. Therefore G is a regular filter on $(t_X L, \mathcal{L}^*)$.

ii) Suppose G is a regular Cauchy filter on $(t_X L, \mathcal{L}^*)$. Since c_L is dense, then $G \cap \mathcal{B}_{(c_L)_*(\lhd^L)} \neq \emptyset$, from Theorem 1.3.16(i). But c_L is an onto dense homomorphism, hence it follows from Theorem 1.3.16(iii) that $c_L(\mathcal{B}_{(c_L)_*(\lhd^L)}) = \mathcal{B}_{\lhd^L}$, thus $c_L(G) \cap \mathcal{B}_{\lhd^L} \neq \emptyset$. This shows that $c_L(G)$ is a Cauchy filter on $(L, \mathcal{L})$. Now, if $y \in G$, then there is $z \in G$ and $(c_L) * (\lhd^L) \in \mathcal{L}^*$ for which $z \ (c_L) * (\lhd^L) \ y$ holds. It follows that $c_L(z) \lhd^L c_L(y)$ and thus $c_L(z) \in c_L(G)$ since c_L is onto. This proves that $c_L(G)$ is indeed a regular filter on $(L, \mathcal{L})$. □

Observation 3.2.3. The following are immediate:

i) For a *Császár* frame $(L,\mathcal{L})$, the homomorphism $c_L : (t_X L, \mathcal{L}^*) \to (L,\mathcal{L})$ is an isomorphism if and only if every filter in X is a Cauchy completely prime filter in L.

ii) A *Császár* frame $(L,\mathcal{L})$ is Cauchy complete if and only if $t_X L = \{(a,\Sigma_a) \mid a \in L\}$.

Theorem 3.2.4. *The Császár frame $(t_X L, \mathcal{L}^*)$ is Cauchy complete.*

Proof:

Let F be a regular Cauchy filter on $(t_X L, \mathcal{L}^*)$ and B a cover of $t_X L$. Then

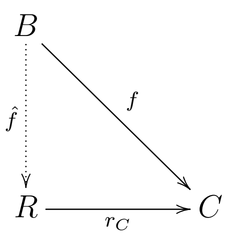
$$B = \{(v, \mathcal{S}_v) \mid v \in V\}, \text{for some } V \subseteq L,$$

and we also know that

$$\bigvee B = (\bigvee V, \mathcal{S}_v) = (e, X).$$

Since c_L preserves regular Cauchy filters, we must have that $c_L(F)$ is also a regular Cauchy filter in $(L,\mathcal{L})$. From $\bigvee B = (\bigvee V, \mathcal{S}_v) = (e, X)$, it follows that $\mathcal{S}_v = X$, and then $c_L(F) \in \mathcal{S}_v$; thus $c_L(F) \cap V \neq \emptyset$. So, we pick $u \in c_L(F) \cap V$ and find a $v \in F$ such that $v \lhd (c_L)_*(u)$ implying that $v \leq (c_L)_*(u)$. Since F is a filter, we must have that $(c_L)_*(u) \in F$ which gives $(c_L)_*(v) = (v, \mathcal{S}_v) \in F \cap B \neq \emptyset$ showing that F is convergent, so that $(t_X L, \mathcal{L}^*)$ is Cauchy complete. □

En route to the main result in this section, we recall the following concept. See, for example, Herrlich ([19]) and Abdujabal ([1]) Given a category $\mathcal{C}$, let $\mathcal{B}$ be a subcategory of $\mathcal{C}$. This means that there is an inclusion functor $I : \mathcal{B} \to \mathcal{C}$. For any $\mathcal{C}$-object C, we call a $\mathcal{B}$-object R together with a morphism $r_C : R \to C$ a *coreflection of* C in $\mathcal{B}$ if for every morphism $f : B \to C$, for $B \in Ob(\mathcal{B})$, factors uniquely through R, that is, there exists a unique $\mathcal{B}$-morphism $\hat{f} : B \to R$ such that $f = r_C \circ \hat{f}$:

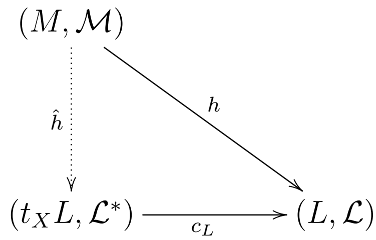


Note that $\hat{f}$ is unique with respect to f and r_C, but (R, r_C) is determined up to isomorphism. Now, here is the main result of this section, namely

Theorem 3.2.5. *The category* $\mathbb{CCC}\boldsymbol{s}\mathbb{F}\boldsymbol{rm}$ *of Cauchy Complete Császár frames and Cauchy homomorphisms is coreflective in the category* $\mathbb{C}\boldsymbol{s}\mathbb{F}\boldsymbol{rm}$ *of Császár frames and Cauchy homomorphisms.*

Proof.

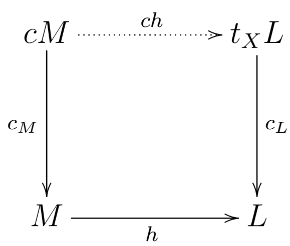
Given a Császár frame $(L, \mathcal{L})$, we know that its completion $t_X L$ is Cauchy complete Császár frame (Theorem 3.2.4) and then by Proposition 3.2.3(i), $c_L : (t_X L, \mathcal{L}^*) \rightarrow (L, \mathcal{L})$ is a Cauchy homomorphism. We take a Cauchy complete Császár frame $(M, \mathcal{M})$ and a Cauchy homomorphism $h : (M, \mathcal{M}) \rightarrow (L, \mathcal{L})$. We want to construct a Cauchy homomorphism $\hat{h} : (M, \mathcal{M}) \rightarrow (t_X L, \mathcal{L}^*)$ such that $h = (c_L) \circ \hat{h}$:



Following Dube *et al* [16], we use the notation ch for the map $(cM, \mathcal{M}^*) \rightarrow (t_X L, \mathcal{L}^*)$ defined by

$$ch(u, \mathcal{S}_u) = (h(u), \bigcup\{\mathcal{S}_{h(v)} \mid v \lhd^{\mathcal{M}} u\}).$$

FACT 1: The map ch makes the following rectangle commutative:



This follows from the fact that, for any $(u, \mathcal{S}_u) \in (cM, \mathcal{M}^*)$, we find that

$$\begin{aligned}(c_L \circ (ch))(u, \mathcal{S}_u) &= c_L[(ch)(u, \mathcal{S}_u)] \\ &= c_L(h(u), \bigcup\{\mathcal{S}_{h(v)} \mid v \lhd^{\mathcal{M}} u\}) \\ &= h(u)\end{aligned}$$

and

$$(h \circ c_M)(u, \mathcal{S}_u) = h[c_M(u, \mathcal{S}_u)] = h(u).$$

which shows that $c_L \circ (ch) = h \circ c_M$.

FACT 2: The map $ch : cM \to t_X L$ is a Cauchy homomorphism. For, suppose that H is a regular Cauchy filter in $(t_X L, \mathcal{L}^*)$. By proposition 3.2.3, we know that $c_L(H)$ is a regular Cauchy filter in $(L, \mathcal{L})$. By Cauchyness of $h : (M, \mathcal{M}) \to (L, \mathcal{L})$, there exists a regular Cauchy filter G in $(M, \mathcal{M})$ for which

$$G \subseteq \; h_*[c_L(H)].$$

It remains to show that

$$(c_M)_*(G) \subseteq (ch)_*(H).$$

If $x \in (c_M)_*(G)$ then $x = (c_M)_*(y)$ with $y \in G$. There exists (by regularity of G) a $z \in G$ such that $z \lhd^{\mathcal{M}} \; y$. Since $G \subset h_*[c_L(H)]$, we have $h(G) \subseteq c_L(H)$ and then $h(z) \in c_L(H)$. This implies that there is an element $(v, \mathcal{S}_v) \in H$ such that

$$h(z) \;\subseteq\; c_L(v, \mathcal{S}_v) = v.$$

Acting $(c_L)_*$ on this equality , we have that

$$(c_L)_*(v) = (c_L)_*(h(z)) \leq (ch)[(c_M)_*(z)].$$

Now let $F \in \mathcal{S}_{h(z)}$ be any filter in $t_X L$. Since $z \lhd^{\mathcal{M}} \; y$ (and therefore $z \leq y$), we must have $h(z) \leq h(y)$ with $h(z) \in F$, so that $h(y) \in F$. The definition of ch ensures that

$$F \in \bigcup \{\mathcal{S}_{h(z)} \mid z \lhd^{\mathcal{M}} \; y\}$$

and so

$$(c_L)_*[h(z)] \leq \; (c_L)_*[h(y)] \;\leq\; (ch)[(c_M)_*(z)].$$

Remember, $(u, \mathcal{S}_U)$ was taken from H and that H is an upset (as a filter), so we must have that

$$(ch)[(c_M)_*(z)] \in H \; or \; (c_M)_*(z) \in (ch)^{-1}(H),$$

which implies that

$$(c_M)_*(G) \;\subseteq\; (ch)^{-1}(H),$$

as was to be shown. □

3.3 Cauchy Complete Uniform Cs$\acute{a}$sz$\acute{a}$r frames

In this section, we will show that the category $\mathbb{CCUC}$s$\mathbb{F}$**rm** of Cauchy complete uniform Császár frames is coreflective in the category $\mathbb{UC}$s$\mathbb{F}$**rm**. The results leading to this result are developed from the following definition, adopted from Chung ([12]). First, if $\lhd^L$ and $\lhd^L_\circ$ are Császár orders in $(L, \mathcal{L})$, we will write

$$\lhd^L \ \subseteq \ (\lhd^L_\circ)^2$$

to mean that:

$$\text{if } u \ \lhd^L \ w \text{ there exists } v \in L \text{ such that } u \ \lhd^L_\circ \ v \ \lhd^L_\circ \ w.$$

Definition 3.3.1. A Császár frame $(L, \mathcal{L})$ is said to be a *uniform Császár frame* if each of $\mathcal{L}$ is symmetric and for any $\lhd^L \in \mathcal{L}$ there exists a $\lhd^L_\circ$ such that

$$\lhd^L \ \subseteq \ (\lhd^L_\circ)^2.$$

Theorem 3.3.2. ([12]). *For any uniform Császár frame $(L, \mathcal{L})$, the pair $(t_X L, \mathcal{L}^*)$ is a Cauchy complete uniform Császár frame.*

Proof:

Suppose $(L, \mathcal{L})$ is a uniform Császár frame. Then $\mathcal{L}$ is symmetric and for any $\lhd^L \in \mathcal{L}$ if $y \ \lhd^L \ z$, then there is a $w \in \mathcal{L}$ with $y \ \lhd^L_\circ \ w \lhd^L_\circ \ z$. By Theorem 3.2.5, the pair $(t_X L, \mathcal{L}^*)$ is Cauchy complete. We must now show that $(t_X L, \mathcal{L}^*)$ is a uniform Császár frame. Suppose $(c_L)_*(y)\ (c_L)_*(\lhd^L)\ (c_L)_*(z)$ in $(t_X L, \mathcal{L}^*)$. Then $y \ \lhd^L \ z$ in $(L, \mathcal{L})$ by Theorem 1.3.10 (v), and then $z^* \ \lhd^L \ y^*$, since $\lhd^L$ is symmetric. Thus it follows that

$$[(c_L)_*(z)]^* \ (c_L)_*(\lhd^L) \ [(c_L)_*(y)]^*.$$

Therefore $(c_L)_*(\lhd^L)$ is symmetric. Now, $y \ \lhd^L \ z$ implies that

$$y \ \lhd^L_\circ \ w \ \lhd^L_\circ \ z \text{ for } \lhd^L_\circ \in \mathcal{L} \text{ and} w \in L.$$

Then we will have $(c_L)_*(y)\ (c_L)_*(\lhd^L_\circ)\ (c_L)_*(w)(c_L)_*(\lhd^L_\circ)(c_L)_*(z)$ since c_L is onto dense (appealing to Theorem 1.3.8 (iv)). This completes the proof. □

Definition 3.3.3 Let $(L, \mathcal{L})$ and $(M, \mathcal{M})$ be uniform Császár frames. A frame homomorphism $h : M \to L$ is said to be a *uniform homomorphism* if for any $\lhd^M \in \mathcal{M}$ there exists $\lhd^L \in \mathcal{L}$ with $\mathcal{B}_{\lhd^L} \leq h(\mathcal{B}_{\lhd^M})$.

Notation: Let $(L, \mathcal{L})$ be a Császár frame and $A, B \subseteq L$. Then A is said to $\lhd-$refine B if for any $a \in A$ there is $b \in B$ with $a \lhd b$. In this case we write $A \lhd B$. When necessary, we may use superscripts (such as $\lhd^L$) to distinguish one Császár order from another.

Lemma 3.3.4 ([12]). *Let $(L, \mathcal{L})$ be a uniform Császár frame. Then for any $\lhd^L \in \mathcal{L}$, there exists $\lhd^L_\circ \in \mathcal{L}$ with $\mathcal{B}_{\lhd^L_\circ} \lhd^L_\circ \mathcal{B}_{\lhd^L}$.*

Proof:

Suppose that $\lhd^L \in \mathcal{L}$. Without loss of generality, we pick $\lhd^L_\circ, \lhd^L_{\circ\circ}$ such that

$$\lhd^L \subseteq (\lhd^L_{\circ\circ})^2 \text{ and } \lhd^L_{\circ\circ} \subseteq (\lhd^L_\circ)^2.$$

Now we take $u \in \mathcal{B}_{\lhd^L_\circ}$. We want to show that $u \in \mathcal{B}_{\lhd^L}$. Suppose then that $v \in L$ and that $u \lhd^L_\circ v$. We then set

$$u^\circ = \bigwedge\{w \in L \mid u \lhd^L_\circ v \lhd^L_\circ w\}$$

We note that

$$u \lhd^L_\circ v \leq u^\circ$$

and therefore $u \lhd^L_\circ u^\circ$. It will be enough to show that $u^\circ \in \mathcal{B}_{\lhd^L}$. Take $m \lhd^L n$ such that $u^\circ \wedge m \neq 0$. Since $\lhd^L \subseteq (\lhd^L_{\circ\circ})^2$ there exists $p \in L$ such that

$$m \lhd^L_{\circ\circ} p \lhd^L_{\circ\circ} n.$$

However, the relation $\lhd^L_{\circ\circ} \subseteq (\lhd^L_\circ)^2$ implies that there are $r, s \in L$ such that

$$m \lhd^L_\circ r \lhd^L_\circ p \text{ and } p \lhd^L_\circ s \lhd^L_\circ n.$$

Consequently, we have

$$m \lhd^L_\circ r \lhd^L_\circ p \lhd^L_\circ s \lhd^L_\circ n.$$

We assert that $u \wedge p \neq 0$: For, if $u \wedge p = 0$, then from the relation (due to symmetry)

$$u \leq p^* \lhd^L_\circ r^* \circ m^*$$

it follows that $u^\circ \leq m^*$ and so $u^\circ \wedge m = 0$. But this contradicts the fact that $u^\circ \wedge m \neq 0$. Therefore, $u \wedge p \neq 0$. From the assumption that $u \in \mathcal{B}_{\lhd_\circ^L}$ it follows that

$$u \leq s \lhd_\circ^L n.$$

Then (by definition of u°), we find that $u^\circ \leq n$ and so $u^\circ \in \mathcal{B}_{\lhd^L}$, hence $\mathcal{B}_{\lhd_\circ^L} \lhd_\circ^L \mathcal{B}_{\lhd^L}$, as asserted. □

Notation 3.3.5. The following result of Chung ([12]) is analogous to the one proved by Hong and Kim ([21]). See Theorem 3.1.2 (earlier). Given a filter F on a uniform Császár frame, F° denotes the filter

$$F^\circ = \{x \in L \mid a \lhd^L x \text{ for some } a \in F\}.$$

Proposition 3.3.6. *Let $(L, \mathcal{L})$ be a uniform Császár frame and F a Cauchy filter on $(L, \mathcal{L})$. Then F° is a regular Cauchy filter on $(L, \mathcal{L})$.*

Proof:

Since $F^\circ = \{x \in L \mid a \lhd_L x \text{ for some } a \in F\}$, it follows that F° is a regular filter. It remains to show that F° is a Cauchy filter. Following the previous result (Lemma 3.3.4), we take $\lhd^L$, $\lhd_\circ^L \in \mathcal{L}$ such that

$$\mathcal{B}_{\lhd_\circ} \lhd_\circ^L \mathcal{B}_{\lhd^L}.$$

Since F is a Cauchy filter, we must have $F \wedge \mathcal{B}_{\lhd_\circ^L} \neq \emptyset$. So, there must be a $u \in F \wedge \mathcal{B}_{\lhd_\circ^L}$ so that $u \in F$ and $u \in \mathcal{B}_{\lhd_\circ^L}$. From $\mathcal{B}_{\lhd_\circ^L} \lhd_\circ^L \mathcal{B}_{\lhd^L}$, we find a $v \in \mathcal{B}_{\lhd^L}$ such that $u \lhd_\circ^L v$. Since F is a filter we must have $v \in F$ (because $u \leq v$). This implies that $v \in F^\circ$, and so $F^\circ \cap \mathcal{B}_{\lhd_\circ^L} \neq \emptyset$; hence F° is a Cauchy filter. □

Lemma 3.3.7. *Every uniform frame homomorphism $h : M \to L$ between uniform frames is a Cauchy homomorphism.*

Proof:

By definition, a proximal Császár frame is symmetric and uniform, so the result follows from Observation 2.3.10. □

We are now ready to prove the main result in this section.

Theorem 3.3.8 ([12]). *The category $\mathbb{CCUC}\boldsymbol{s}\mathbb{F}\boldsymbol{rm}$ of Cauchy complete uniform Császár frames and Cauchy frame homomorphisms is coreflective in the category $\mathbb{UC}\boldsymbol{s}\mathbb{F}\boldsymbol{rm}$ of uniform Császár frames and uniform Cauchy frame homomorphisms.*

Proof

Suppose $(L, \mathcal{L})$ is a uniform Császár frame and let $c_L : (t_X L, \mathcal{L}^* \to (L, \mathcal{L})$ be its Cauchy completion; thus $(t_X L, \mathcal{L}^*)$ is a Cauchy complete uniform Császár frame and c_L is uniform because c_L is onto dense (by Theorem 2.3.4). Let $(M, \mathcal{M})$ be a Cauchy complete uniform Császár frame and let $h : (M, \mathcal{M}) \to (L, \mathcal{L})$ be a uniform frame homomorphism. Since h is a uniform frame homomorphism, it follows from Theorem 2.3.9 that it is a Cauchy frame homomorphism. By an argument similar to that used in Theorem 3.2.6, there is a unique Cauchy homomorphism $ch : cM \to t_X L$ such that

$$c_L \circ ch \ = \ h \circ c_M.$$

We want to show that ch is uniform. We take $(c_M)_*(\lhd^M) \in (c_M)_*(\mathcal{M})$, for some $\lhd^M \in \mathcal{M}$. By assumption $(M, \mathcal{M})$ is a (Cauchy complete) uniform Császár frame, so we pick $\lhd^M_\circ \in \mathcal{M}$ (guaranteed by Lemma 3.3.4) such that

$$\mathcal{B}_{\lhd^M_\circ} \ \lhd^M_\circ \ \mathcal{B}_{\lhd^M}.$$

But h is a uniform frame homomorphism, so we find a $\lhd^L \in \mathcal{L}$ such that

$$\mathcal{B}_{\lhd^L} \ \leq \ h(\mathcal{B}_{\lhd^M_\circ}).$$

We will work back to $t_X L$ by showing that

$$(c_L)_*(\mathcal{B}_{\lhd^L}) \ \leq \ ch \circ (c_M)_*(\mathcal{B}_{\lhd^M}).$$

To this end, we take $(c_L)_*(u) \in (c_L)_*(\mathcal{B}_{\lhd^L})$, so that $u \in \mathcal{B}_{\lhd^L}$. Since $\mathcal{B}_{\lhd^L} \ \leq \ h(\mathcal{B}_{\lhd^M_\circ})$, we find a $v \in \mathcal{B}_{\lhd^M_\circ}$ such that $u \ \leq \ h(v)$. From $v \in \mathcal{B}_{\lhd^M_\circ} \ \lhd^M_\circ \ \mathcal{B}_{\lhd^M}$, we pick $x \in \mathcal{B}_{\lhd^M}$ such that $v \ \lhd^M \ x$. Now take $F \in \Sigma_u$. Now the relations $v \ \lhd^M \ x$ and $\leq \ h(v)$ ensure that

$$F \in \cup\{\Sigma_{h(z)} \mid z \ \lhd^M \ x\}$$

and

$$(c_L)_*(u) \ \leq \ ch \circ (c_M)_*(x)$$

But this means that

$$(c_L)_*(\mathcal{B}_{\lhd^L}) \;\leq\; ch \circ (c_M)_*(\mathcal{B}_{\lhd^M}).$$

Since c_M and c_L are dense onto frame homomorphisms, we invoke Theorem 1.3.16(ii) so that

$$\mathcal{B}_{(c_L)_*(\lhd^L)} \;\leq\; (c_L)_*(\mathcal{B}_{\lhd^L})$$

which yields

$$\mathcal{B}_{(c_L)_*(\lhd^L)} \;\leq\; ch(\mathcal{B}_{(c_M)_*(\lhd^M)}).$$

Therefore $c_L : (t_X L, \mathcal{L}^*) \to (L, \mathcal{L})$ is the $\mathbb{CCUC}$**s**$\mathbb{F}$**rm**-coreflection of $(L, \mathcal{L})$ in $\mathbb{UC}$**s**$\mathbb{F}$**rm**.
□

3.4 Cauchy Complete Proximal Császár frames

For simplicity, we recall that $\mathcal{B}_{\lhd^L}$ denotes the set of all elements of L which are $\lhd^L$-small. In addition, we have $c_L : \mathcal{L}^* \to L$ where

$$\mathcal{L}^* \;=\; \{(c_L)_*(\lhd^L) \mid \lhd^L \in \mathcal{L}\}.$$

The following result [13] extends that of Proposition 3.2.2 proved earlier.

Theorem 3.4.1. *For any proximal Császár frame $(L, \mathcal{L})$, the pair $(t_X L, \mathcal{L}^*)$ is a Cauchy complete proximal Császár frame.*

Proof:

Suppose F is a regular Cauchy filter on $(t_X L, \mathcal{L}^*)$. Since c_L preserves regular Cauchy filters (by Theorem 3.2.5), it follows that $c_L(F)$ is a regular Cauchy filter on L. We take a basic cover U of $t_X L$ so that

$$V \;=\; \{(u, \Sigma_u) \mid u \in U\},$$

for some $U \subseteq L$. We have that

$$\bigvee V \;=\; (\bigvee U, \Sigma_U) \;=\; (e_L, X),$$

which implies that $\Sigma_U = X$ and $c_L(F) \in \Sigma_U$ and then

$$c_L(F) \cap U \neq \emptyset;$$

and so we pick $u \in c_L(F) \cap U$. Now, for the right adjoint $(c_L)_*$, we find a $v \in F$ such that

$$v \leq (c_L)_*(u).$$

Consequently, we arrive at

$$(c_L)_*(u) = (u, \Sigma_u) \in F \cap u$$

making $F \cap U \neq \emptyset$, thus F is convergent as was to be shown. □

The following result, analogous to Chapter 1 (Theorem 1.3.15) on Cs*á*sz*á*r frames, provides an interplay between dense onto frame homomorphisms and *small sets* [13]. Importantly, we show that to any Cs*á*sz*á*r order on a uniform proximal Cs*á*sz*á*r frame corresponds a Cs*á*sz*á*r order whose "small set" is related to that of original Cs*á*sz*á*r order.

Lemma 3.4.2. *Let $(L, \mathcal{L})$ be a proximal Császár frame. Then the following results hold:*

i) *If L is also uniform, then for any $\lhd^L \in \mathcal{L}$, there is $\lhd_\circ \in \mathcal{L}$ with $\mathcal{B}_{\lhd_\circ} \lhd_\circ \mathcal{B}_{\lhd^L}$.*

ii) *If $h : M \to L$ is an onto dense frame homomorphism and $\lhd$ is a Császár order on L, then $\mathcal{B}_{h_*(\lhd^L)} \leq h_*(\mathcal{B}_{\lhd^L})$.*

Proof:

i) Suppose that $\lhd^L \in \mathcal{L}$ and, without loss of generality, find $\lhd_1, \lhd_\circ \in \mathcal{L}$ such that

$$\lhd^L \subseteq \lhd_1^2, \text{ and } \lhd_1 \subseteq \lhd_\circ^2.$$

We pick $u \in \mathcal{B}_{\lhd_\circ}$ and take $w \in L$ such that $u \lhd_\circ w$ (by $\lhd_\circ$-smallness). We set

$$u^\circ = \bigwedge\{v \in L \mid u \lhd_\circ w \lhd_\circ v\}$$

and note that $u \lhd_\circ u^\circ$. We claim that $u^\circ \in \mathcal{B}_{\lhd^L}$.

Suppose then that $p \lhd_1^L q$ with $u^\circ \wedge p \neq 0$. From $\lhd_1 \subseteq \lhd_\circ^2$, we find $m, n, l \in L$ such that

$$p \lhd_\circ m \lhd_\circ n \lhd_\circ l \lhd_\circ q.$$

Now $u \wedge n \neq 0$: For, if not (see Remark 1.3,12), and by symmetry of $\lhd_\circ$

$$u \leq n^* \lhd_\circ m^* \lhd_\circ p^*$$

from which we find that $u^\circ \leq p^*$ and so (easily) $u^\circ \wedge p = 0$, a contradiction to our assumption that $u^\circ \wedge p \neq 0$. Since u was chosen from $\mathcal{B}_{\lhd_\circ}$, we find that

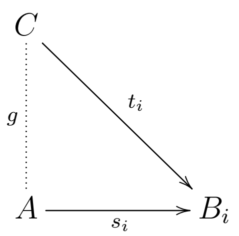
$$u \leq l \lhd_\circ q \textit{ so that } u^\circ \leq q.$$

Therefore, $u^\circ \in \mathcal{B}_{\lhd^L}$, making $\mathcal{B}_{\lhd_\circ} \subseteq \mathcal{B}_{\lhd^L}$ as desired.

ii) We refer to Theorem 1.3.15. □

The following definition is adopted from [28].

Definition 3.4.3. Given a functor $U : \mathcal{C} \to \mathcal{D}$ and a source $(s_i : A \to B_i)_{i \in I}$, we say that (s_i) is U-initial if whenever $(t_i : C \to B_i)_{i \in I}$ is any source and f is a $\mathcal{D}$-morphism such that $U(s_i) \circ f = U(t_i)$ for all i, then there is only one $\mathcal{C}$-morphism $g : C \to A$ such that $U(g) = f$ and $s_i \circ g = t_i$, for all $i \in I$, that is, the following triangles commute, one for each $i \in I$:



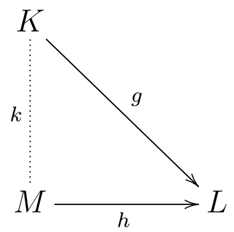
Remark. U-initial morphisms have been studied extensively in the setting of forgetful functors $U : \mathcal{C} \to \underline{\mathcal{C}}$, where in "mapping" from $\mathcal{C}$ to $\underline{\mathcal{C}}$, the functor U "leaves something behind". See, for instance, Herrlich [19]. We are interested in the following result of Chung [13].

Lemma 3.4.4. *If $U : \mathbb{PC}\boldsymbol{s}\mathbb{F}\boldsymbol{rm} \to \mathbb{F}\boldsymbol{rm}$ is a forgetful functor, then every surjection in the category of proximal Császár frames $\mathbb{PC}\boldsymbol{s}\mathbb{F}\boldsymbol{rm}$ is U-initial.*

Proof:

Let $h : (M, \mathcal{M}) \to (L, \mathcal{L})$ be a surjection in $\mathbb{PC}\mathbf{s}\mathbb{F}\mathbf{rm}$, let $g : (K, \mathcal{K}) \to (L, \mathcal{L})$ be a uniform continuous homomorphism and let $k : K \to M$ be a frame homomorphism (in $\mathbb{F}\mathbf{rm}$) such that (we refer to the triangle below)

$$U(h) \circ k \;=\; h \circ k \;=\; g.$$



Claim i): The frame homomorphism k is continuous:

suppose $u \ \lhd^K \ v$ for any $\lhd^K \in \mathcal{K}$. Take any $\lhd^K_\circ \in \mathcal{K}$ such that

$$u \ \lhd^K_\circ \ r \ \lhd^K_\circ \ s \ \lhd^K_\circ \ v \text{ for } a, b \in K.$$

Since K is a proximal Cs*á*sz*á*r frame, then it is regular and from Definition 2.1.7(i) it follows that

$$u \ \prec \ r \ \prec \ s \ \prec v.$$

Using the fact that k is a frame homomorphism and M is a proximal Cs*á*sz*á*r frame, we have that

$$k(u) \ \lhd^M_\circ \ k(r) \ \lhd^M_\circ \ k(s) \ \lhd^M_\circ \ k(v).$$

Applying h we get

$$h(k(u)) \ h(\lhd^M_\circ) \ h(k(r)) \ h(\lhd^M_\circ) \ h(k(s)) \ h(\lhd^M_\circ) \ h(k(v)).$$

But h is uniform hence continuous, therefore we have

$$h(k(u)) \ \lhd^L_\circ \ h(k(r)) \ \lhd^L_\circ \ h(k(s)) \ \lhd^L_\circ h(k(v))$$

Since h is a surjection, then it is a dense onto homomorphism, thus we have

$$k(u) \ h_*(\lhd^L_\circ) \ k(r) \ h_*(\lhd^L_\circ) \ k(s) \ h_*(\lhd^L_\circ) \ k(v)$$

Therefore it follows that $k(u) \ h_*(\lhd^L) \ k(v)$. This proves that k is continuous.

Claim ii): The continuous frame homomorphism k is uniform: To this end, we start with $\lhd^K \in \mathcal{K}$. By Lemma 3.3.4, we pick $\lhd^K_\circ \in \mathcal{K}$ such that

$$\mathcal{B}_{\lhd^K_\circ} \ \lhd^K_\circ \ \mathcal{B}_{\lhd^K}.$$

Since g is uniform by assumption, we find $\lhd^L \in \mathcal{L}$ satisfying

$$\mathcal{B}_{\lhd^L} \ \leq \ g(\mathcal{B}_{\lhd^K_\circ}).$$

Applying the right adjoint h_*, we find that

$$h_*(\mathcal{B}_{\lhd^L}) \ \leq \ h_*(g(\mathcal{B}_{\lhd^K_\circ}))$$

Now for any $u \in \mathcal{B}_{\lhd^K_\circ}$, we find from $\mathcal{B}_{\lhd^K_\circ} \ \lhd^K_\circ \ \mathcal{B}_{\lhd^K}$ that there is a $v \in \mathcal{B}_{\lhd^K}$ such that

$$h_*(g(u)) \ \leq \ k(v)$$

from which it follows that

$$h_*(g(\mathcal{B}_{\lhd^K_\circ})) \leq \ k(\mathcal{B}_{\lhd^K}).$$

By hypothesis h is surjective (and so dense onto), therefore Theorem 1.3.15(ii) implies that

$$\mathcal{B}_{h_*(\lhd^K_\circ)} \ \leq \ k(\mathcal{B}_{\lhd^K}).$$

Therefore k is uniform. □

Theorem 3.4.5. ([13]). *The category Cauchy complete proximal Császár frames* $\mathbb{CPC}\boldsymbol{s}\mathbb{F}\boldsymbol{rm}$ *of Cauchy proximal Császár frames is coreflective in the category* $\mathbb{PC}\boldsymbol{s}\mathbb{F}\boldsymbol{rm}$ *of proximal Császár frames and Cauchy homomorphisms.*

Proof:

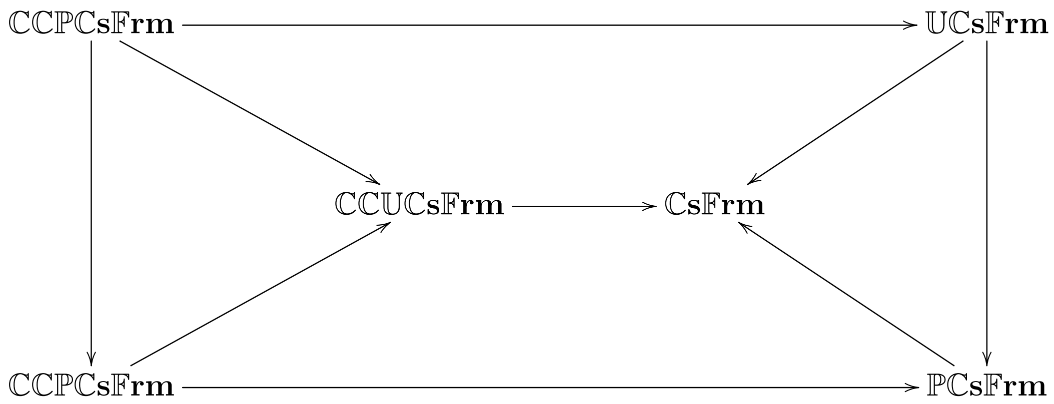
Let $(L, \mathcal{L})$ be any proximal Császár frame. Then $c_L : (t_X L, \mathcal{L}^*) \to (L, \mathcal{L}$ is a surjection and hence a uniform continuous homomorphism. Take any Cauchy complete proximal Császár frame $(M, \mathcal{M})$ and a uniform continuous homomorphism $h : M \to L$. Define $h_c : cM \to t_X L$ by

$$(h_c)(a\Sigma_a) = (h(a), \bigvee\{\Sigma_{h(x)} : x \lhd_M a\}).$$

It then follows that h_c is a homomorphism with $h \circ c_M \ = \ c_L \circ h_c$. Suppose $l \ = \ h_c \circ (c_M)_*$. Since $h : M \to L$ is a uniform continuous homomorphism, it is a Cauchy homomorphism and hence l is a frame homomorphism. Then we have that l is a uniform continuous homomorphism. But c_L is dense and thus a monomorphism (the underlying frames are regular). Therefore c is unique with this property. This implies that c_L is a coreflection of $(L, \mathcal{L})$ in the category $\mathbb{PC}\mathbf{s}\mathbb{F}\mathbf{rm}$. □

From Observation 2.3.10, Theorem 3.2.6, Theorem 3.3.6 and Theorem 3.4.5, the following is immediate:

Proposition 3.4.6. *The FIVE subcategories we have come across thus far relate as per the following diagram (the arrows indicate inclusion functor):*



Chapter 4

Compactification and Connectedness in Cs$\acute{a}$sz$\acute{a}$r frames

4.1 Compactification of proximal Cs$\acute{a}$sz$\acute{a}$r frames

Proximal Császár frames, being proximal, are completely regular, and so, they have compactifications. For background on compactifications, we follow Banaschewski [5]. See also Banaschewski and Pultr [8]. In this section, we follow the construction of Banaschewski and Mulvey [7] to construct the compactification of a proximal Császár frame. First, by a *compactification* of a frame L we mean a dense onto frame homomorphism $h : M \to L$, where M is a compact regular frame. Ideals are dual to filters, thus: a non-empty proper subset I of a frame L is said to be an *ideal* of L if:

i) whenever $u,\ v\ \in I$ then $u\ \vee\ v \in I$, and

ii) whenever $u\ \in I$ and $v\ \leq\ u$, then $v\ \in I$.

Construction

An ideal I of a proximal Császár frame $(M, \mathcal{M})$ is said to be *strongly regular* if whenever $u \in I$ there exists $v \in I$ such that $u \lhd^M v$ for some $\lhd^M \in \mathcal{M}$. We shall denote by $\mathcal{C}_{\mathcal{M}}M$ the set of all strongly regular ideals of M.

We will need the following result

Observation 4.1.1. *Let $(M, \mathcal{M})$ be a Császár frame and let $\lhd^M \in \mathcal{M}$.*

i) If $x \leq y \lhd^M z$ then $x \lhd^M z$: Together with $x \leq y \lhd^M z$ and $z \leq z$, Definition 1.3.1 ensures that $x \lhd^M z$.

ii) If $y \lhd y$ and $u \lhd^M v$, then $x \wedge u \lhd^M y \wedge v$: Under the hypothesis, we have

$$x \wedge u \leq x \lhd^M y \text{ and } x \wedge u \leq u \lhd^M v$$

which by i) imply that

$$x \wedge u \lhd^M y \text{ and } x \wedge u \lhd^M v.$$

Since $\mathcal{M}$ is a meet-sublattice of $M \times M$, we much have

$$x \wedge u \lhd^M y \wedge v.$$

□

Lemma 4.1.2. *$\mathcal{C}_{\mathcal{M}}M$ is a compact frame.*

Proof.

a) First, we observe that the *bottom element* and *top element* in $\mathcal{C}_{\mathcal{M}}M$ are respectively $\{0_M\}$ and $\{M\}$. Recall that for ideals (even strongly regular ones) $I, J \in \mathcal{C}_{\mathcal{M}}M$, we have

$$I \wedge J = \{x \wedge y \mid x \in I, y \in J\}.$$

Now, if $u \in I \wedge J$ then $u = x \wedge y$, so that there exists $u_x \in I$ and $u_y \in J$ such that

$$x \lhd^M u_x \text{ and } y \lhd^M u_y.$$

Consequently, we have

$$u = x \wedge y \lhd^M u_x \wedge u_y \in I \wedge J,$$

showing that strongly regular ideals are closed under finite meets. To see that $I \vee J \in \mathcal{C}_{\mathcal{M}}M$ for any $I, J \in \mathcal{C}_{\mathcal{M}}M$, take $x \in I \vee J$ so that $x = u \vee v$ for some $u \in I$ and $v \in J$. By definition, we pick $u_x \in I,\ v_x \in J$ such that

$$u \lhd^M u_x \text{ and } v \lhd^M v_x.$$

Then

$$x = u \vee v \lhd^M u_x \vee v_x \in I \vee J,$$

which establishes that $I \vee J$ is a strongly regular ideal. Next, we consider $\bigcup_i I_i$ where each $I_i \in \mathcal{C}_{\mathcal{M}}M$. It is easily shown that for any $x \in \bigcup_i I_i$ there is a $y \in \bigcup_i I_i$ with $x \ \lhd^M \ y$, for some $\lhd^M \in \mathcal{M}$. Given a strongly regular ideal I and a collection $\{J_i\}_i$ of strongly regular ideals in $\mathcal{C}_{\mathcal{M}}M$, let us consider

$$\bigvee_i J_i \ = \ \{\bigvee E \mid E \ finite, \ E \subseteq \bigcup_i J_i\}.$$

We claim that

$$I \cap (\bigvee_i J_i) \ = \ \bigvee_i (I \cap J_i) :$$

Note that $J_i \ \leq \ \bigvee_i J_i$ and $J_i \cap I \ \subseteq \ I$, so it easily follows that

$$\bigvee_i (J_i \cap I) \ \leq \ (\bigvee_i J_i) \cap I.$$

For the opposite implication, we take $u \in I \cap (\bigvee_i J_i)$, say

$$u \ = \ u_1 \vee u_2 \vee \ \ldots \ \vee u_n.$$

Since $u_j \ \leq \ u$, the definition of an ideal implies that $u \ = \ u_1 \vee u_2 \vee \ \ldots \ \vee u_n \in \bigvee_i (I \cap J_i)$, and so

$$(\bigvee_i J_i) \cap I \ \leq \ \bigvee_i (J_i \cap I)$$

whence the generalised distributivity property follows:

$$(\bigvee_i J_i) \cap I \ = \ \bigvee_i (J_i \cap I)$$

Consequently, $\mathcal{C}_{\mathcal{M}}M$ is a frame. To see that $\mathcal{C}_{\mathcal{M}}M$ is compact, let us take a cover $\{J_i\}_i$ for $C_{\mathcal{M}}M$, say $\bigvee_i J_i \ = \ M$, since $\{M\}$ is the top element in $\mathcal{C}_{\mathcal{M}}M$. Since $e_M \in M$, it follows that there are finitely many $u_j \in J_{i_j}$ fuch that

$$u_1 \vee u_2 \vee \ \ldots \ \vee u_n \ = \ e_M \in \bigvee_{i_j}^{n} J_{i_j}.$$

But then we must have $M \ = \ \bigvee_{i_j}^n J_{i_j}$, making $\mathcal{C}_{\mathcal{M}}M$ a compact frame. □

Lemma 4.1.3. *$\mathcal{C}_{\mathcal{M}}M$ is a proximal Császár frame.*

Proof.

We start with the Császár part: we will say (and write)

$$I \ \lhd_{\mathcal{C}}^{M} \ J \text{ if and only if for every } x \in I \text{ there exists a } y \in J \text{ such that } x \ \lhd^{M} \ y,$$
$$\text{for some } \lhd^{M} \in \mathcal{M}.$$

We claim that: $\lhd_{\mathcal{C}}^{M}$ is a Császár order on $\mathcal{C}_{\mathcal{M}}M$:

(CO_1): We have $\{O_M\} \ \lhd_{\mathcal{C}}^{M} \ \{O_M\}$ and $\{L\} \ \lhd_{\mathcal{C}}^{M} \ \{L\}$ since $0_M \ \lhd^{M} \ 0_M$ and $e_M \ \lhd^{M} \ e_M$.sss

(CO_2): Suppose that $I \ \lhd_{\mathcal{C}}^{M} \ J$. To see that $I \ \leq \ J$, we take $x \in I$ and $y \in J$ such $x \ \lhd^{M} \ y$ which implies that $x \ \leq \ y$, thus $I \ \leq \ J$.

(CO_3): We start with $I \ \leq \ A \ \lhd_{\mathcal{C}}^{M} \ B \ \leq \ J$. For any $x \in I$, there exist $y \in A$ $z \in B$ and $t \in J$ such that

$$x \ \leq \ y \ \leq^{M} \ z \ \leq \ t.$$

Then $x \ \lhd^{M} \ t$, showing that $I \ \lhd_{\mathcal{C}}^{M} \ J$. By construction, $\mathcal{C}_{\mathcal{M}}M$ is regular, being a collection of strongly regular ideals.

It remains to show that $\mathcal{C}_{\mathcal{M}}M$ is strong and symmetric. Let us denote by $\mathcal{M}_{\mathcal{C}}$ the Cs*á*sz*á*r structure on $\mathcal{C}_{\mathcal{M}}M$. Well, for symmetry, we assume that $x \ \lhd_{\mathcal{C}}^{M} \ y$, for some $\lhd_{\mathcal{C}}^{M} \in \mathcal{M}_{\mathcal{C}}$, and $x \in I$ and $y \in J$. Then, by definition, for some $\lhd^{M} \in \mathcal{M}$ (which is symmetric because M is proximal),

$$x \ \lhd^{M} \ y \text{ and so } y^* \ \lhd^{M} \ x^*; \text{ thus } y^* \ \lhd_{\mathcal{C}}^{M} \ x^*,$$

which shows that $J^* \lhd_{\mathcal{C}}^{M} \ I^*$, where $J^* \ = \{u^* \mid u \in J\}$; thus $\mathcal{C}_{\mathcal{M}}M$ is symmetric. Finally, for strongness of $\mathcal{C}_{\mathcal{M}}M$, suppose that $\lhd_{\mathcal{C}}^{M} \in \mathcal{M}_{\mathcal{C}}$ and let $I \ \lhd_{\mathcal{C}}^{M} \ J$, and let $\lhd^{M}$ be the associated Cs*á*sz*á*r order in $\mathcal{M}$. Since M is strong (being proximal), we find an order $\lhd_{\circ}^{M} \in \mathcal{M}$ such that

$$u \ \lhd^{M} \ v \Longrightarrow u \ \lhd_{\circ}^{M} \ w \ \lhd_{\circ}^{M} \ v,$$

for some $w \in M$. There is a $\lhd_{\mathcal{C}_\circ}^{M}$ generating the $\lhd_{\circ}^{M}$ so that $\lhd_{\mathcal{C}}^{M} \ \subseteq \ (\lhd_{\mathcal{C}_\circ}^{M})^2$, so $\lhd_{\mathcal{C}_\circ}^{M}$ is strong as desired. □

Lemma 4.1.4. *The map*

$$\nu_M : \mathcal{C}_{\mathcal{M}}M \to M, \ I \mapsto \bigvee I,$$

(that is, the join map ν_M) is a dense onto proximal map.

Proof.

We first note that for a map

$$\rho_M : M \to \mathcal{C}_{\mathcal{M}}M,\ u \mapsto \downarrow u,$$

the map ν_M is onto because for each $u \in M$ it holds that

$$\nu_M \circ (\rho_M(u)) \;=\; \nu_M(\downarrow u) \;=\; u,$$

We also see that for any $I \in \mathcal{C}_{\mathcal{M}}M$, it holds that

$$(\rho_M \circ \nu_M)(I) \;=\; \rho_M(\bigvee I) \;= \downarrow(\bigvee I) \;\geqq I$$

which means that ν_M is a left adjoint of ρ_M; consequently it must preserve all updirected joins. So, if $I_1 \;\; \lhd_{\mathcal{M}_{\mathcal{C}}} \;\; J_1$ and $I_2 \;\lhd_{\mathcal{C}}^{M}\;\; J_2$, then (see Definition 2.3.1)

$$\nu_M(I_1 \vee I_2) \;=\; \nu_M(I_1) \vee \nu_M(I_2) \;\lhd_{\mathcal{M}}\; \nu_M(J_1) \vee \nu_M(J_2).$$

In addition, if

$$J \;=\; \bigvee\{I \in C_{\mathcal{M}}M \in \mid I \;\; \lhd_{\mathcal{C}}^{M} J\},$$

it easily follows that this left adjoint satisfies

$$\nu_M(J) \;=\; \bigvee\{\nu_M(I) \mid I \;\; \lhd_{\mathcal{C}}^{M} J\}.$$

To show that $\rho_M : \mathcal{C}_{\mathcal{M}}M \to M$ preserves finite meets, we take $I, J \in \mathcal{C}_{\mathcal{M}}M$, and note that

$$\begin{aligned}
\nu_M(I \wedge J) \quad &= \quad \bigvee I \wedge \bigvee J \\
&= \quad \bigvee\{u \wedge v \mid u \in I,\ v \in J\} \\
&\leq \quad \bigvee\{w \mid w \in I \cap J\} \\
&= \quad \nu_M(I \cap J) \\
&\leq \quad \nu_M(I) \wedge \nu_M(J).
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
\nu_M(I) \wedge \nu_M(J) \quad &= \quad (\bigvee I) \wedge (\bigvee J) \\
&\leq \quad \bigvee(I \cap J) \\
&= \quad \nu_M(I \wedge J);
\end{aligned}$$

whence,

$$\nu_M(I \wedge J) \;=\; \nu_M(I) \wedge \nu_M(J).$$

For denseness, we proceed thus: let $\nu_M(I) \;=\; 0_M$ and note that then

$$\bigvee I \;=\; 0_M,$$

can only be true if $I \;=\; \{0_M\}$, the bottom element of $\mathcal{C}_{\mathcal{M}}M$. It is also true that

$$\nu_M(\{L\}) \;=\; \bigvee\{L\} \;=\; e_M.$$

We have therefore shown that ν_M is a dense onto proximal map. □

Putting these results together, noting also that $\bigvee\{L\} \;=\; e_M$, we have proved that

Proposition 4.1.5. *The pair $(\mathcal{C}_{\mathcal{M}}M,\ \nu_M)$ is a compactification of a proximal Császár frame $(M, \mathcal{M})$.* □

By definition we have $\mathbb{PC}\mathbf{s}\mathbb{F}\mathbf{rm} \;\subseteq\; \mathbb{R}\mathbf{eg}\mathbb{F}\mathbf{rm}$, so we derive the following:

Corollary 4.1.6. *The proximal frame homomorphism $\nu_M : \mathcal{C}_{\mathcal{M}}M \to M$ is a monomorphism.*

Proof. Since $\mathcal{C}_{\mathcal{M}}M$ ia proximal, it is regular. Since ν_M is dense (onto), it follows from Murugan in [27] (see Lemma 1.1.5) that this morphism ν_M is a monomorphism. □

Analogous to Banaschewski and Mulvey in [7], there is more if M is compact in the sense of the following.

Proposition 4.1.7. *In our construction, if M compact, then for any strongly regular ideal $I \in \mathcal{C}_{\mathcal{M}}M$, it holds that*

$$x \in I \text{ if and only if } x \lhd_{\mathcal{M}} \bigvee I,$$

for all $x \in M$. Moreover, $\nu_M : \mathcal{C}_{\mathcal{M}}M \to M$ is an isomorphism in $\mathbb{PC}\boldsymbol{s}\mathbb{F}\boldsymbol{rm}$.

Proof.

We need only find an inverse for ν_M. We claim that $\rho_M : M \to \mathcal{C}_{\mathcal{M}}M$ is actually the desired inverse. Since we saw in Lemma 4.1.4 that $I \;\leq\; (\nu_M \circ \rho_M)(I)$, it is enough if we can show that $(\rho_M \circ \nu_M)(I) \;\leq\; I$.

ii) Since $\mathcal{C}_{\mathcal{M}}M$ is compact, then for any $I \subseteq \mathcal{C}_{\mathcal{M}}M$ with $\bigvee I = e$ there exists a finite $U \subseteq I$ with $\bigvee U = e$. From the definition of ρ_M we have that $\rho_M(I) = \bigvee I$, it implies that $\bigvee I$ is a subset of M and $\bigvee(\bigvee I) = \bigvee(e) = e$. We also have that $\bigvee U \subseteq \bigvee I$ and $\bigvee(\bigvee U) = \bigvee(e) = e$. Therefore M is compact.

We close this section by relating compactifications with strong inclusions as advocated by Banaschewski in [5]. The definition of a *strong inclusion* was given in Chapter 2 (Definition 2.1.4).

Proposition 4.1.8. *If $h : N \to (M, \mathcal{M})$ is a compactification of M, then the relation $\lhd^h \subseteq \mathcal{P}(N) \times \mathcal{P}(N)$ defined by*

$$u \lhd^h v \textit{ if and only if there exists } s \textit{ and } t \textit{ in } N \textit{ with } h(s) = u \textit{ and } h(t) = v$$

is a strong inclusion on M.

Proof.

i) Suppose $x \leq u \lhd^h v \leq y$. Then there exists $s, t \in N$ such that $h(s) = u$ and $h(t) = v$. We then have $x \leq h(s) \lhd^h h(t), h(t) \leq y$ which then shows that $x \lhd^h y$.

ii) Suppose $u \lhd^h v$ and $u \lhd^h w$. Then there are $r, s, t \in N$ such that $h(r) = u, h(s) = v$ and $h(t) = w$. We then have that

$$h(r) \lhd^h h(s) \text{ and } h(r) \lhd^h h(t)$$

Since h is a compactification and hence a proximal homomorphism, it follows that:

$$\begin{aligned} h(r) &= h(r \wedge r) \\ &= h(r) \wedge h(r) \\ &\lhd^h h(s) \wedge h(t) \\ &= h(s \wedge t). \end{aligned}$$

Therefore $u \lhd^h v \wedge w$.

On the other hand, suppose $u \lhd^h w$ and $v \lhd^h w$. Then we have

$$h(r) \lhd^h h(t) \text{ and } h(s) \lhd^h h(t) \text{ for } r, s, t \in N.$$

Since h is a proximal homomorphism, we have that

$$h(r \vee s) \lhd^h h(t) \vee h(t) = h(t)$$

This implies that $u \wedge v \lhd^h w$ as was to be shown.

(iii) Suppose $u \lhd^h v$. It immediately follows that $u \prec^h v$ since $\prec^h$ is coarser than $\lhd^h$.

(iv) Suppose $u \lhd^h v$. Since h is a compactification, then h is a proximal homomorphism and thus implies that N and M are proximal frames. By property (PF_5) there exists $a \in N$ such that $u \lhd^h a \lhd^h v$.

(v) Suppose $u \lhd^h v$. Since M is a proximal frame, then by (PF_6) it follows that $v^* \lhd^h u^*$.

(vi) For any $u \in N$, by (PF_7) there exists $v \in N$ such that $u = \bigvee\{v \in N \mid u \lhd^h v\}$. Therefore $\lhd^h$ is regular. □

4.2 $\mathcal{L}$-Connectedness of Cs*á*sz*á*r frames

When Cs*á*sz*á*r introduced syntopogenous space, he did not treat connectedness. However, Sieber and Pervin in [32] noted this omission and then introduced it via separated sets and proceeded to show which familiar properties of connectedness are do-able in syntopogenous spaces. It is in this regard that this section is aimed at translating their approach into the setting of Cs*á*sz*á*r frames, which are attributed to the work of [12]. This work is influenced by remarks made by Baboolal in his paper [3] where connectedness is as looked at in relation to some earlier work of Whyburn which we will not touch on here. In addition, we want to point out that connectedness is an important concept in pointfree topology. For the purpose of this work, we cite the work of [11], [3] and [4] who respectively introduced connectedness in frames, introduced and classified various properties of connectedness in uniform spaces, and its relation of other topological notions such as Property S. In fact, there are many weaker forms of connectedness that have been shown in pointfree topology which we will not mention in this exposition.

Definition 4.2.1. Let $(L, \mathcal{L})$ be a Cs*á*sz*á*r frame.

(i) Two elements $u, v \in L$ are said to be $\mathcal{L}$-*separated* if there exists a Császár order $\lhd^L \in \mathcal{L}$ such that $u \ \lhd^L \ v^*$ and $v \ \lhd^L \ u^*$.

(ii) An element w in $(L, \mathcal{L})$ is said to be $\mathcal{L}$-*connected* if whenever $w = u \vee v$, u and v are $\mathcal{L}$-separated, then $w = u$ or $w = v$.

(iii) The frame $(L, \mathcal{L})$ is $\mathcal{L}$-*connected* if its top element, e_L, is $\mathcal{L}$-connected.

(iv) A frame $(L, \mathcal{L})$ is *locally connected* if each element is a join of $\mathcal{L}$-connected elements.

Remark 4.2.2. Our definition is influenced by what has been done in fuzzy syntopogeneous spaces [25]. Since $\mathcal{L}-$separated are disjoint, it is clear that an $\mathcal{L}-$connected element is also connected in a familiar sense.

Remark 4.2.3. Recall that in Chapter 1 we showed that if $h : M \to L$ is a frame homomorphism where $(M, \lhd^M)$ is a Császár frame, then $(L, h(\lhd^M))$ is a Császár frame where $h(\lhd^M)$ is given by

$$xh(\lhd^M)y \text{ in } L \text{ if and only if there exist } a, b \in M \text{ such that } x \leq h(a),\ a \lhd^M b \text{ and } h(b) \leq y$$

We use this result to show that

Lemma 4.2.4. *For any frame homomorphism $h : (M, \mathcal{M}) \to L$ from Császár frame M, it holds that if u and v are $\mathcal{M}$-separated in M, then $h(u)$ and $h(v)$ are separated relative to the induced Császár order $h(\lhd^M)$ for some $\lhd^M \ \in \mathcal{M}$.*

Proof: If u and v are $\mathcal{M}$-separated then

$$u \lhd^M v^* \text{ and } v \ \lhd^M \ u^*$$

for some $\lhd^M \in \mathcal{M}$. By the remark above, we have that

$$h(u)\ h(\lhd^M)\ h(v^*) \text{ and } h(v)\ h(\lhd^M)\ h(u^*).$$

By [31]), we know that

$$h(v^*) \ \leq \ [h(v)]^*.$$

So we find that

$$h(u)\ h(\lhd^M)\ h(v^*)\ \leq\ [h(v)]^* \text{ and } h(v)\ h(\lhd^M)\ h(u^*)\ \leq\ [h(u)]^*.$$

By a familiar property, we must have

$$h(u)\ h(\lhd^M)\ [h(u)]^* \text{ and } h(v)\ h(\lhd^M)\ [h(v)]^*,$$

which shows that $h(u)$ and $h(v)$ are separated relative to $h(\lhd^M)$. □

In [32], Sieber and Pervin showed that analogous to the topological setting, $\mathcal{L}$-connectedness is related to a constant continuous function on a discrete space. We follow the approach by Baboolal and Banaschewski in [3] and show that $\mathcal{L}$-connectedness is also related to a certain factorisation.

Theorem 4.2.5. *A frame L is connected if and only if each homomorphism $h : \boldsymbol{4} \to L$ factors through the unique homomorphism $\bar{h} : \boldsymbol{2} \to L$.*

Proof:

Let L be connected. Suppose x and y are the non-zero element of $\mathbf{4}$. The connectedness of L implies that $h(x) \vee h(y) = e$ and $h(x) \wedge h(y) = 0$ for any frame homomorphism $h : \mathbf{4} \to L$. It then follows that $h(x) = e$ or $h(y) = e$ and we can define $\bar{h} : \mathbf{4} \to \mathbf{2}$ by $\bar{h}(x) = 1$ or $\bar{h}(y) = 0$ and this shows that h factors through the homomorphism $\bar{h} : \mathbf{2} \to L$.

Conversely. Suppose that $h : 4 \to L$ factors through a unique homomorphism $h : 2 \to L$. Take any $s, t \in L$ such that $s \vee t = e$ and $s \wedge t = 0$. We get $h : \mathbf{4} \to L$ by letting $h(w) = s$ and $h(z) = t$. The fact that h factors trough $\bar{h}$, we have $s = e$ or $t = e$. This completes the proof. □

Corollary in [32] can be cast into Cs*á*sz*á*r frames as follows, see also [11] in Lemma 3.1.

Theorem 4.2.6. *For any dense frame homomorphism $h : (M, \mathcal{M}) \to (L, \mathcal{L})$, if L is $\mathcal{L}$-connected, then so is M.*

Proof:

We assume that $h : (M, \mathcal{M}) \to (L, \mathcal{L})$ is dense onto frame homomorphism with $h(u)$ and $h(v)$ $\mathcal{L}-$separated for some $u, v \in M$. We claim that

$$u \ \lhd^M \ v^* \text{ and } v \ \lhd^M \ u^*.$$

By definition we have

$$h(u) \ \lhd^L \ h(v^*) \text{ and } h(v) \ \lhd^L \ h(u^*).$$

Let us pick $a, b \in M$ satisfying

$$h(u) \ \leq \ h(a), \ a \ \lhd^M \ b \text{ and } h(b) \ \leq \ [h(v)]^*.$$

We set $s = h(a)$ and $t = h(b)$. Similarly, for $v \ \lhd^M \ u^*$, take $x, y \in M$ with

$$h(v) \ \leq \ h(x), \ x \ \lhd^M \ y \text{ and } h(y) \ \leq \ [h(u)]^*.$$

By definition of $h(\lhd^M)$, we have that $s \ h(\lhd^M) \ t$. Since $h(u) \leq s$ is clear, we remain to show that $h_*(t) \leq v^*$. Since h is onto, we must have that

$$\begin{aligned} h[v \wedge h_*(t)] &= h(v) \wedge h \circ h_*(t) \\ &= h(v) \wedge t \\ &\leq h(v) \wedge [h(v)]^* \\ &= 0, \end{aligned}$$

thus $h[v \wedge h_*(t)] = 0$. But h is dense, so we must have

$$v \wedge h_*(t) \ = \ 0$$

from which it follows that $h_*(t) \ \leq \ v^*$. This completes the proof that u and v are $\mathcal{M}$-separated. □

Definition 4.2.7 ([3]). A family $\{u_i \in (L, \mathcal{L}) \mid i \in I\}$ is said to be *chained* if $u_i \wedge u_j \neq 0$ for any $i \neq j$.

Proposition 4.2.8 *For any pairwise $\mathcal{L}$-connected family in $(L, \mathcal{L})$, the join $u = \bigvee_I u_i$ is $\mathcal{L}$-connected.*

Proof:

Let $\{u_i \mid i \in I\}$ be a pairwise $\mathcal{L}$-connected family in $(L, \mathcal{L})$. This implies that u_i is $\mathcal{L}$-connected for each $i \in I$. Take $a, b \in L$ with $\bigvee u_i = a \vee b$ for $i \in I$, where a and b are $\mathcal{L}$-separated. Then

$$\begin{aligned} \bigvee u_i &= \bigvee (a_i \wedge b_i) \\ &= \bigvee (0) \\ &= 0. \end{aligned}$$

Therefore $u = \bigvee u_i$ is $\mathcal{L}$-connected. $\square$

Bibliography

[1] H. A. Abujabal and S. Zaidi. On characterization of reflection and coreflection in categories. *Proyecciones (Antofagasta, On line)*, 18(2):175–182, 1999.

[2] G. R. Apfel. Strict extensions in pointfree topology. Master's thesis, University of Cape Town, 2013.

[3] D. Baboolal. Local connectedness made uniform. In *Papers in Honour of Bernhard Banaschewski*, pages 377–390. Springer, 2000.

[4] D. Baboolal and B. Banaschewski. Compactification and local connectedness of frames. *Journal of Pure and Applied Algebra*, 70(1-2):3–16, 1991.

[5] B. Banaschewski. Compactification of frames. *Mathematische Nachrichten*, 149(1): 105–115, 1990.

[6] B. Banaschewski. Uniform completion in pointfree topology. In *Topological and Algebraic Structures in Fuzzy Sets*, pages 19–56. Springer, 2003.

[7] B. Banaschewski and C. Mulvey. Stone-čech compactification of locales ii. *Journal of pure and applied Algebra*, 33(2):107–122, 1984.

[8] B. Banaschewski and A. Pultr. Cauchy points of uniform and nearness frames. *Quaestiones Mathematicae*, 19(1-2):101–127, 1996.

[9] G. Bezhanishvili and J. Harding. Proximity frames and regularization. *Applied Categorical Structures*, 22(1):43–78, 2014.

[10] G. Bezhanishvili, N. Bezhanishvili, and J. Harding. Modal operators on compact regular frames and de vries algebras. *Applied Categorical Structures*, 23(3):365–379, 2015.

[11] X. Chen. On the local connectedness of frames. *Journal of pure and applied algebra*, 79 (1):35–43, 1992.

[12] S. H. Chung. Cauchy completion of császár frames. *J. Korean Math. Soc*, 42(2):291–304, 2005.

[13] S. H. Chung. On homomorphisms on csaszar frames. *Communications of the Korean Mathematical Society*, 23(3):453–459, 2008.

[14] A. Császár and K. Császár. *Foundations of general topology*, volume 35. Pergamon, 1963.

[15] C. Dowker. Mappings of proximity structures. *General Topology and its Relations to Modern Analysis and Algebra*, pages 139–141, 1962.

[16] T. Dube, M. Mugochi, and I. Naidoo. Čech-completeness in pointfree topology. *Quaestiones Mathematicae*, 37(1):49–65, 2014.

[17] C. L. Flax. Syntopogenous structures and real-compactness. Master's thesis, University of Cape Town, 1972.

[18] J. L. Frith. *Structured frames.* PhD thesis, University of Cape Town, 1986.

[19] H. Herrlich and G. E. Strecker. *Category theory: an introduction*, volume 1. Heldermann, 1979.

[20] S. S. Hong. Convergence in frames. *Kyungpook Mathematical Journal*, 35(1):85–91, 1995.

[21] S. S. Hong and Y. K. Kim. Cauchy completions of nearness frames. *Applied Categorical Structures*, 3(4):371–377, 1995.

[22] J. R. Isbell. *Uniform spaces.* Number 12. American Mathematical Soc., 1964.

[23] J. R. Isbell. Atomless parts of spaces. *Mathematica Scandinavica*, 31(1):5–32, 1973.

[24] P. T. Johnstone et al. The point of pointless topology. *Bulletin (New Series) of the American Mathematical Society*, 8(1):41–53, 1983.

[25] A. Katsaras and C. Petalas. On fuzzy syntopogenous structures. *Journal of mathematical analysis and applications*, 99(1):219–236, 1984.

[26] M. Muraleetharan. Generalisations of filters and uniform spaces. Master's thesis, Rhodes University, 1997.

[27] U. G. Murugan. Completion of uniform and metric frames. Master's thesis, 1996.

[28] L. D. Nel. Initially structured categories and cartesian closedness. *Canadian Journal of Mathematics*, 27(6):1361–1377, 1975.

[29] J. Picado and A. Pultr. Cover quasi-uniformities in frames. *Topology and its Applications*, 158(7):869–881, 2011.

[30] J. Picado and A. Pultr. Frames and locales: topology without points. vol. 28. *Basel: Birkhäuser. DOI*, 10:978–3, 2012.

[31] A. Pultr. Frames. In *Handbook of algebra*, volume 3, pages 791–857. Elsevier, 2003.

[32] J. Sieber and W. Pervin. Connectedness in syntopogenous spaces. *Proceedings of the American Mathematical Society*, 15(4):590–595, 1964.