BALANCED IDEALS IN COZERO PARTS OF FRAMES

by

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Declaration

I declare that the dissertation hereby submitted to the University of Limpopo, for the degree of Master of Science in Mathematics has not previously been submitted by me for a degree at this or any other university; that it is my work in design and in execution, and that all materials contained herein has been duly acknowledged.

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List of notation

We use the following standard notation throughout the dissertation.

$\wp(X)$ - the power set of X

$\emptyset$ - set with no elements

$\mathbb{N}$ - set of all positive integers

$\mathbb{Z}$ - set of all integers

$\mathcal{R}L$ - ring of continuous functions frame L

$\mathrm{Cov}(L)$ - cover of L

$\mathrm{FCov}(L)$ - finite cover of L

RegFrm - regular frames

NFrm - nearness frames

βX - Stone-Čech compactification of X

$\mathrm{Coz}\, L$ - cozero parts of L

Abstract

We study balanced filters and balanced z-filters considered by Carlson in [20] and [21] in topological spaces. We consider closed filters which are open-generated and open filters which are closed-generated. We show that a closed filter is open-generated precisely if it is a minimal balanced closed filter and that an open filter is closed-generated precisely when it is a minimal balanced open filter. For a completely regular topological space X, we study balanced z-filters and show that there is a one-to-one correspondence between the nonempty closed sets of βX and the balanced z-filter on X. By dualising closed filters we obtain ideals which then enables us to put some of the results in the context of frames. Dube in [28] has shown that a frame is normal if and only if its closed-generated filters are precisely the stably closed-generated ones. By dualisation we show that a frame is extremally disconnected if and only if its open-generated ideals are precisely the stably open-generated ones. We show that there is one-to-one correspondence between points of βL and the balanced ideals of Coz L. Furthermore we study nearness frames and show that the locally finite nearness frames strictly contain the Pervin nearness frames and the two coincide if the locally finite nearness frames are totally bounded. For perfect extension $h : M \to L$ of L, we show that a point p of M is a remote point if and only if $I_p = \{a \in L \mid h_*(a) \leq p\}$.

Keywords: filters, balanced filters, closed-generated filters, remote points, frames, nearness frames, balanced ideals, open-generated ideals.

Contents

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Chapter 1

Introduction

1.1 A brief history on filters

A filter is a special subset of a partially ordered set. For example, the power set of some set, partially ordered by set inclusion, is a filter. Filters appear in order and lattice theory, but can also be found in topology from where they originate. The dual notion of a filter is an ideal. Filters were introduced by Henri Cartan in 1937 and subsequently used by Bourbaki in their book Topologie Generale as an alternative to the similar notion of a net developed in 1922 by E.H. Moore and H.L. Smith.

Filters and nets were introduced as generalisation of sequences in topological spaces. In literature we encounter filters more than nets. Balanced filters were considered by J.W. Carlson [21] in 1984. Carlson showed in the same article that there is a one-to-one correspondence between the nonempty closed sets in the Stone-Čech compactification βX and the balanced closed filters in X. Filters on $\mathbb{N}$ serve as examples of balanced closed filters (see for instance, Lemma 6.1 in [20]). As a special case Carlson [20] showed that there exists a one-to-one correspondence between the nonempty closed subsets of $\beta\mathbb{N}$ and the filters on $\mathbb{N}$. It is Carlson who introduced the notion of balanced z-filters in classical topology in 1985. Carlson [21] further showed that there is a one-to-one correspondence between the nonempty closed subsets of the Stone-Čech compactification βX and the balanced z-filters. In this dissertation we study the dual notion of balanced z-filters in point-free setting known as balanced ideals in cozero parts of frames.

In [28], Dube introduced the notion of balanced filters in frames and showed that the balanced filters are precisely the ones containing dense elements. It had been shown in Corollary 5.5 in [18] that there is a one-to-one correspondence between the nonempty closed sets in βX and the balanced closed filters in X. In [20], Carlson had shown as a special case that there exists a one-to-one correspondence between the nonempty closed subsets of $\beta\mathbb{N}$ and the filters on $\mathbb{N}$. Filters on $\mathbb{N}$ are balanced closed filters (see for instance, Lemma 6.1 in [20]); where $\mathbb{N}$ is the set of natural numbers and $\beta\mathbb{N}$ is the Stone-Čech compactification of the set of natural numbers.

1.2 Synopsis of the dissertation

We study the notion of balanced ideals of cozero parts of completely regular frames. We show that there is a one-to-one correspondence between the points of a Stone-Čech compactification of a frame and the balanced ideals of its cozero part. We study balanced filters that correspond to remote points and those that correspond to p-points.

Chapter 1 is essentially introductory. Here we present the relevant definitions, pertaining to frames and outline the relevant background for the other chapters.

In chapter 2, we show that closed filters which are open-generated are precisely the minimal balanced closed ones, and the open filters which are closed-generated are precisely the minimal balanced open ones. We show that there is one-to-one correspondence between the closed sets of βX and balanced z-filters on X. We also give some brief background on nearness spaces.

In chapter 3, we present some of the results of chapter 2 in the pointfree setting. Some results have been presented in the pointfree context by Dube in [28]. Here we dualise some of his results, for instance, Dube [28] has shown that a frame is normal if and only if its closed-generated filters are precisely the stably closed-generated ones. In dualising Dube's statement we show that a frame is extremally disconnected if and only if its open-generated ideals are precisely the stably open-generated ones. We show that there is a one-to-one correspondence between the points of βL and the balanced ideals of Coz L. Furthermore, we study locally finite nearness frames, Pervin nearness frames and finite nearness frames and show that the Pervin nearness frames are contained in locally finite nearness frames and locally finite nearness are in turn contained in fine nearness frames.

In chapter 4, we study remote points and show that if $h : M \to L$ is a perfect extension and $p \in Pt(M)$, then p is a remote point if and only if I_p is a balanced ideal of L, where $I_p = \{a \in L \mid h_*(a) \leq p\}$. We show also that for any $I \in \beta L$, the closed quotient $\beta L \to \uparrow I$ is round if and only if there is only one ideal J of Coz L such that $I = \bigvee\{r(a) \mid a \in J\}$. We end this chapter by defining z_∞-ideals and nicely balanced ideals of $\mathcal{R}_\infty(L)$ which is the frame analogue of z_∞-ideals and nicely balanced ideals of $C_\infty(X)$ defined by Ghosh in [37].

1.3 Preliminaries

1.3.1 Partially ordered set

Let A be a nonempty set. A binary relation on A is a subset R of the cartesian product $A \times A$. We write $(a, b) \in R$ as $a \backsim b$. A binary relation in lattices and ordered sets is said to be:

(1) Reflexive if for all $a \in A, a \sim a$,

(2) Antisymmetric if for all $a,\ b \in A$,

$$a \backsim b, b \backsim a \Rightarrow a = b$$

(3) Transitive if for all $a, b, c \in A$,

$$a \backsim b, b \backsim c \Rightarrow a \backsim c$$

A partial order on a nonempty set P is a binary relation $\leq$ on P that is reflexive, antisymmetric and transitive, specifically, for $x, y, z \in P$,

(i) Reflexive

$$x \leq x,$$

(ii) Antisymmetry

$$x \leq y, y \leq x \Rightarrow x = y,$$

(iii) Transitivity

$$x \leq y, y \leq z \Rightarrow x \leq z.$$

The pair $(P, \leq)$ is called *a partially ordered set or poset*, although it is often said that P is a poset, when the order relation is understood. If $x \leq y$, we say that x is less than or equal to y or y is greater than or equal to x. We also say that x is contained in y or that y contains x. If $x \leq y$ but $x \neq y$, we write $x < y$ or x and y are incomparable or parallel, denoted by $a \parallel b$.

If S and T are subsets of a poset P, then $S \leq T$ means that $s \leq t$ for all $s \in S, t \in T$. If $T = \{t\}$, then $S \leq \{t\}$ is written $S \leq t$ and similarly for $s \leq T$.
If X is a nonempty set, then the power set $\wp(X)$ of X is the set of all subsets of X. It is well known that $\wp(X)$ is a poset under set inclusion.

1.3.2 Maximal and minimal elements

Maximal and minimal elements can be defined in posets.

Let $(P, \leq)$ be a partially ordered set.

(1) A maximal element is an element $m \in P$ that is not contained in any other element of P, that is,

$$p \in P, m \leq p \Rightarrow m = p.$$

A maximal (largest or greatest) element m in P is an element that contains every element of P, that is,

$$p \in P \Rightarrow p \leq m.$$

We will generally denote the largest element by 1 and call it the unit element.

(2) A minimal element is an element $n \in P$ that does not contain any other element of P, that is,

$$p \in P, p \leq n \Rightarrow p = n.$$

A minimum (smallest or least) element n in P is an element contained in all other elements of P, that is,

$$p \in P \Rightarrow n \leq p.$$

We will generally denote the smallest element by 0 and call it the zero element. A partially ordered set is bounded if it has both a 0 and a 1.

If a poset P has a smallest element 0, then any cover of 0 is called an atom or point of P. The set of all atoms of a poset P is denoted by $\mathcal{A}(P)$. A poset with 0 is atomic if every nonzero element contains an atom. If P has a 1, then any element covered by 1 is called a coatom or copoint of P.

1.3.3 Upper and lower bounds

Upper and lower bounds can be defined in a poset.

Let $(P, \leq)$ be a partially ordered set and let $S \subseteq P$.

(1) An upper bound for S is an element $x \in P$ for which

$$S \leq x.$$

The set of all upper bounds for S is denoted by S^u. We abbreviate $\{s\}^u$ by s^u. If S^u has a least element, it is called the join or least upper bound or supremum of S and is denoted by $\bigvee S$. The join of a finite set $S = \{a_1, ..., a_n\}$ is denoted by

$$a_1 \vee ... \vee a_n.$$

(2) A lower bound for S is denoted by S^ℓ. We abbreviate $\{s\}^\ell$ by s^ℓ. If S^ℓ has a greatest element, it is called the meet or greatest lower bound or infimum of S and is denoted by $\bigwedge S$. The meet of a finite set $S = \{a_1, ..., a_n\}$ is denoted by

$$a_1 \wedge ... \wedge a_n.$$

1.3.4 Lattices

Let P be a poset. Then P is said to be a *lattice* if every pair of elements of P has a meet and a join. P is said to be a *complete lattice* if P is closed under arbitrary meets and arbitrary joins.

Examples of Lattices

(1) Any totally ordered set is a lattice, but not necessarily a complete lattice. For example, the set $\mathbb{Z}$ of integers under the natural order is a lattice, but not a complete lattice.

(2) If S is a nonempty set, then the power set $\wp(S)$ is a complete lattice under the usual inclusion ordering.

A lattice L is distributive if it satisfies the distributive laws: For all $a, b, c \in L$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Theorem 1.3.1. [44] If either of the distributive laws holds for all elements of a lattice L, then so does the other.

Proof. Suppose that the first distributive law holds. Then applying it to the right side of the second distributive law and using absorption gives

$$\begin{aligned}(a \vee b) \wedge (a \vee c) &= [(a \vee b) \wedge a] \vee [(a \vee b) \wedge c] \\ &= a \vee [(a \vee b) \wedge c] \\ &= a \vee [(a \wedge c) \vee (b \wedge c)] \\ &= a \vee (b \wedge c)\end{aligned}$$

which shows that the second law holds. □

1.3.5 Frames

A *frame* is a complete lattice L in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

holds for all $a \in L$ and $S \subseteq L$.

Examples of frames

(1) All finite distributive lattices are frames.

(2) All complete Boolean algebras are frames.

(3) For any topological space X, the collection of open subsets of $X, \mathcal{D}X$, is a frame, where for any arbitrary subset $\{V_i\}_{i\in\triangle} \subseteq \mathcal{D}X$

$$\bigwedge_{i\in\triangle} V_i = \text{int}(\bigcap_{i\in\triangle} V_i)$$

with int being the topological interior operator and

$$\bigvee V_i = \bigcup V_i.$$

We call $\mathcal{D}X$ the frame of open sets of X.

(4) Every complete chain is a frame.

A *frame homomorphism* is a map $h : L \to M$ between frames which preserves finite meets, including the top element, and arbitrarily joins, including the bottom element. Associated with any frame homomorphism $h : L \to M$ is its right adjoint $h_* : M \to L$ given by

$$h_*(b) = \bigvee\{a \in L \mid h(a) \leq b\}.$$

The *pseudocomplement* of an element x of L is the element

$$x^* = \bigvee\{y \in L \mid x \wedge y = 0\}.$$

It is the largest element that misses x. In general, $x \leq x^{**}$ and x is a regular element in case $x = x^{**}$. We say x is:

(a) complemented if $x \vee x^* = 1$,

(b) dense if $x^* = 0$,

(c) a point (or prime) if $x \neq 1$ and $a \wedge b \leqq x$ implies $a \leqq x$ or $b \leqq x$.

Let L be a frame. We call $D \subseteq L$ a *downset* if $x \in D$ and $y \leq x$ implies $y \in D$, and $U \subseteq L$ an *upset* if $u \in U$ and $u \leq v$ implies $v \in U$. For any $a \in L$, we write

$$\downarrow a = \{x \in L \mid x \leq a\},$$

which is a downset, and

$$\uparrow a = \{x \in L \mid a \leq x\},$$

which is an upset. We note that $\downarrow a$ is a frame whose bottom element is $0 \in L$ and top element a. Similarly, $\uparrow a$ has $1 \in L$ as the top element and a as its bottom element. These frames are in fact the quotients of L via the maps $L \to \uparrow a$ and $L \to \downarrow a$, given respectively by $x \mapsto a \vee x$ and $x \mapsto a \wedge x$. These quotient are known as the *closed quotients* and *open quotients*, respectively.

An element a is said to be *rather below* an element b denoted by $a \prec b$ if there is a *separating* element $c \in L$ such that $c \wedge a = 0$ and $c \vee b = 1$.

An element a is said to be *completely below* an element b denoted $a \prec\prec b$ if there is a scale $((c_q) \mid q \in Q \cap [0,1])$ such that $c_0 = a,\ c_1 = b$ and $c_r \prec c_s$ whenever $r \leq s$.

A frame L is said to be *regular* if for every $a \in L$,

$$a = \bigvee\{x \in L \mid x \prec a\}.$$

A frame L is said to be *completely regular* if for every $a \in L$,

$$a = \bigvee\{x \in L \mid x \prec\prec a\}.$$

1.3.6 Cozero parts of frames

A *cozero element of* L is an element of the form $\mathrm{coz}\,\psi$ for some $\psi \in \mathcal{R}L$, where $\mathcal{R}L$ is the ring of continuous functions on the frame L. It is shown in [8] that $a \in L$ is a cozero element if and only if there exists a sequence (a_n) such that $a_n \prec\prec a$ for each n and $a = \bigvee a_n$. The cozero part of L, denoted by $\mathrm{Coz}\,L$, is the regular sub-σ-frame consisting of all the cozero elements of L. We refer to [8] for general properties of cozero elements and cozero parts of frames. A frame homomorphism $h : L \to M$ is *coz-onto* if for every $c \in \mathrm{Coz}\,M$ there exists $d \in \mathrm{Coz}\,L$ such that $h(d) = c$. It is almost *coz-codense* if for $c \in \mathrm{Coz}\,L$ such that $h(c) = 1$, there exists $d \in \mathrm{Coz}\,L$ such that $h(d) = 0$ and $c \vee d = 1$.

A frame is completely regular if and only if it is generated by its cozero part. General properties of cozero elements and cozero parts of frames can be found in [8], [9] and [11]. Here we highlight the following:

(a) If $a \in \mathrm{Coz}\, L$; there is a sequence (c_n) in $\mathrm{Coz}\, L$ such that $c_n \prec\!\!\prec c_{n+1}$ for each n, and $a = \bigvee c_n$.

(b) If $a \prec\!\!\prec b$, there is a cozero element c such that $a \prec\!\!\prec c \prec\!\!\prec b$.

(c) If $a \prec\!\!\prec b$, there is a cozero element c such that $a \wedge c = 0$ and $c \vee b = 1$.

(d) If $c, d \in \mathrm{Coz}\, L$ and $c \prec d$ in $\mathrm{Coz}\, L$, then $c \prec\!\!\prec d$.

The properties of the *cozero map* coz: $\mathcal{R}L \to L$, given by

$$\mathrm{coz}\, \varphi = \bigvee \{\varphi(p, 0) \vee \varphi(0, q) \mid p, q \in \mathbb{Q}\} = \varphi((-, 0) \vee (0, -)),$$

that we shall frequently use are:

(a) $\mathrm{coz}\, \gamma\delta = \mathrm{coz}\, \gamma \wedge \mathrm{coz}\, \delta$,

(b) $\mathrm{coz}\, (\gamma + \delta) \leqq \mathrm{coz}\, \gamma \vee \mathrm{coz}\, \delta$,

(c) $\mathrm{coz}\, (\gamma + \delta) = \mathrm{coz}\, \gamma \vee \mathrm{coz}\, \delta, \text{if } \gamma, \delta \geqq 0$,

(d) $\mathrm{coz}\, \delta = 0$ if and only if $\delta = 0$,

(e) φ is invertible if and only if $\mathrm{coz}\, \varphi = 1$.

1.3.7 The Stone-Čech compactification

A compactification $h : M \to L$ of L is large if whenever $h(a) = 1$, then $M \twoheadrightarrow \downarrow a$ is the Stone-Čech compactification of $\downarrow a$.

Note that if h is as in the definition and $h(a) = 1$, then $M \twoheadrightarrow \downarrow a$ is indeed a compactification, for if $a \wedge x = 1$, then $h(x) \wedge h(a) = 1$, implying that $h(x) = 1$, and hence $x = 1$ by codensity of h. Thus, $M \twoheadrightarrow \downarrow a$ is a dense onto homomorphism with compact domain. Consequently, in light of [2, Corollary 8.2.7], $h : M \twoheadrightarrow L$ is a large compactification of L if and only if $M \twoheadrightarrow \downarrow a$ is a C^*-quotient map for every $a \in M$ with $h(a) = 1$.

Not every compactification is large.

Example 1.3.1. Let X be a locally compact space which is not pseudocompact, and let K be its one-point compactification. Then K is not a large compactification of X. If it were, then, in light of X being open in K, K would be the Stone-Čech compactification of X, which would mean that K is the only compactification of X, and hence X would be pseudocompact.

Every frame has a large compactification, namely, its Stone-Čech compactification. This will be apparent after proving the following lemma. In the proof we use the well-known fact that if $g : L \twoheadrightarrow M$ is a dense onto frame homomorphism and $a \prec\!\!\prec b$ in L, then $g_*g(a) \leq b$. Indeed, the ontoness of g implies that $g(g_*g(a) \wedge a^*) = 1$, hence $g_*(a) \wedge a^* = 1$ by denseness. Hence $g_*g(a) \leq a^{**} \leq b$.

Lemma 1.3.1. [32] Suppose $\beta L \twoheadrightarrow L$ factorizes as $\beta L \xrightarrow{h} M \xrightarrow{g} L$ with h onto. Then $\beta L \xrightarrow{h} M$ is a compactification isomorphic to $\beta M \twoheadrightarrow M$.

Proof. The homomorphism h is easily checked to be dense onto, so that $\beta L \xrightarrow{h} M$ is indeed a compactification. To prove the latter assertion, it suffices, by [2, Corollary 8.2.7], to show that h is a C^* - quotient map. We apply [2, Theorem 8.2.6]. So let a and b be cozero elements of M such that $a \vee b = 1$. Find cozero elements of L such that $g(u) \vee g(v) = 1$, and so $rg(u) \vee rg(v) = 1$. But $r = h_*g_*$, so $h_*(g_*g(u)) \vee h_*(g_*g(v)) = 1$. A straight forward calculation shows that g is dense onto. Consequently, $g_*g(u) \leq a$ and $g_*g(v) \leq b$, and hence $h_*(a) \vee h_*(b) = 1$, as desired. □

Now consider the Stone-Čech compactification of L, and let $I \in \beta L$ with $\bigvee I = 1$. Then $\beta L \twoheadrightarrow L$ factorises as $\beta L \twoheadrightarrow \downarrow I \longrightarrow L$, where the first map is the quotient map $J \mapsto J \wedge I$, and the second is the join map. Therefore, by the last lemma, $\beta L \twoheadrightarrow \downarrow I$ is isomorphic to $\beta(\downarrow I) \twoheadrightarrow \downarrow I$. Consequently, $\beta L \twoheadrightarrow L$ is a large compactification of L.

An ideal J of L is *completely regular* if for each $x \in J$ there exists $y \in J$ such that $x \prec\!\!\prec y$. For a completely regular frame L, the frame of its completely regular ideals is denoted by βL. The join map $\beta L \to L$ is dense onto, and βL (together with the join map) is referred to as the Stone-Čech compactification of L. We denote the right adjoint of the join map $\beta L \to L$ by r (using subscripts if there is more than one frame under consideration), and recall (from [4], for instance) that for any $a \in L$ and $I \in \beta L$:

(a) $r(a) = \{x \in L \mid x \prec\!\prec a\}$,

(b) $r(a^*) = r(a)^*$,

(c) $I^* = r((\bigvee I)^*)$,

(d) $r(a) \prec I$ if and only if $a \in I$,

(e) r *preserves* $\prec\!\prec$.

Lemma 1.3.2. For $a, b \in \mathrm{Coz}\, L$, we have $r_L(a \vee b) = r_L(a) \vee r_L(b)$.

Proof. We have

$$\begin{aligned} r_L(a) \vee r_L(b) &= \bigvee \{I_1 \in \beta L \mid j(I_1) \leq a\} \vee \bigvee \{I_2 \in \beta L \mid j(I_2) \leq b\} \\ &= \bigvee \bigvee \{I_1 \vee I_2 \in \beta L \mid j(I_1 \vee I_2) \leq a \vee b\} \\ &\geq \bigvee \{I_1 \vee I_2 \in \beta L \mid j(I_1 \vee I_2) \leq a \vee b\} \\ &= \bigvee \{J \in \beta L \mid j(J) \leq a \vee b\} \text{ where } J = I_1 \vee I_2 \\ &= r_L(a \vee b) \end{aligned}$$

Thus $r_L(a) \vee r_L(b) \geq r_L(a \vee b)$.

On the other hand, we have $a \leq a \vee b$ and $b \leq a \vee b$. So that $r_L(a) \leq r_L(a \vee b)$ and $r_L(b) \leq r_L(a \vee b)$. Therefore

$$r_L(a) \vee r_L(b) \leq r_L(a \vee b) \vee r_L(a \vee b) = r_L(a \vee b).$$

Hence $r_L(a \vee b) = r_L(a) \vee r_L(b)$. □

1.3.8 Nearness frames

In this section we lay out the necessary terminology for these structured frames. Let L be a frame and $A, B \in \mathrm{Cov}(L)$. We say A *refines* B and write $A \leq B$ if for every $a \in A$, there exists $b \in B$ such that $a \leq b$. We write $F\mathrm{Cov}(L)$ for the collection of all covers of L refined by some finite cover.

The *star* of $x \in L$ with respect to a cover A of L is the element

$$Ax = \bigvee \{a \in A \mid a \wedge x \neq 0\}.$$

Further, we write $AB = \{Ax \mid x \in B\}$, which is a cover of L if A and B are covers. We say A *star-refines* B, written $A \leq^* B$, if $AA \leq B$.

Given a collection $\mu \subseteq \mathrm{Cov}(L)$, we say $x \in L$ is $\mu -$ *strongly below* $y \in L$, written $x \lhd_\mu y$ (or simply $x \lhd y$) if there is a cover $A \in \mu$ such that $Ax \leq y$. We shall frequently use the following properties of the relation $\lhd$.

(1) If $x \lhd y, a \leq x$ and $y \leq b$, then $a \lhd b$.

(2) If $x \lhd y$ and $a \lhd b$, then $x \wedge a \lhd y \wedge b$ and $x \vee a \lhd y \vee b$.

(3) If μ is a uniformity, then $x \lhd y$ implies $x \lhd z \lhd y$ for some $z \in L$.

We may now state the definition of nearness frames.

Definition 1.3.1. A nonempty collection $\mu \in \mathrm{Cov}(L)$ is called a *nearness* on L if the following hold:

(n1) Whenever $A \in \mu$ refines $B \in \mathrm{Cov}\ (L)$, then $B \in \mu$.

(n2) Whenever $A, B \in \mu$, then $A \wedge B \in \mu$.

(n3) Every $x \in L$ can be expressed as

$$x = \bigvee \{y \in L \mid y \lhd_\mu x\}.$$

This property is referred to as the admissibility property.

In the case where μ is a nearness on L, we refer to $\lhd_\mu$ as the uniformly below relation on L, often times dropping the index and simply writing $\lhd$ when the nearness on L is understood. The pair (L, μ) is called a nearness frame, and members of μ are called uniform covers.

A map $h : (L, \mu) \to (M, \eta)$ between nearness frames is called a *uniform frame homomorphism* if it is frame homomorphism and for every $A \in \mu, h[A] \in \eta$.

Chapter 2

Filters in topological spaces

The notion of a filter was introduced by H. Cartan [24] in 1937. Although recently Ashaea and Yousif [1] indicated that the notion of a filter was first encountered by Riesz [51]. Filters and nets play a vital role in describing topological properties in more abstract spaces. Sequences were found to be only sufficient to describe topological properties in metric spaces and topological spaces having a countable base for the topology. Because nets are not user friendly to work with many authors preferred filters. A good survey of filters can be found in Bourbaki [16] and [25]. Carlson in [20] and [21] studied balanced filters and balanced z-filters. In this chapter we are following Carlson [20] and [21] in studying balanced filters and balanced z-filters.

2.1 Open and closed filters

Definition 2.1.1. A *filter* $\mathcal{F}$ is a nonempty collection of subsets of a topological space X satisfying

(i) $\emptyset \notin \mathcal{F}$.

(ii) $A \in \mathcal{F}$ and $A \subseteq B \Rightarrow B \in \mathcal{F}$.

(iii) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$.

If the collection in $\mathcal{F}$ consists entirely of open sets, then $\mathcal{F}$ is called an *open filter*. It is called a *closed filter* if it consists entirely of closed sets. A filter $\mathcal{F}$ is said to be a *prime* *filter* if it is a filter and satisfies the following

$$A \cup B \in \mathcal{F} \Rightarrow A \in \mathcal{F} \text{ or } B \in \mathcal{F}.$$

Furthermore, it is a prime open (closed) filter if it is an open (closed) filter.
Recall that a topological space (X, τ) is said to be a T_1-space if for any two distinct points $x \neq y$ in X, there are open sets U and V such that $x \in U$ and $y \in V$.

The following definition will be useful in the sequel.

Definition 2.1.2. Let (X, τ) be a T_1-space. Let $\mathcal{C}$ denote the collection of all closed subsets in X. Let $\alpha \subseteq \wp(X)$. Set

(1) $\mathcal{F}(\alpha) = \{F \mid F \text{ is closed and } X \smallsetminus F \notin \alpha\}$.

(2) $\mathcal{O}(\alpha) = \{O \mid O \text{ is open and } X \smallsetminus O \notin \alpha\}$.

(3) $\mathcal{G}(\alpha) = \{F \mid F \text{ is closed and there exists } A \in \alpha \text{ with } A \subseteq F\}$.

(4) $\mathcal{S}(\alpha) = \{O \mid O \text{ is open and there exists } A \in \alpha \text{ with } A \subseteq O\}$.

(5) $\sec(\alpha, \tau) = \{O \in \tau \mid O \cap A \neq \emptyset \text{ for each } A \in \alpha\}$.

(6) $\sec(\alpha, \mathcal{C}) = \{F \in \mathcal{C} \mid F \cap A \neq \emptyset \text{ for each } A \in \alpha\}$.

Now we are ready for the following useful lemma, it is culled in [21] and here we include the proof.

Lemma 2.1.1. [21] Let $\mathcal{U}$ be a nonempty collection of open sets and $\mathcal{V}$ a nonempty collection of closed sets in a T_1- topological space X.

(1) $\sec(\mathcal{U}, \mathcal{C}) \subseteq \mathcal{F}(\mathcal{U})$.

(2) $\sec(\mathcal{V}, \tau) \subseteq \mathcal{O}(\mathcal{V})$.

(3) $\sec(\mathcal{U}, \tau) = \mathcal{O}(\mathcal{G}(\mathcal{U}))$.

(4) $\sec(\mathcal{V}, \mathcal{C}) = \mathcal{F}(\mathcal{S}(\mathcal{V}))$.

Proof. (1) Let $B \in \sec(\mathcal{U}, \mathcal{C})$. Then, by definition, $B \in \mathcal{C}$ and $B \cap A \neq \emptyset$ for each $A \in \mathcal{U}$. Hence B is closed. It remains to show that $X \setminus B \notin \mathcal{U}$. Suppose on the contrary that $X \setminus B \in \mathcal{U}$. Since $X \setminus B$ is open and B intersects all the open sets in $\mathcal{U}$, it follows that $(X \setminus B) \cap B \neq \emptyset$ which is a contradiction. Therefore $X \setminus B \notin \mathcal{U}$. Therefore $B \in \mathcal{F}(\mathcal{U})$ and hence $\sec(\mathcal{U}, \mathcal{C}) \subseteq \mathcal{F}(\mathcal{U})$.

(2) Let $D \in \sec(\mathcal{V}, \tau)$. Then, by definition, $D \in \tau$ and $D \cap A \neq \emptyset$ for each $A \in \mathcal{V}$. Since D is open, it follows that its complement $X \setminus D$ is closed. We show that $X \setminus D \notin \mathcal{V}$. If $X \setminus D \in \mathcal{V}$, then $(X \setminus D) \cap D \neq \emptyset$ because D intersects all closed sets in $\mathcal{V}$ which is a contradiction. Therefore $X \setminus D \notin \mathcal{V}$. Thus $D \in \mathcal{O}(\mathcal{V})$ and hence $\sec(\mathcal{U}, \tau) \subseteq \mathcal{O}(\mathcal{V})$.

(3) Let $B \in \sec(\mathcal{U}, \tau)$. Then $B \in \tau$ and $B \cap A \neq \emptyset$ for each $A \in \mathcal{U}$. Then $X \setminus B$ is closed and $X \setminus B \notin \mathcal{U}$. Now $B \cap A \neq \emptyset$ and is not contained in $X \setminus B$ for all $A \in \mathcal{U}$. This implies that $X \setminus B \notin \mathcal{G}(\mathcal{U})$. B is open and $X \setminus B \notin \mathcal{G}(\mathcal{U})$ and hence $B \in \mathcal{O}(\mathcal{G}(\mathcal{U}))$ so $\sec(\mathcal{U}, \tau) \subseteq \mathcal{O}(\mathcal{G}(\mathcal{U}))$. On the other hand D is open and $X \setminus D \notin \mathcal{G}(\mathcal{U})$. Therefore $X \setminus D$ is a closed set and there is no subset $A \in \mathcal{U}$ such that $A \subseteq (X \setminus D)$. Therefore $D \cap A \neq \emptyset$ for every $A \in \mathcal{U}$. Hence $D \in \sec(\mathcal{U}, \tau)$. So $\mathcal{O}(\mathcal{G}(\mathcal{U})) \subseteq \sec(\mathcal{U}, \tau)$ and hence equality.

(4) Let $D \in \sec(\mathcal{V}, \mathcal{C})$. Then $D \in \mathcal{C}$ and $D \cap A \neq \emptyset$ for each $A \in \mathcal{V}$. Then $X \setminus D$ is open and $X \setminus D \in \mathcal{V}$. Now $D \cap A \neq \emptyset$ for each $A \in \mathcal{V}$ and is not contained in $X \setminus D$ for all $A \in \mathcal{V}$. This implies that $X \setminus D \notin \mathcal{S}(\mathcal{V})$. D is closed and $X \setminus D \notin \mathcal{S}(\mathcal{V})$ and hence $D \in \mathcal{F}(\mathcal{S}(\mathcal{V}))$. So $\sec(\mathcal{V}, \mathcal{C}) \subseteq \mathcal{F}(\mathcal{S}(\mathcal{V}))$. Conversely, let $B \in \mathcal{F}(\mathcal{S}(\mathcal{V}))$. Then B is closed and $X \setminus B \notin \mathcal{S}(\mathcal{V})$. Therefore $X \setminus B$ is a closed set and there is no subset $A \in \mathcal{V}$ such that $A \subseteq (X \setminus B)$. Therefore $B \cap A \neq \emptyset$ for every $A \in \mathcal{V}$. Hence $B \in \sec(\mathcal{V}, \mathcal{C})$. So $\mathcal{F}(\mathcal{S}(\mathcal{V})) \subseteq \sec(\mathcal{V}, \mathcal{C})$ and hence equality. $\square$

Theorem 2.1.1. [21] Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be nonempty collections of open sets, and $\mathcal{F}_1$ and $\mathcal{F}_2$ nonempty collections of closed sets. Then

(1) $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathcal{G}(\mathcal{O}_1) \subseteq \mathcal{G}(\mathcal{O}_2)$.

(2) $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathcal{F}(\mathcal{O}_2) \subseteq \mathcal{F}(\mathcal{O}_1)$.

(3) $\mathcal{F}_1 \subseteq \mathcal{F}_2$ implies $\mathcal{S}(\mathcal{F}_1) \subseteq \mathcal{S}(\mathcal{F}_2)$.

(4) $\mathcal{F}_1 \subseteq \mathcal{F}_2$ implies $\mathcal{O}(\mathcal{F}_2) \subseteq \mathcal{O}(\mathcal{F}_1)$.

Proof. (1) Let $P \in \mathcal{G}(\mathcal{O}_1)$. The set P is closed and there exists $A \in \mathcal{O}_1$ with $A \subseteq P$. Then $A \in \mathcal{O}_2$ because $\mathcal{O}_1 \subseteq \mathcal{O}_2$. Therefore $P \in \mathcal{G}(\mathcal{O}_2)$ and hence $\mathcal{G}(\mathcal{O}_1) \subseteq \mathcal{G}(\mathcal{O}_2)$.

(2) It is immediate from the fact that $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies that $X \setminus \mathcal{O}_2 \subseteq X \setminus \mathcal{O}_1$.

(3) Let $K \in \mathcal{S}(\mathcal{F}_1)$. By definition K is open and there exists $A \in \mathcal{F}_1$ with $A \subseteq K$. Then $A \in \mathcal{F}_2$ because $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Therefore $K \in \mathcal{S}(\mathcal{F}_2)$ and hence $\mathcal{S}(\mathcal{F}_1) \subseteq \mathcal{S}(\mathcal{F}_2)$.

(4) $\mathcal{O}(\mathcal{F}_2) = \{D \mid D \text{ is open and } X \setminus D \notin \mathcal{F}_2\}$. Let $D \in \mathcal{O}(\mathcal{F}_2)$. D is open and $X \setminus D \notin \mathcal{F}_2$. Now $X \setminus D \notin \mathcal{F}_1$, since $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Therefore $D \in \mathcal{O}(\mathcal{F}_1)$ and hence $\mathcal{O}(\mathcal{F}_2) \subseteq \mathcal{O}(\mathcal{F}_1)$. □

Definition 2.1.3. A minimal prime open filter is a prime open filter that is minimal in the collection of prime open filters.

Minimal prime closed filters are defined similarly.

Definition 2.1.4. Let S be a set. A collection of subsets $F \subseteq \wp(S)$ is an ultrafilter on S if for every subset $A \subseteq S$, F contains either A or its complement A^c.

Now we are ready for the following.

Theorem 2.1.2. [21] Let $\mathcal{K}$ be a nonempty collection of closed sets and $\mathcal{P}$ a nonempty collection of open sets.

(1) $\mathcal{K}$ is a prime closed filter if and only if $\mathcal{O}(\mathcal{K})$ is an open prime filter.

(2) $\mathcal{K}$ is a minimal prime closed filter if and only if $\mathcal{O}(\mathcal{K})$ is an open ultrafilter.

(3) $\mathcal{K}$ is a closed ultrafilter if and only if $\mathcal{O}(\mathcal{K})$ is a minimal prime open filter.

(4) $\mathcal{P}$ is a prime open filter if and only if $\mathcal{F}(\mathcal{P})$ is a prime closed filter.

(5) $\mathcal{P}$ is a minimal prime open filter if and only if $\mathcal{F}(\mathcal{P})$ is a closed ultrafilter.

(6) $\mathcal{P}$ is an open ultrafilter if and only if $\mathcal{F}(\mathcal{P})$ is a minimal prime closed filter.

(7) $\mathcal{K} = \mathcal{F}(\mathcal{O}(\mathcal{K}))$.

(8) $\mathcal{P} = \mathcal{O}(\mathcal{F}(\mathcal{P}))$.

Proof. (1) We first show that $\mathcal{O}(\mathcal{K})$ is a filter. Since $\mathcal{K}$ is a prime closed filter and all filters contain X, it follows that $X \in \mathcal{K}$.

(a) $\emptyset \notin \mathcal{O}(\mathcal{K})$, because if it were in $\mathcal{O}(K)$ we would have $X \setminus \emptyset \notin \mathcal{K}$ i.e, $X \notin \mathcal{K}$ which contradict the fact that $\mathcal{K}$ is a filter.

(b) Suppose $A \subseteq B$ and $A \in \mathcal{O}(\mathcal{K})$. We must show that $B \in \mathcal{O}(\mathcal{K})$. If $B \notin \mathcal{O}(\mathcal{K})$, then we would have $X \setminus B \in \mathcal{K}$. But $A \subseteq B$ implies $X \setminus B \subseteq X \setminus A$, so we would have $X \setminus A \in \mathcal{K}$, which would imply $A \notin \mathcal{O}(\mathcal{K})$, a contradiction. Therefore $B \in \mathcal{O}(K)$.

(c) Let $A, B \in \mathcal{O}(\mathcal{K})$. Since $\mathcal{K}$ is prime $(X \setminus A) \cup (X \setminus B) \notin \mathcal{K} = X \setminus (A \cap B) \notin \mathcal{K}$. Therefore $A \cap B \in \mathcal{O}(\mathcal{K})$. For primeness, let A and B be sets such that $A \cup B \in \mathcal{O}(\mathcal{K})$. Then $X \setminus (A \cup B) \notin \mathcal{K}$ but $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B) \notin \mathcal{K}$. Therefore either $X \setminus A \notin \mathcal{K}$ or $X \setminus B \notin \mathcal{K}$. That is, either $A \in \mathcal{O}(\mathcal{K})$ or $B \in \mathcal{O}(\mathcal{K})$.

Conversely, suppose $\mathcal{O}(\mathcal{K})$ is an open prime filter. We want to show that $\mathcal{K}$ is a prime closed filter. We first show that $\mathcal{K}$ is a filter.

(a) $X \setminus \emptyset = X \in \mathcal{O}(\mathcal{K})$, so $\emptyset \notin \mathcal{K}$.

(b) Suppose $A \subseteq B$ and $A \in \mathcal{K}$. We must show that $B \in \mathcal{K}$. If we had $B \notin \mathcal{K}$, then we would have $X \setminus B \in \mathcal{O}(\mathcal{K})$. But $A \subseteq B$ implies $X \setminus B \subseteq X \setminus A$, so we would have $X \setminus A \in \mathcal{O}(\mathcal{K})$, which would imply $A \notin \mathcal{K}$, a contradiction.

(c) Let $A, B \in \mathcal{K}$. Then $X \setminus A \notin \mathcal{O}(\mathcal{K})$ and $X \setminus B \notin \mathcal{O}(\mathcal{K})$. Since $\mathcal{O}(\mathcal{K})$ is prime, it follows that $(X \setminus A) \cup (X \setminus B) \notin \mathcal{O}(\mathcal{K})$ which implies $X \setminus (A \cap B) \notin \mathcal{O}(\mathcal{K})$. Therefore $A \cap B \in \mathcal{K}$. For primeness, let $A \cup B \in \mathcal{K}$. Then $X \setminus (A \cup B) \notin \mathcal{O}(\mathcal{K})$. But $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B) \notin \mathcal{O}(\mathcal{K})$. Therefore either $X \setminus A \notin \mathcal{O}(\mathcal{K})$ or $X \setminus B \notin \mathcal{O}(\mathcal{K})$. That is, either $A \in \mathcal{K}$ or $B \in \mathcal{K}$ which shows that $\mathcal{K}$ is prime.

(2) Given that K is a minimal prime closed filter, we need to show that $\mathcal{O}(\mathcal{K})$ is an open ultrafilter. We have already seen from (1) that $\mathcal{O}(K)$ is an open filter. Now we show that $\mathcal{O}(K)$ is an ultrafilter. To this end, let $A \subset X$. Then $A \cup (X \setminus A) = X \in \mathcal{K}$. By primeness of $\mathcal{K}$ either $A \in \mathcal{K}$ or $X \setminus A \in \mathcal{K}$ but not both. If $X \setminus A \notin \mathcal{K}$, then $A \in \mathcal{O}(\mathcal{K})$ if $A \notin \mathcal{K}$, then $X \setminus (X \setminus A) \notin \mathcal{K}$ showing that $X \setminus A \in \mathcal{O}(\mathcal{K})$ and hence by Definition 2.1.4 $\mathcal{O}(\mathcal{K})$ is an ultrafilter.

Conversely, suppose that $\mathcal{O}(\mathcal{K})$ is an open ultrafilter. We need to show that $\mathcal{K}$ is a minimal prime closed filter. We have already seen from (1) that $\mathcal{K}$ is a closed filter.

For primeness, suppose $A \cup B \in \mathcal{K}$, Then $X \setminus (A \cup B) \notin \mathcal{O}(\mathcal{K})$. Now $X \cap (A \cup B)^c = X \cap (A^c \cap B^c) = (X \cap A^c) \cap (X \cap B^c) = (X \setminus A) \cap (X \setminus B) \notin \mathcal{O}(\mathcal{K})$. Then $X \setminus A \notin \mathcal{O}(\mathcal{K})$ or $X \setminus B \notin \mathcal{O}(\mathcal{K})$ implies $A \in \mathcal{K}$ or $B \in \mathcal{K}$. $X \setminus (X \setminus A) = A \in \mathcal{K}$ or $X \setminus (X \setminus B) = B \in \mathcal{K}$. So $\mathcal{K}$ is prime. For minimality, suppose G is a prime closed filter contained in $\mathcal{K}$. That is, there is a closed set $B \in \mathcal{K}$ and $B \notin G$. That is $X \setminus B \notin \mathcal{O}(\mathcal{K})$ and $B \notin \mathcal{O}(\mathcal{K})$ contradicting that $\mathcal{O}(\mathcal{K})$ is an ultrafilter. Hence $\mathcal{K}$ must be a minimal prime closed filter.

(3) Suppose that $\mathcal{K}$ is a closed ultrafilter. We have already seen from (1) that $\mathcal{O}(\mathcal{K})$ is a filter. Now we want to show that $\mathcal{O}(K)$ is a minimal prime open filter. Let A and B be open sets such that $A \cup B \in \mathcal{O}(\mathcal{K})$. Then $X \setminus (A \cup B) \notin \mathcal{K}$. But then $X \setminus (A \cup B) = X \cap (A \cup B)^c = (X \cap A^c) \cap (X \cap B^c) = (X \setminus A) \cap (X \setminus B) \notin \mathcal{K}$. Therefore either $X \setminus A \notin \mathcal{K}$ or $X \setminus B \notin \mathcal{K}$. That is, either $A \in \mathcal{O}(\mathcal{K})$ or $B \in \mathcal{O}(\mathcal{K})$ which shows that $\mathcal{O}(\mathcal{K})$ is prime. Lastly we show that $\mathcal{O}(\mathcal{K})$ is a minimal prime open filter. Suppose there is a prime open filter $\mathcal{G}$ contained in $\mathcal{O}(\mathcal{K})$, that is, there is an open set $U \in \mathcal{O}(\mathcal{K})$ such that $U \notin \mathcal{G}$. Then $X \setminus U \notin \mathcal{K}$ and $U \notin \mathcal{K}$ contradicting that $\mathcal{K}$ is an ultrafilter. Hence $\mathcal{O}(\mathcal{K})$ must be a minimal prime open filter.

Conversely, suppose that $\mathcal{O}(\mathcal{K})$ is a minimal prime open filter. We show that $\mathcal{K}$ is a closed ultrafilter. Let B be a subset of X. Then $B \cup (X \setminus B) = X \in \mathcal{O}(\mathcal{K})$ and by primeness of $\mathcal{O}(\mathcal{K})$ either $B \in \mathcal{O}(\mathcal{K})$ or $X \setminus B \in \mathcal{O}(\mathcal{K})$ but not both. Therefore either $X \setminus B \notin \mathcal{K}$ or $B \notin \mathcal{K}$ but not both. That is either $X \setminus B \in \mathcal{K}$ or $B \in \mathcal{K}$ and hence $\mathcal{K}$ is a closed ultrafilter.

(4) The proof is similar to (1).

(5) The proof is similar to (2).

(6) The proof is similar to (3).

(7) Let $A \in \mathcal{K}$. Then A is closed and $X \setminus A$ is open, so $X \setminus A \notin \mathcal{K}$ so that $X \setminus A \in \mathcal{O}(\mathcal{K})$. Now $X \setminus (X \setminus A) = A$ is closed and $X \setminus (X \setminus A) = A \notin \mathcal{O}(\mathcal{K})$. That is, $A \in \mathcal{F}(\mathcal{O}(\mathcal{K}))$ and hence $\mathcal{K} \subseteq \mathcal{F}(\mathcal{O}(\mathcal{K}))$.

Conversely, let $B \in \mathcal{F}(\mathcal{O}(\mathcal{K}))$ and $X \setminus B$ is open. This shows that $X \setminus B \notin \mathcal{F}(\mathcal{O}(\mathcal{K}))$ and hence $X \setminus B \in \mathcal{O}(\mathcal{K})$ since $X \setminus B \notin \mathcal{K}$. Since $X \setminus B$ is open and $\mathcal{K}$ is a closed set in X, it follows that $X \setminus (X \setminus B) = B$ is closed and hence $B \in \mathcal{K}$. So $\mathcal{F}(\mathcal{O}(\mathcal{K})) \subseteq \mathcal{K}$ and hence $\mathcal{K} = \mathcal{F}(\mathcal{O}(\mathcal{K}))$.

(8) Let $A \in \mathcal{P}$. Then A is open and $X \smallsetminus A$ is closed, so $X \smallsetminus A \notin \mathcal{P}$ so that $X \smallsetminus A \in \mathcal{F}(\mathcal{P})$. Now $X \smallsetminus (X \setminus A) = A$ is open and $X \smallsetminus (X \smallsetminus A) = A \notin \mathcal{F}(\mathcal{P})$ so that $A \in \mathcal{O}(\mathcal{F}(\mathcal{P}))$. Hence $P \subseteq \mathcal{O}(\mathcal{F}(\mathcal{P}))$.

Conversely, let $B \in \mathcal{O}(\mathcal{F}(\mathcal{P}))$. Then B is open and so $X \smallsetminus B$ is closed. This shows that $X \smallsetminus B \notin \mathcal{O}(\mathcal{F}(\mathcal{P}))$ and hence $X \smallsetminus B \in \mathcal{F}(\mathcal{P})$ because $X \smallsetminus B \notin \mathcal{P}$. Since $X \smallsetminus B$ is closed and $\mathcal{P}$ is a collection of open sets in X. It follows that $X \smallsetminus (X \smallsetminus B) = B$ is open and hence $B \in \mathcal{P}$ so $\mathcal{O}(\mathcal{F}(\mathcal{P}) \subseteq \mathcal{P})$ and hence equality. □

The following theorem plays a vital role in the discussion, and for completeness we include the proof.

Theorem 2.1.3. [21] Let $\mathcal{M}$ be an open ultrafilter and $\mathcal{N}$ a closed ultrafilter. Then:

(1) $\mathcal{S}(\mathcal{N}) = \mathcal{O}(\mathcal{N})$.

(2) $\mathcal{G}(\mathcal{M}) = \mathcal{F}(\mathcal{M})$.

Proof. (1) Let $B \in \mathcal{S}(\mathcal{N})$. Then B is open and there exists $A \in \mathcal{N}$ with $A \subseteq B$. Then $X \smallsetminus B \subseteq X \smallsetminus A$ and $X \smallsetminus A \notin \mathcal{N}$ so that $X \smallsetminus B \notin \mathcal{N}$. Hence $B \in \mathcal{O}(\mathcal{N})$. Thus $\mathcal{S}(\mathcal{N}) \subseteq \mathcal{O}(\mathcal{N})$.

Conversely, let $A \in \mathcal{O}(\mathcal{N})$. Then A is open and $X \smallsetminus A \notin \mathcal{N}$. Therefore $X \smallsetminus A$ is closed and $X \smallsetminus A \notin \mathcal{N}$. Since $\mathcal{N}$ is a closed ultrafilter, it follows that there is a closed set $U \in \mathcal{N}$ such that $(X \smallsetminus A) \cap U = \emptyset$. This implies that $U \subseteq A$ because A is the largest set that does not intersect $X \smallsetminus A$. Therefore $A \in \mathcal{S}(\mathcal{N})$ so that $\mathcal{O}(\mathcal{N}) \subseteq \mathcal{S}(\mathcal{N})$ and hence equality.

(2) Let $A \in \mathcal{G}(\mathcal{M})$. Then A is closed and there exists $B \in \mathcal{M}$ with $A \subseteq B$. Then $X \smallsetminus B \subseteq X \smallsetminus A$ and $X \smallsetminus A \notin \mathcal{M}$ so that $X \smallsetminus B \notin \mathcal{M}$. Hence $B \in \mathcal{F}(\mathcal{M})$. Thus $\mathcal{G}(\mathcal{M}) \subseteq \mathcal{F}(\mathcal{M})$.

Conversely, let $B \in \mathcal{M}$. Then B is closed and $X \smallsetminus B \notin \mathcal{M}$. Since $\mathcal{M}$ is a closed ultrafilter, it follows that there is a closed set $U \in \mathcal{M}$ such that $(X \smallsetminus B) \cap U = \emptyset$. This implies that $U \subseteq B$ because B is the largest set that does not intersect $X \smallsetminus B$. Therefore $B \in \mathcal{G}(\mathcal{M})$ so that $\mathcal{F}(\mathcal{M}) \subseteq \mathcal{G}(\mathcal{M})$ and hence equality. □

Definition 2.1.5. A point x is adherent to a set A if every neighborhood of x meets A.

The set of points adherent to a set A is called the adherence (closure) $\overline{A}$ of A.

An adherent point of A is an isolated point of A if there is a neighborhood of A which contains no point of A other than x; otherwise is a point of accumulation (limit point) of A.

Definition 2.1.6. Let $\mathcal{F}$ be a closed filter on X and $x \in X$. We say x is a cluster point of $\mathcal{F}$ if for all neighborhoods U of x there exists $F \in \mathcal{F}$ such that $F \subseteq U$. If $\mathcal{O}$ is an open filter on X, and $x \in X$, we say $\mathcal{O}$ converges to x if every open neighborhood of X belongs to $\mathcal{O}$.

It is known that a point $p \in X$ of a topological space X is a cluster point of an ultrafilter $\mathcal{F}$ if and only if $\mathcal{F}$ converges to p. In other words, for ultrafilters their cluster points are precisely their limit points, and *adherence* is equivalent to *convergence*.

Theorem 2.1.4. [21] Let $\mathcal{F}$ be a closed filter and $\mathcal{O}$ be an open filter.

(1) $x \in \mathrm{adh}\mathcal{F}$ implies $x \in \mathrm{adh}S(\mathcal{F})$.

(2) $G(\mathcal{O}) \to x$ implies $\mathcal{O} \to x$.

(3) $x \in \mathrm{adh}\mathcal{O}$ if and only if $x \in \mathrm{adh}\mathcal{G}(\mathcal{O})$.

(4) $\mathcal{F} \to x$ if and only if $S(\mathcal{F}) \to x$.

Proof. (1) Let $F \in S(\mathcal{F})$ and U be any neighborhood of x. We want to show that $F \cap U \neq \emptyset$. Now $x \in \mathrm{adh}\mathcal{F}$, so every $G \in \mathcal{F}$ meets every neighborhood of x, in particular $G \cap U \neq \emptyset$. Since $F \in S(\mathcal{F})$, it follows that there exists $A \in \mathcal{F}$ with $A \subseteq F$. Since $\mathcal{F}$ adheres to x and $A \in \mathcal{F}$, it follows that $A \cap U \neq \emptyset$. Hence $F \cap U \neq \emptyset$ because $A \subseteq F$.

(2) Let U be a neighborhood of x. By hypothesis $\mathcal{G}(\mathcal{O}) \to x$, that is, every neighborhood of x belongs to $\mathcal{G}(\mathcal{O})$, in particular $U \in \mathcal{G}(\mathcal{O})$. That is, U is closed and there exists $A \in \mathcal{O}$ with $A \subseteq U$. Since $\mathcal{O}$ is a filter and $U \supseteq A \in \mathcal{O}$, it follows that $U \in \mathcal{O}$. Hence $\mathcal{O} \to x$.

(3) Suppose that $x \in \mathrm{adh}\mathcal{O}$ and let $K \in \mathcal{G}(\mathcal{O})$. Let U be a neighborhood of x. We want to show that $K \cap U \neq \emptyset$. Since $K \in \mathcal{G}(\mathcal{O})$, it follows that there exists $A \in \mathcal{O}$ such that $A \subseteq K$. By hypothesis, $x \in \mathrm{adh}\mathcal{O}$, so $A \cap U \neq \emptyset$ which implies that $K \cap U \neq \emptyset$, as was to be shown.

Conversely, suppose $x \in \text{adh}\mathcal{G}(\mathcal{O})$. Let U be a neighborhood of x. We want to show that $U \in \mathcal{O}$. By hypothesis, $x \in \text{adh}\mathcal{G}(\mathcal{O})$, that is, for every neighborhood U of x there is an $F \in \mathcal{O}$ such that $F \subseteq U$. Now $\mathcal{O}$ is a filter, so $U \in \mathcal{O}$ as required.

(4) Suppose $\mathcal{F} \to x$ and let U be a neighborhood of x. By hypothesis $\mathcal{F} \to x$, that is, every neighborhood of x belongs to $\mathcal{F}$, in particular $U \in \mathcal{F}$. That is, U is open and there exists $A \in \mathcal{F}$ with $A \subseteq U$. Since $\mathcal{F}$ is a filter and $U \supseteq A \in \mathcal{F}$, it follows that $U \in \mathcal{F}$. Hence $\mathcal{S}(\mathcal{F}) \to x$.

Conversely, let U be a neighborhood of x. By hypothesis $\mathcal{S}(\mathcal{F}) \to x$, that is, every neighborhood of x belongs to $\mathcal{S}(\mathcal{F})$, in particular $U \in \mathcal{S}(\mathcal{F})$. That is U is open and there exists $A \in \mathcal{F}$ with $A \subseteq U$. Since $\mathcal{F}$ is a filter and $U \supseteq A \in \mathcal{F}$, it follows that $U \in \mathcal{F}$. Hence $\mathcal{F} \to x$. □

Definition 2.1.7. Let X be a topological space. An *open grill* is a nonempty collection $\mathcal{G}$ of open sets satisfying:

(1) $\emptyset \notin \mathcal{G}$.

(2) $O \in \mathcal{G}, Q$ open and $Q \supseteq O$ implies $Q \in \mathcal{G}$.

(3) For open sets O and Q we have that $O \cup Q \in \mathcal{G}$ if and only if $O \in \mathcal{G}$ or $Q \in \mathcal{G}$.

Closed grills are defined similarly. Let $\mathcal{G}$ be a closed grill and $\mathcal{H}$ be an open grill. Set:

(1) $\mathcal{O}(\mathcal{G}) = \{O \mid O \text{ is open and } \mathrm{X} \setminus O \notin \mathcal{G}\}$.

(2) $\mathcal{F}(\mathcal{H}) = \{F \mid F \text{ is closed and } X \setminus F \notin \mathcal{H}\}$.

Easily prime open (closed) filters are open (closed) grills and the operators $\mathcal{F}$ and $\mathcal{O}$ will be used on prime open and closed filters. In a topological space prime open filters are not necessarily open ultrafilters and prime closed filters are not necessary closed ultrafilters. Similarly, open grills are not necessarily the union of the family of open ultrafilters and this is also true for closed grills and closed filters. However for prime filters we have the following theorem.

Theorem 2.1.5. [21]

(1) Every open filter equals the intersection of the family of prime open filters that contain it.

(2) Every closed filter equals the intersection of the family of prime closed filters that contain it.

(3) Every open grill is precisely the union of a family of prime open filters.

(4) Every closed grill is precisely the union of a family of prime closed filters.

The following theorem appears in [21] and [36]. Here we include the proof for the sake of completeness.

Theorem 2.1.6. [21], [36] Let X be a T_1-topological space. The following statements are equivalent:

(1) Every prime open filter is an open ultrafilter.

(2) Every prime closed filter is a closed ultrafilter.

(3) Each closed ultrafilter $\mathcal{F}$ satisfies the following equivalent properties:

  (a) $F \in \mathcal{F}$ implies there exists $\mathcal{G} \in \mathcal{F}$ and $\mathcal{O}$ open with $\mathcal{G} \subseteq \mathcal{O} \subseteq \mathcal{F}$.

  (b) $F \in \mathcal{F}$ implies that $\text{int}(F) \in \mathcal{S}(\mathcal{F})$.

  (c) $\mathcal{F} = \mathcal{G}(\mathcal{S}(\mathcal{F}))$.

(4) X has the discrete topology.

Proof. (1) $\Rightarrow$ (2): Let $\mathcal{K}$ be a prime closed filter and let $U \subseteq X$. Then $\mathcal{O}(\mathcal{K})$ is a prime open filter and hence an open ultrafilter by hypothesis. Then $\mathcal{O}(\mathcal{K})$ is minimal with this condition and hence $\mathcal{K}$ is a closed ultrafilter.

(2) $\Rightarrow$ (3): We show that the following conditions are equivalent:

(a) $\Rightarrow$ (b): Let $F \in \mathcal{F}$. Find $G \in \mathcal{F}$ by hypothesis and an open set $\mathcal{O}$ such that $G \subseteq \mathcal{O} \subseteq F$. Put $\mathcal{O} = \text{int}(F)$, hence $G \subseteq \text{int}(F) \subseteq F$. Now $\text{int}(F)$ is open and there exists $G \in \mathcal{F}$ such that $G \subseteq \text{int}(F)$. Hence $\text{int}(F) \in S(\mathcal{F})$.

(b) $\Rightarrow$ (c): Let $U \in \mathcal{F}$. Then U is closed and $\text{int}(U) \in \mathcal{S}(\mathcal{F})$. Therefore $\text{int}(U)$ is open and there exists $A \in \mathcal{F}$ with $A \subseteq \text{int}(U)$. Now $A \subseteq \text{int}(U) \subseteq U$. Hence $A \in U$ and therefore $U \in \mathcal{G}(\mathcal{S}(\mathcal{F}))$. Hence $\mathcal{F} \subseteq \mathcal{G}(\mathcal{S}(\mathcal{F}))$.

Let $B \in \mathcal{G}(\mathcal{S}(\mathcal{F}))$. Then by definition B is closed and there exists $D \in \mathcal{S}(\mathcal{F})$ with $D \subseteq B$. Now $D \in \mathcal{S}(\mathcal{F})$ implies that D is open and there exists $E \in \mathcal{F}$ with $E \subseteq D$. Now $E \subseteq D \subseteq B \Rightarrow E \subseteq B$. This means that $B \in \mathcal{F}$ showing that $\mathcal{G}(\mathcal{S}(\mathcal{F})) \subseteq \mathcal{F}$ and hence equality.

(c) $\Rightarrow$ (a): Let $F \in \mathcal{F}$. Then $F \in \mathcal{G}(\mathcal{S}(\mathcal{F}))$. Then F is closed and there is $A \in \mathcal{S}(\mathcal{F})$ such that $A \subseteq F$. By definition A is open and there is $B \in \mathcal{F}$ such that $B \subseteq A$. Hence there exist $B \in \mathcal{F}$ and A open with $B \subseteq A \subseteq F$.

(3) $\Rightarrow$ (4): In a discrete topology every subset of X is open. To this end, let $A \subseteq X$. We want to show that A is open. Since $\mathcal{F}$ is a closed ultrafilter, it follows that either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$. If $X \setminus A \in \mathcal{F}$, then $X \setminus A$ is closed and so its complement A is open. If $A \in \mathcal{F}$, then A is closed and $\text{int}(A) \in \mathcal{S}(\mathcal{F})$ with $\text{int}(A) \subseteq F$. $\text{Int}(A) \in \mathcal{S}(\mathcal{F})$, $\text{int}(A)$ is open and there $F \in \mathcal{F}$ with $F \subseteq \text{int}(A)$. In particular, $A \in \mathcal{F}$ with $A \subseteq \text{int}(A)$. That is, $A = \text{int}(A)$ and hence A is open.

(4) $\Rightarrow$ (1): Let $\mathcal{G}$ be a prime open filter. Then by hypothesis U and $X \setminus U$ are open subsets of X. $U \cup (X \setminus U) = X \in \mathcal{G}$ and by primeness of $\mathcal{G}$ either $U \in \mathcal{G}$ or $X \setminus U \in \mathcal{G}$. Therefore $\mathcal{G}$ is an ultrafilter. $\square$

Theorem 2.1.7. [21] Let $\mathcal{P}$ be a prime open filter and $\mathcal{K}$ a prime closed filter. Then:

(1) $\mathcal{F}(\mathcal{P}) \to x$ implies $x \in \text{adh}\mathcal{P}$.

(2) $\mathcal{K} \to x$ implies $x \in \text{adh}\mathcal{O}(\mathcal{K})$.

(3) $x \in \text{adh}\mathcal{K}$ if and only if $\mathcal{O}(\mathcal{K}) \to x$.

(4) $\mathcal{P} \to x$ if and only if $x \in \text{adh}\mathcal{F}(\mathcal{P})$.

Proof. (1) Let U be a neighborhood of x. Since $\mathcal{F}(\mathcal{P}) \to x$, it follows that $U \in \mathcal{F}(\mathcal{P})$. That is, U is closed and $X \setminus U \notin \mathcal{P}$. Now $U \cup (X \setminus U) = X \in \mathcal{P}$ and by primeness of $\mathcal{P}$, $U \in \mathcal{P}$. Since $\mathcal{P}$ is a filter, so every member of $\mathcal{P}$ meets U. So indeed $x \in \text{adh}\mathcal{P}$.

(2) Let U be a neighborhood of x. Since $\mathcal{K} \to x$, it follows that $U \in \mathcal{K}$. That is, U is open and $X \smallsetminus U \notin \mathcal{K}$. Now $U \cup (X \smallsetminus U) = X \in \mathcal{K}$ and by primeness of $\mathcal{K}$, $U \in \mathcal{K}$. Since $\mathcal{K}$ is a filter, so every member of $\mathcal{K}$ meets U. So indeed $x \in \mathrm{adh}\mathcal{O}(\mathcal{K})$.

(3) Let U be a neighborhood of x. We need to show that $U \in \mathcal{O}(\mathcal{K})$. By hypothesis, $B \cap U \neq \emptyset$ for every $B \in \mathcal{K}$. This shows that $X \smallsetminus U$ misses some members of $\mathcal{K}$ because U intersects all members of $\mathcal{K}$. Therefore $X \smallsetminus U \notin \mathcal{K}$ and since U is an open neighborhood of x, it follows that $U \in \mathcal{O}(\mathcal{K})$.

Conversely, suppose that $\mathcal{O}(\mathcal{K}) \to x$. Let U be a neighborhood of x. Since $\mathcal{O}(\mathcal{K}) \to x$ every neighborhood U of x belongs to $\mathcal{O}(\mathcal{K})$. That is, U is open and $X \smallsetminus U \notin \mathcal{K}$. Since $\mathcal{K}$ is a prime closed filter is a closed ultrafilter by Theorem 2.1.6 (2), it follows that $U \in \mathcal{K}$ and hence meets every members of $\mathcal{K}$. So $x \in \mathrm{adh}\mathcal{K}$.

(4) Let U be a neighborhood of x. Since $\mathcal{P} \to x$, it follows that $U \in \mathcal{P}$. That is U is open and $X \smallsetminus U \notin \mathcal{P}$. Now $U \cup (X \smallsetminus U) = X \in \mathcal{P}$ and by primeness of $\mathcal{P}$, $U \in \mathcal{P}$. Since $\mathcal{P}$ is a filter, so every member of $\mathcal{P}$ meets U and hence $x \in \mathrm{adh}\mathcal{F}(\mathcal{P})$.

Conversely, suppose $x \in \mathrm{adh}\mathcal{F}(\mathcal{P})$. Let U be a neighborhood of x. We need to show that $U \in \mathcal{P}$. By hypothesis, $B \cap U \neq \emptyset$ for every $B \in \mathcal{P}$. This shows that $X \smallsetminus U$ misses some members of $\mathcal{P}$. Therefore $X \smallsetminus U \notin \mathcal{P}$ and since U is an open neighborhood of x, it follows that $U \in \mathcal{O}(\mathcal{K})$. $\square$

2.2 Balanced filters

Open filters are not necessarily the intersection of open ultrafilters that contain them and the statement is also true for closed filters. This motivates for the following definitions.

Definition 2.2.1. Let $\mathcal{O}$ be an open filter. Set

$$b(\mathcal{O}) = \bigcap \{M \mid M \text{ is an open ultrafilter and } \mathcal{O} \subseteq M\}.$$

An *open filter* $\mathcal{O}$ is said to be *balanced* provided $\mathcal{O} = b(\mathcal{O})$, that is, $\mathcal{O}$ is equal to the intersection of all of the open ultrafilters that contain it.

Definition 2.2.2. Let $\mathcal{F}$ be a closed filter. Set

$$b(\mathcal{F}) = \bigcap \{N \mid N \text{ is a closed ultrafilter and } \mathcal{F} \subseteq N\}.$$

A closed filter $\mathcal{F}$ is said to be *balanced* provided $\mathcal{F} = b(\mathcal{F})$, that is, $\mathcal{F}$ is equal to the intersection of family of closed ultrafilters that contain it.

Definition 2.2.3.

(1) A closed filter $\mathcal{F}$ is said to be *open-generated* if there exists an open filter $\mathcal{O}$ such that $\mathcal{F} = \mathcal{G}(\mathcal{O})$.

(2) An open filter $\mathcal{O}$ is said to be *closed-generated* if there exists a closed filter $\mathcal{F}$ such that $\mathcal{O} = \mathcal{S}(\mathcal{F})$.

We give the lemma below without proof.

Lemma 2.2.1. [18], [21] Let $\mathcal{O}$ be an open filter and $\mathcal{F}$ a closed filter on a T_1-topological space X.

(1) $\sec(\mathcal{O}) = \bigcup\{\mathcal{M} \mid \mathcal{M}$ an open ultrafilter and $\mathcal{O} \subseteq \mathcal{M}\}$.

(2) $\sec^2(\mathcal{O}) = \bigcap\{\mathcal{M} \mid \mathcal{M}$ an open ultrafilter and $\mathcal{O} \subseteq \mathcal{M}\}$.

(3) $\sec(\mathcal{F}) = \bigcup\{\mathcal{N} \mid \mathcal{N}$ a closed ultrafilter and $\mathcal{F} \subseteq \mathcal{N}\}$.

(4) $\sec^2(\mathcal{F}) = \bigcap\{\mathcal{N} \mid \mathcal{N}$ an closed ultrafilter and $\mathcal{F} \subseteq \mathcal{N}\}$.

From the Lemma we deduce immediately that $b(\mathcal{F}) = \sec^2(\mathcal{F})$. The following theorem and its proof is taken verbatim as in [46].

Theorem 2.2.1. [21],[47] Let $\mathcal{F}$ be an open filter on X and

$$\mathcal{G} = \bigcap\{\mathcal{U} \mid \mathcal{U} \text{ is an open ultrafilter with } \mathcal{F} \subseteq \mathcal{U}\}.$$

Then

$$\mathcal{G} = \{U \mid U \text{ is open and } \operatorname{int}(U) \in \mathcal{F}\} = \mathcal{F} \vee \mathcal{D}$$

where $\mathcal{D} = \{U \mid U$ is open and dense$\}$.

Proof. Since $\mathcal{D}$ is contained in every open ultrafilter, then $\mathcal{F} \vee \mathcal{D} \subseteq \mathcal{G}$. Let U be open in X such that $\operatorname{int}(\operatorname{cl}(U)) \notin \mathcal{F}$. Then $X \smallsetminus \operatorname{cl}(U) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. For some open ultrafilter $\mathcal{U}, \mathcal{U} \supseteq \mathcal{F} \cup \{X \smallsetminus \operatorname{cl}(U)\}$. This shows

$$\mathcal{G} \subseteq \{U \mid U \text{ open in } X \text{ and } \operatorname{int}(\operatorname{cl}(U)) \in \mathcal{F}\}.$$

Now, let U be open in X such that $\operatorname{int}(\operatorname{cl}(U)) \in \mathcal{F}$. Then $X \setminus \operatorname{bd} U \in \mathcal{D}$ and $\operatorname{int}(\operatorname{cl}(U)) \cap X \setminus \operatorname{bd} U = U \in \mathcal{F} \vee \mathcal{D}$. Hence,

$$\{U \mid U \text{ is open in } X \text{ and } \operatorname{int}(\operatorname{cl}(U)) \in \mathcal{F}\} \subseteq \mathcal{F} \vee \mathcal{D}.$$

This completes the proof. $\square$

Theorem 2.2.2. [21],[47] Let X be a T_1-topological space and set

$$\mathcal{S} = \bigcap\{\mathcal{M} \mid \mathcal{M} \text{ an open ultrafilter on } X\}.$$

Let $\mathcal{O}$ be an open filter on X. Then:

(1) $\mathcal{S}$ is the smallest balanced open filter on X.

(2) $\mathcal{S} = \{O \mid O \text{ is an open dense set}\}$.

(3) $b(\mathcal{O}) = \{Q \in \tau \mid \text{there exists } O \in \mathcal{O} \text{ with } O \subseteq \overline{Q}\}$.

(4) $b(\mathcal{O})$ is the smallest balanced open filter containing $\mathcal{O}$.

(5) $b(\mathcal{O}) = \mathcal{O} \vee \mathcal{S}$.

Proof. (1) $\mathcal{S}$ is the intersection of all open ultrafilters, it follows that $\mathcal{O} \subseteq \mathcal{M}$, where $\mathcal{M}$ is an open ultrafilter. Thus $\mathcal{S} = b(\mathcal{O})$ and hence $\mathcal{S}$ is the smallest balanced open filter on X.

(2) By definition, $U \in \mathcal{S}$ if U is an open ultrafilter, so U is an open set in X. U is an open ultrafilter, so it is not contained in any other open set other than X, so $\overline{U} = X$.

(3) Now by definition of $b(\mathcal{O})$, $Q \in b(\mathcal{O})$ if Q is an open ultrafilter and $\mathcal{O} \subset Q$. Thus every $O \in \mathcal{O}$ also belongs to Q. $Q \in \tau$ because is open. Thus

$$b(\mathcal{O}) = \{Q \in \tau \mid \text{there exists } O \in \mathcal{O} \text{ with } O \subseteq \overline{Q}\}$$

because $Q \subseteq \overline{Q}$, as required.

(4) It follows immediately from the definition of a balanced filter.

(5) This is Theorem 2.2.1. $\square$

The following theorems appears in [21] without proofs. Here we include the proof for completeness.

Theorem 2.2.3. [21] Let $\mathcal{O}$ be an open filter. The following statements are equivalent.

(1) $\mathcal{O}$ is balanced and prime.

(2) $O \cup Q \in \sec^2(\mathcal{O})$ implies $O \in \mathcal{O}$ or $Q \in \mathcal{O}$.

Proof. (1) $\Rightarrow$ (2): Let $O \cup Q \in \sec^2(\mathcal{O})$. We show that either $O \in \mathcal{O}$ or $Q \in \mathcal{O}$. Then by definition $\sec^2(\mathcal{O}) = b(\mathcal{O})$ and because $\mathcal{O}$ is a balanced, $b(\mathcal{O}) = \mathcal{O}$. Hence if $O \cup Q \in \sec^2(\mathcal{O})$, then $O \cup Q \in \mathcal{O}$ and by primeness of $\mathcal{O}$, $O \in \mathcal{O}$ or $Q \in \mathcal{O}$ as required.

(2) $\Rightarrow$ (1): Let $O \cup Q \in \mathcal{O}$. Since $\mathcal{O} \subseteq b(\mathcal{O}) = \sec^2(\mathcal{O})$, it follows that $O \cup Q \in \sec^2(\mathcal{O})$ and hence by (2) $O \in \mathcal{O}$ or $Q \in \mathcal{O}$. So Q is prime. It remains to show that $\mathcal{O}$ is balanced. Let $U \in b(\mathcal{O})$. Then $X \smallsetminus U \notin b(\mathcal{O})$. But $U \cup (X \smallsetminus U) \in b(\mathcal{O}) = \sec^2(\mathcal{O})$. By (2) $U \in \mathcal{O}$ or $X \smallsetminus U \in \mathcal{O}$. But $X \smallsetminus U \notin \mathcal{O}$ since $\mathcal{O} \subseteq b(\mathcal{O})$. Therefore $U \in \mathcal{O}$ and hence $b(\mathcal{O}) \subseteq \mathcal{O}$. The other containment is trivial and hence $\mathcal{O} = b(\mathcal{O})$. Thus $\mathcal{O}$ is balanced. $\square$

Theorem 2.2.4. [21] Let $\mathcal{F}$ be a closed filter. The following statements are equivalent:

(1) $\mathcal{F}$ is a balanced closed filter.

(2) For each closed set $G \notin \mathcal{F}$ there exists an open set $O \supseteq G$ such that $F \subseteq O$ for each $F \in \mathcal{F}$.

(3) If F is a closed set and each open O that contains F belongs to $\mathcal{S}(\mathcal{F})$ then $F \in \mathcal{F}$.

Proof. (1) $\Rightarrow$ (2): Let G be a closed set such that $G \notin \mathcal{F}$. Put $E = \cap\{F \mid F \in \mathcal{F}\} \neq \emptyset$ and $E \in \mathcal{F}$. We have two possibilities $E \cap G = \emptyset$ or $E \cap G \neq \emptyset$. If $E \cap G = \emptyset$, then $G \subseteq X \smallsetminus E$ and $X \smallsetminus E$ is open. So $F \subsetneq X \smallsetminus E$. If $E \cap G \neq \emptyset$. Put $K = E \cap G \notin \mathcal{F}$ because $\mathcal{F}$ is a closed filter and $G \notin \mathcal{F}$. But $E \cap G \subseteq E \in \mathcal{F}$ with $E \cap G \notin \mathcal{F}$, this then contradicts that $E \notin \mathcal{F}$ and hence $E \cap G = \emptyset$.

(2) $\Rightarrow$ (3): Let $\mathcal{F}$ be a closed set and an open set $O \supseteq F$ be such that $O \in \mathcal{S}(\mathcal{F})$. Therefore there is $A \in \mathcal{F}$ such that $A \subseteq O$. Suppose $F \notin \mathcal{F}$, then, by hypothesis, there exists an open set O such that $F \subsetneq O$ for each $F \in \mathcal{F}$. This shows that $O \notin \mathcal{S}(\mathcal{F})$ which is a contradiction. Hence $F \in \mathcal{F}$.

(3) $\Rightarrow$ (1): Let $\mathcal{F}$ be closed and every open set O containing F belongs to $\mathcal{F}$. Then by (3) $F \in \mathcal{F}$. So

$$\mathcal{F} = \{F \mid F \text{ is closed and } F \subseteq O \text{ where } O \in \mathcal{S}(\mathcal{F})\}$$

$$\mathcal{F} = \{F \mid F \text{ is closed and } F \subseteq \overline{O} \text{ where } O \in \mathcal{S}(\mathcal{F})\}$$

$$\mathcal{F} = \cap\{\overline{O} \mid \overline{O} \text{ is a closed ultrafilter and } F \subseteq \overline{O}\}$$

$$\mathcal{F} = b(\mathcal{F})$$

as required. □

Recall that a space X is normal if and only if for each pair A, B of disjoint closed subsets of X, there is a pair U, V of disjoint open subsets of X so that $A \subseteq U$ and $B \subseteq V$. A normal T_1-space is called a T_4-space.

Theorem 2.2.5. [21] Let X be a T_1-topological space and $\mathcal{F}$ a closed filter on X.

(1) $b(\mathcal{F}) = \mathcal{F} \cup \{G \mid G$ is closed and for each $O \in t$ with $O \supseteq G$ there exists $F \in \mathcal{F}$ with $F \subseteq O\}$.

(2) If X is normal then each balanced prime closed filter is a closed ultrafilter.

Proof. (1) Follows immediately from the definition of the balanced filter.

(2) Let $\mathcal{F}$ be a balanced prime closed filter. We want to show that $\mathcal{F}$ is an ultrafilter. To this end, let $A \subseteq X$. Since X is a T_1-space it follows that both A and $X \smallsetminus A$ are disjoint closed subsets of X. By normality of X find disjoint open subsets U and V such that $A \subseteq U$ and $X \smallsetminus A \subseteq V$. Since X is closed and $X \in \mathcal{F}$, it follows that $A \cup (X \smallsetminus A) = X \in \mathcal{F}$. So $A \cup (X \smallsetminus A) \in \mathcal{F}$ and by primeness of $\mathcal{F}$ either $A \in \mathcal{F}$ or $X \smallsetminus A \in \mathcal{F}$. Since A was arbitrary chosen, it follows that $\mathcal{F}$ is a closed ultrafilter. □

Theorem 2.2.6. [21] Let $\mathcal{F}$ be a closed filter and $\mathcal{O}$ an open filter. Then:

(1) $\mathcal{S}(\mathcal{F}) = \mathcal{S}(b(\mathcal{F}))$.

(2) $\mathcal{G}(\mathcal{O}) = \mathcal{G}(b(\mathcal{O}))$.

Proof. (1) Let $B \in \mathcal{S}(\mathcal{F})$. Then B is open and there exists $A \in b(\mathcal{F})$ with $A \subseteq B$. Then $X \smallsetminus B \subseteq X \smallsetminus A$ and $X \smallsetminus A \notin b(\mathcal{F})$ so that $X \smallsetminus B \notin b(\mathcal{F})$. Hence $B \in \mathcal{S}(\mathcal{F})$. Then $\mathcal{S}(\mathcal{F}) \subseteq S(b(\mathcal{F}))$.

Conversely, let $B \in \mathcal{S}(b(\mathcal{F}))$. Then B is open and there exists $A \in b(\mathcal{F})$ with $A \subseteq B$. Then $X \setminus B \subseteq X \setminus A$ and $X \setminus A \notin b(\mathcal{F})$ so that $X \setminus B \notin b(\mathcal{F})$. Therefore $B \in \mathcal{S}(b(\mathcal{F}))$ so that $\mathcal{S}(b(\mathcal{F})) \subseteq \mathcal{S}(\mathcal{F})$. Hence equality.

(2) Let $A \in \mathcal{G}(\mathcal{O})$. Then A is closed and there exists $B \in b(\mathcal{O})$ with $B \subseteq A$. Then $X \setminus A \subseteq X \setminus B$ and $X \setminus B \notin \mathcal{G}(\mathcal{O})$ so that $X \setminus A \notin \mathcal{G}(\mathcal{O})$. Hence $B \in \mathcal{G}(\mathcal{O})$. Thus $\mathcal{G}(\mathcal{O}) \subseteq \mathcal{G}(b(\mathcal{O}))$.

Conversely, let $B \in \mathcal{G}(b(\mathcal{O}))$. Then B is closed and there exists $A \in b(\mathcal{O})$ with $B \subseteq A$. Then $X \setminus A \subseteq X \setminus B$ and $X \setminus B \notin b(\mathcal{O})$ so that $X \setminus A \notin b(\mathcal{O})$. Therefore $A \in \mathcal{G}(b(\mathcal{O}))$ so that $\mathcal{G}(b(\mathcal{O})) \subseteq \mathcal{G}(\mathcal{O})$. Hence equality. □

2.3 Open-generated closed filters and closed-generated open filters

We notice from Carlson [21] in the beginning of section 4 that from Theorem 2.2.6, we have that an open envelope of a closed filter equals the open envelope of the smallest balanced closed filter that contains the filter, that is, $\mathcal{S}(\mathcal{F}) = \mathcal{S}(b(\mathcal{F}))$ for each closed filter $\mathcal{F}$. We also have that a closed envelope of an open filter equals the closed envelope of the smallest balanced open filter that contains the filter, that is, $\mathcal{G}(\mathcal{O}) = \mathcal{G}(b(\mathcal{O}))$ for each open filter $\mathcal{O}$. This motivates the following definitions.

Definition 2.3.1. A closed filter $\mathcal{F}$ is said to be *open-generated* if there exists an open filter $\mathcal{O}$ such that $\mathcal{F} = \mathcal{G}(\mathcal{O})$.

Definition 2.3.2. An open filter $\mathcal{O}$ is said to be *closed-generated* if there exists a closed filter $\mathcal{F}$ such that $\mathcal{O} = \mathcal{S}(\mathcal{F})$.

A *minimal prime closed filter* is a prime closed filter that is minimal in the collection of prime closed filters. A *minimal prime open filter* is a prime open filter that is minimal in the collection of prime open filters.

Theorem 2.3.1. [21] A closed filter $\mathcal{F}$ is open-generated if and only if $\mathrm{cl}(\mathrm{int}(F)) \in \mathcal{F}$ for each $F \in \mathcal{F}$.

Proof. Suppose that a filter $\mathcal{F}$ is open-generated. That is, there is an open filter $\mathcal{O}$ such that $\mathcal{F} = \mathcal{G}(\mathcal{O})$. But then

$$\mathcal{G}(\mathcal{O}) = \{F \mid F \text{ is closed and there exists } A \in \mathcal{O} \text{ with } A \subseteq F\}.$$

Now A is open and $A \subseteq F$, so $A \subseteq \mathrm{int}(F) \subseteq F$ implies that $\overline{A} \subseteq \mathrm{cl}(\mathrm{int}(F)) = F \in \mathcal{F}$. Hence $\mathrm{cl}(\mathrm{int}(F)) \in \mathcal{F}$ for each $F \in \mathcal{F}$ as required.

Conversely, suppose that $\mathrm{cl}(\mathrm{int}(F)) \in \mathcal{F}$ for each $F \in \mathcal{F}$. Now for each $F \in \mathcal{F}$ we have $\mathrm{int}(F)$ is open, so there exists an open filter $\mathcal{O}$ such that $\mathrm{int}(F) \in \mathcal{O}$. Also $\mathrm{int}(F) \subseteq F$. Indeed $\mathcal{F} = \mathcal{G}(\mathcal{O})$ and hence $\mathcal{F}$ is open-generated. □

Remark 2.3.1. The dual statement of the preceding theorem does not hold because $\mathrm{cl}(\mathcal{O}) \supseteq \mathcal{O}$.

Now we are ready for the following theorem.

Theorem 2.3.2. [21] Let $\mathcal{F}$ be a closed filter and $\mathcal{O}$ an open filter.

(1) If $\mathcal{F}$ is prime then $\mathcal{S}(\mathcal{F}) \subseteq \mathcal{O}(\mathcal{F})$ and $\mathcal{S}(\mathcal{F}) = \mathcal{O}(\mathcal{F})$ if and only if $\mathcal{F}$ is a closed ultrafilter.

(2) If $\mathcal{O}$ is prime then $\mathcal{G}(\mathcal{O}) \subseteq \mathcal{F}(\mathcal{O})$ and $\mathcal{G}(\mathcal{O}) = \mathcal{F}(\mathcal{O})$ if and only if $\mathcal{O}$ is a open ultrafilter.

(3) If $\mathcal{S}(\mathcal{F})$ is an open ultrafilter then $\mathcal{F}$ is a closed ultrafilter.

(4) If $\mathcal{G}(\mathcal{O})$ is a closed ultrafilter then $\mathcal{O}$ is an open ultrafilter.

(5) If X is normal then $\mathcal{S}(\mathcal{F}) = \mathcal{S}(\mathcal{G}(\mathcal{S}(\mathcal{F})))$.

(6) If X is normal then $\mathcal{O}$ is closed-generated if and only if $\mathcal{O} = \mathcal{S}(\mathcal{G}(\mathcal{O}))$.

Proof. (1) Let $B \in \mathcal{S}(\mathcal{F})$. Then B is open and there exists $V \in \mathcal{F}$ with $V \subseteq B$. Now $X \setminus B$ is closed and $X \setminus B \notin \mathcal{F}$. So $B \in \mathcal{O}(\mathcal{F})$ and hence $\mathcal{S}(\mathcal{F}) \subseteq \mathcal{O}(\mathcal{F})$. Suppose that $\mathcal{S}(\mathcal{F}) = \mathcal{O}(\mathcal{F})$. Let $A \subseteq X$. If A is open, then $X \setminus A$ is closed. We show that $X \setminus A \in \mathcal{F}$. Suppose on contrary that $X \setminus A \notin \mathcal{F}$. Then $A \in \mathcal{O}(\mathcal{F}) = \mathcal{S}(\mathcal{F})$. Find $B \in \mathcal{F}$ with $B \subseteq A$. By Definition 2.1.1 of a filter, $A \in \mathcal{F}$ contradicting that $\mathcal{F}$ is a closed filter. If A

is closed, then $X \setminus A$ is open. We show that $A \in \mathcal{F}$. Suppose on contrary that $A \notin \mathcal{F}$. Now $X \setminus (X \setminus A) = A \in \mathcal{F}$, so that $X \setminus A \in \mathcal{O}(\mathcal{F}) = \mathcal{S}(\mathcal{F})$. By definition of $\mathcal{S}(\mathcal{F})$ there is a $D \in \mathcal{F}$ such that $D \subseteq X \setminus A$. Again by Definition 2.1.1 of a filter $X \setminus A \in \mathcal{F}$ contradicting that $\mathcal{F}$ is a closed filter.

Conversely, suppose that $\mathcal{F}$ is a closed ultrafilter. Let $A \subseteq X$. If A is open then $X \setminus A$ is closed and $X \setminus A \in \mathcal{F}$. If A is closed, then $A \in \mathcal{F}$. We want to show that $\mathcal{S}(\mathcal{F}) = \mathcal{O}(\mathcal{F})$ and it suffices to show that $\mathcal{O}(\mathcal{F}) \subseteq \mathcal{S}(\mathcal{F})$. Let $B \in \mathcal{O}(\mathcal{F})$. Then B is open and $X \setminus B \notin \mathcal{F}$. Further $X \setminus B$ is closed and since $\mathcal{F}$ is a closed ultrafilter, it follows that there is a closed set $U \in \mathcal{F}$ such that U misses $X \setminus B$. Then $U \subseteq B$ because B is the largest set that misses $X \setminus B$. That is, $U \in \mathcal{F}$ with $U \subseteq B$. Therefore $B \in \mathcal{S}(\mathcal{F})$. Hence $\mathcal{O}(\mathcal{F}) \subseteq \mathcal{S}(\mathcal{F})$ and hence equality.

(2) The proof is similar to (1).

(3) Let $\mathcal{S}(\mathcal{F})$ be an open ultrafilter. Recall that

$$\mathcal{S}(\mathcal{F}) = \{O \mid O \text{ is open and there exists } A \in \mathcal{F} \text{ with } A \subseteq O\}.$$

For any $B \subseteq X$, we have either $B \in \mathcal{S}(\mathcal{F})$ or $X \setminus B \in \mathcal{S}(\mathcal{F})$. If $B \in \mathcal{S}(\mathcal{F})$, then B is open and there exists $A \in \mathcal{F}$ such that $A \subseteq B$, then $X \setminus B$ is closed. Since $A \subseteq B \subseteq \overline{B}$, it follows that $\overline{B} \in \mathcal{F}$. If $X \setminus B \in \mathcal{S}(\mathcal{F})$. Then $X \setminus B$ is open and there exists $A \in \mathcal{F}$ such that $A \subseteq (X \setminus B)$. Since $A \subseteq (X \setminus B) \subseteq \overline{X \setminus B}$, it follows that $\overline{X \setminus B} \in \mathcal{F}$. Hence either $\overline{B} \in \mathcal{F}$ or $\overline{X \setminus B} \in \mathcal{F}$, so $\mathcal{F}$ is a closed ultrafilter.

(4) The proof is similar to (3).

(5) Let $A \in \mathcal{S}(\mathcal{F})$. Then A is open and there exists $B \in \mathcal{F}$ such that $B \subseteq A$. Then $X \setminus A$ is closed and $B \cap (X \setminus A) = \emptyset$. By normality of X there exists disjoint open sets U and V such that $B \subseteq U$ and $(X \setminus A) \subseteq V$. Now

$$\mathcal{G}(\mathcal{S}(\mathcal{F})) = \{X \setminus A \mid X \setminus A \text{ is closed and there exists } X \setminus B \in \mathcal{S}(\mathcal{F}) \text{ such that } X \setminus B \subseteq X \setminus A\}$$

and

$$\mathcal{S}(\mathcal{G}(\mathcal{S}(\mathcal{F}))) = \{A \mid A \text{ is open and there exists } X \setminus V \in \mathcal{G}(\mathcal{S}(\mathcal{F})) \text{ such that } X \setminus V \subseteq A\}.$$

So $\mathcal{S}(\mathcal{F}) \subseteq \mathcal{S}(\mathcal{G}(\mathcal{S}(\mathcal{F})))$. Then the other containment is trivial. So $\mathcal{S}(\mathcal{F}) = \mathcal{S}(\mathcal{G}(\mathcal{S}(\mathcal{F})))$.

(6) Suppose that $\mathcal{O}$ is closed-generated. That is, there exists a closed filter $\mathcal{F}$ such that every closed set K in $\mathcal{F}$ is a subset of an open set U in $\mathcal{O}$. Now $\mathcal{O}$ is an open filter, so it is a collection of open subsets of X. That is

$$\mathcal{G}(\mathcal{O}) = \{V \mid V \text{ is open and } X \setminus V \notin \mathcal{O}\}.$$

Now $\mathcal{O}$ is closed-generated, so there exists a closed set $H \subseteq V$. Now H and $X \setminus V$ are disjoint closed subset of X, so by normality of X there exists disjoint open sets U and W such that $H \subseteq U$ and $X \setminus V \subseteq W$.

$$\mathcal{G}(\mathcal{O}) = \{X \setminus V \mid X \setminus V \text{ is closed and there exists } A \in \mathcal{O} \text{ with } A \subseteq X \setminus V\}.$$

Now

$$\mathcal{S}(\mathcal{G}(\mathcal{O})) = \{V \mid V \text{ is open and there exists } X \setminus W \in \mathcal{G}(\mathcal{O}) \text{ with } X \setminus W \subseteq V\}.$$

So $\mathcal{O} \subseteq \mathcal{S}(\mathcal{G}(\mathcal{O}))$. Then the other containment is trivial. So $\mathcal{O} = \mathcal{S}(\mathcal{G}(\mathcal{O}))$.

Conversely, suppose that $\mathcal{O} = \mathcal{S}(\mathcal{G}(\mathcal{O}))$. Let $U \in \mathcal{O}$. We need to show that there is a closed set $H \subseteq U$. By hypothesis $U \in \mathcal{S}(\mathcal{G}(\mathcal{O}))$ by definition U is open and there is a closed set $K \in \mathcal{G}(\mathcal{O})$ such that $K \subseteq U$. If $\mathcal{O}$ is an open filter then $\mathcal{G}(\mathcal{O})$ is a closed filter and hence $\mathcal{O}$ is closed-generated. $\square$

Theorem 2.3.3. [21] Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be closed filters and $\mathcal{O}_1$ and $\mathcal{O}_2$ be open filters. Then

1. $\mathcal{S}(\mathcal{F}_1) = \mathcal{S}(\mathcal{F}_2)$ if and only if $b(\mathcal{F}_1) = b(\mathcal{F}_2)$.

2. $\mathcal{G}(\mathcal{O}_1) = \mathcal{G}(\mathcal{O}_2)$ if and only if $b(\mathcal{O}_1) = b(\mathcal{O}_2)$.

Proof. We only show (1) and (2) is done similarly. Let $K \in b(\mathcal{F}_1)$ be a closed set. We show that $K \in b(\mathcal{F}_2)$. Then by Theorem 2.2.4 each open filter $\mathcal{O}$ that contains K belongs to $S(\mathcal{F}_1)$ implies that each open filter $\mathcal{O}$ that contains K belongs to $S(\mathcal{F}_2)$, implies that $K \in b(\mathcal{F}_2)$. Therefore $b(\mathcal{F}_1) \subseteq b(\mathcal{F}_2)$ and by symmetry $b(\mathcal{F}_2) \subseteq b(\mathcal{F}_1)$ and hence equality.

Conversely, Let $U \in S(\mathcal{F}_1)$, then there exists $A \in \mathcal{F}_1$ with $A \subseteq U$. Then by Theorem 2.2.4 $U \in b(\mathcal{F}_1)$. So $U \in S(\mathcal{F}_2)$ because $b(\mathcal{F}_1) = b(\mathcal{F}_2)$. Therefore $U \in S(\mathcal{F}_2)$. By symmetry $S(\mathcal{F}_2) \subseteq S(\mathcal{F}_1)$ so $S(\mathcal{F}_1) = S(\mathcal{F}_2)$.

$\square$

Definition 2.3.3. Let $\mathcal{N}$ be a specified type of filter. A nonempty collection $\mathcal{C}$ will be called a balanced collection of type $\mathcal{N}$ provided:

(1) if $\mathcal{F} \in \mathcal{N}$ and $\mathcal{F} \supseteq \bigcap \mathcal{C}$ then $\mathcal{F} \in \mathcal{C}$, and

(2) if $\mathcal{F} \in \mathcal{N}$ and $\mathcal{F} \subseteq \bigcup \mathcal{C}$ then $\mathcal{F} \in \mathcal{C}$.

$\mathcal{N}$ will denote: open ultrafilters, closed ultrafilters, minimal prime open filters, or minimal prime closed filters.

Recall that an open filter $\mathcal{O}$ is called a minimal balanced open filter provided there exists a balanced collection $\mathcal{P}$ of minimal prime open filters such as $\mathcal{O} = \bigcap \mathcal{P}$. Minimal balanced closed filters are defined similarly. Every balanced open (closed) filter is the intersection of a balanced collection of open (closed) ultrafilters. Also, if $\{\mathtt{N}_\alpha \mid \alpha \in \Omega\}$ is a balanced collection of closed ultrafilters then $\mathcal{S}(\{\mathtt{N}_\alpha \mid \alpha \in \Omega\}) = \bigcap \{\mathcal{S}(\mathtt{N}_\alpha) \mid \alpha \in \Omega\}$.

The theorem below is culled in [21] and the author showed (1) and indicated that (2) can be shown similarly. Here we show (2).

Theorem 2.3.4. [21] Let X be a topological space. Let $\mathcal{F}$ be a closed filter and $\mathcal{O}$ an open filter. Then:

(1) $\mathcal{F}$ is an open-generated closed filter if and only if $\mathcal{F}$ is a minimal balanced closed filter.

(2) $\mathcal{O}$ is a closed-generated open filter if and only if $\mathcal{O}$ is a minimal balanced open filter.

Proof. (2) Let $\mathcal{O}$ be a closed-generated open filter. Then there exists a closed filter $\mathcal{F}_1$ such that $\mathcal{O} = G(\mathcal{F}_1)$. Let $\mathcal{F} = b(\mathcal{F}_1)$. Then, $\mathcal{F}$ is a balanced closed filter and by Theorem 2.3.3, $\mathcal{O} = G(\mathcal{F}_1) = G(b(\mathcal{F}_1)) = G(\mathcal{F})$. Since $\mathcal{F}$ is balanced, the family

$$\{\mathtt{N} \mid \mathcal{F} \subseteq \mathtt{N}, \mathtt{N} \text{ is a closed ultrafilter}\}$$

is a balanced collection of closed ultrafilters. Let Ω be an index set for this collection. Then $\mathcal{F} = \bigcap \{\mathtt{N}_\alpha \mid \alpha \in \Omega\}$ and

$$\mathcal{O} = G(\mathcal{F}) = \bigcap \{G(\mathtt{N}_\alpha) \mid \alpha \in \Omega\} = \bigcap \{\mathcal{O}(\mathtt{N}_\alpha) \mid \alpha \in \Omega\}.$$

Now $\{\mathcal{O}(\mathtt{N}_\alpha) \mid \alpha \in \Omega\}$ is a balanced collection of minimal prime open filters and thus $\mathcal{O}$ is a minimal balanced open filter.

Conversely, if $\mathcal{O}$ is a minimal balanced open filter there exists a balanced collection of minimal prime open filters $\{\mathcal{P}_\alpha \mid \alpha \in \Omega\}$ such that $\mathcal{O} = \bigcap\{\mathcal{P}_\alpha \mid \alpha \in \Omega\}$. Now $\{\mathcal{F}(\mathcal{P}_\alpha) \mid \alpha \in \Omega\}$ is a balanced collection of closed ultrafilters and $\mathcal{F} = \bigcap\{\mathcal{F}(\mathcal{P}_\alpha) \mid \alpha \in \Omega\}$ is a balanced closed filter. Now,

$$G(\mathcal{F}) = \bigcap\{G(\mathcal{F}(\mathcal{P}_\alpha)) \mid \alpha \in \Omega\} = \bigcap\{\mathcal{O}(\mathcal{F}(\mathcal{P}_\alpha)) \mid \alpha \in \Omega\} = \bigcap\{\mathcal{P}_\alpha \mid \alpha \in \Omega\} = \mathcal{O}.$$

Thus, $\mathcal{O}$ is a closed-generated open filter. □

2.4 Extensions

In this section we consider extensions of a topological space X. There are several extensions of a topological space X, Here we will consider Hausdorff extensions. We pay more attention to the simple and strict extensions resulting from the filter trace of Y on X.

Definition 2.4.1. A space Y is an *extension* of a topological space X, if X is a dense subspace of Y. If Y possesses some topological property P, then Y is a P- extension of X.

Let Y be a Hausdorff extension of X. For $y \in Y$,

$$\mathcal{O}^y = \{U \cap X \mid y \in U, U \in \tau_Y\}$$

is called the neighborhood filter trace of Y on X (see [3]). Then $\mathcal{O}^y$ is an open filter on X and for each $x \in X$, $\mathcal{O}^x = N_x$. For an open subset U of X,

$$oU = \{y \in Y \mid U \in \mathcal{O}^y\}.$$

Now the sets

$$\{U \cup \{y\} \mid U \in \mathcal{O}^y, y \in Y\} \text{ and } \{oU \mid U \text{ open in } X\}$$

form bases for topologies on Y and the resulting new spaces are denoted by Y^+ and $Y^\sharp$, respectively. The spaces Y^+ and $Y^\sharp$ are also Hausdorff extensions of X, and the topology of $Y^\sharp$ is coarser than the topology of Y, also the topology of Y is coarser than that of Y^+. Y is called a simple extension if $Y = Y^+$ and Y is called a strict extension if $Y = Y^\sharp$

(see [46]). The space is H-closed if X is closed in every Hausdorff space in which it is embedded, or equivalently, if every open filter is fixed. Let

$$X^* = X \cup \{\mathcal{U} \mid \mathcal{U} \text{ is a free open ultrafilter on } X\}.$$

For each open subset U of X, let

$$\mathcal{O}_U = U \cup \{\mathcal{U} \in X^* \setminus X \mid U \in \mathcal{U}\}.$$

Then X^* with the topology generated by the open basis $\{\mathcal{O}_U \mid U \text{ open in } X\}$ is an H-closed extension of X denoted by σX and called the Fomin extension of X. Again X^* with the topology generated by open basis

$$\{U \mid U \text{ open in } X\} \cup \{U \;\cup\; \{\mathcal{U}\} \mid U \text{ open in } X,\ U \in \mathcal{U},\ \mathcal{U} \in X^* \setminus X\}$$

is an H-closed extension [40] of X denoted by κX called the Katětov extension of X. Next recall that a cover (or covering) of a space X is a collection $\mathcal{A}$ of subsets of X whose union is all of X. A subcover of a cover $\mathcal{A}$ is a subcollection $\mathcal{B}$ of $\mathcal{A}$ which is a cover. An open cover of X is a cover consisting of open sets, and other adjectives applying to subsets of X apply similarly to covers. A topological space X is said to be compact if every open cover of X has a finite subcover.

Again recall that a function $f : Y \to Z$ is

(i) *irreducible* if f is onto and no proper closed subset of Y is mapped onto Z,

(ii) *compact* if $f^{-1}(z)$ is compact for each $z \in Z$, and

(iii) *perfect* if f is closed and compact (see [47]).

Theorem 2.4.1. [46] Let Y be an extension of X. Then for an open subset V of Y:

(a) $X \cap o(V \cap X) = V \cap X$;

(b) $V \subseteq o(V \cap X) \subseteq \mathrm{cl}_Y(V \cap X) = \mathrm{cl}_Y(V)$;

(c) $\mathrm{int}_Y\, \mathrm{cl}_Y(V) = \mathrm{int}_Y\, \mathrm{cl}_Y(V \cap X) = o(\mathrm{int}_X\, \mathrm{cl}_X(V \cap X))$.

The following theorem is consequences of the preceding theorem [47] and was not proved.

Theorem 2.4.2. [47] If $\mathcal{U}$ is an open ultrafilter on X and V is an open set such that $\mathrm{int}(\mathrm{cl}(V)) \in \mathcal{U}$, then $V \in \mathcal{U}$.

Proof. Let $\mathcal{F}$ be an open filter such that $V \in \mathcal{F}$.

$$\mathcal{G} = \{U \mid U \text{ is open and } \mathrm{int}(\mathrm{cl}(U)) \in \mathcal{F}\}.$$

Then by Theorem 2.2.1

$$\mathcal{G} = \bigcap \{\mathcal{U} \mid \mathcal{U} \text{ is an open ultrafilter on } X \text{ and } \mathcal{F} \subseteq \mathcal{U}\}.$$

If V is open and $\mathrm{int}(\mathrm{cl}(V)) \in \mathcal{U}$, then by Theorem 2.2.1, $V \in \mathcal{F} \vee \mathcal{D}$. Hence $V \in \mathcal{F}$, but $\mathcal{F} \subset \mathcal{U}$, so $V \in \mathcal{U}$. □

Lemma 2.4.1. [47]

(1) For each open set U, $\pi(G_U) \setminus X = O_U \setminus U$ and $\pi^{-1}(O_U \setminus U) = G_U \setminus pX$.

(2) $\pi : (\theta X \setminus pX) : \theta X \setminus pX \to \sigma X \setminus X$ is a homeomorphism.

Now we are ready for the following theorem which is taken verbatim in [47].

Theorem 2.4.3. [47] Suppose there exists a continuous function from σX onto an H-closed extension Y of X that leaves X pointwise fixed. Let $O \subseteq Y \setminus X$. The following are equivalent:

(1) O is closed in Y.

(2) O is compact.

(3) $\bigcap\{\mathcal{O}^y \mid y \in O\}$ is a free open filter and if $\bigcap\{\mathcal{O}^y \mid y \in O\}$ meets $\mathcal{O}^z$ for some $z \in Y \setminus X$ then $z \in O$.

Proof. Let $f : \sigma X \to Y$ be continuous function that leaves X point-wise fixed and $\mathcal{O} = \cap\{\mathcal{O}^y : y \in O\}$.
(1) $\Rightarrow$ (2): By Lemma 2.4.1 (2), it suffices to show that $\pi^{-1}(f^{-1}(O))$ is closed in θX since θX is compact. By Lemma 2.4.1 (2), $\pi^{-1}(f^{-1}(O))$ is closed in $\theta X \setminus pX$. Let $q \in pX$ and $\mathcal{N} \subseteq q$ where $x \in X$. Now $O_V \cap f^{-1}(O) = \emptyset$ for some $V \in \mathcal{N}_x$, implying $G_V \cap \pi^{-1}(f^{-1}(O)) = \emptyset$. Thus, $\pi^{-1}(f^{-1}(O))$ is closed in θX.

(2) $\Rightarrow$ (3): Clearly $\mathcal{O}$ is an open filter on X. Let $x \in X$. There are disjoint open sets R and S in Y such that $x \in R$ and $F \subseteq S$. Now $x \in R \cap X$, $S \cap X \in \mathcal{O}$, and $(R \cap X) \cap (S \cap X) = \emptyset$; so, $\mathcal{O}$ is free. Suppose $z \in Y \setminus X$ and $z \notin O$. There are disjoint open sets R and S in Y such that $z \in R$ and $F \subseteq S$. Now $R \cap X \in \mathcal{O}^z$ and $S \cap X \in \mathcal{O}$, implying $\mathcal{O}$ does not meet $\mathcal{O}^z$.

(3) $\Rightarrow$ (1): Since $\mathcal{O}$ is a free open filter, the closure of O in Y is contained in $Y \setminus X$. Let $z \in Y \setminus (X \cup O)$. Then $\mathcal{O}^z$ does not meet $\mathcal{O}$; so, there are disjoint open sets $\mathcal{V} \in \mathcal{O}^z$ and $V \in \mathcal{O}$. Now $o(\mathcal{V}) \cap o(V) = \emptyset$, $z \in o(\mathcal{V})$, and $O \subseteq o(V)$. So, $z \notin \mathrm{cl}_y O$. Hence, O is closed in Y. $\square$

If Y is the Fomin extension of X in the preceding theorem and since identity map on Y is continuous it follows that condition (3) to say that $\{\mathcal{O}^y \mid y \in O\}$ is a balanced collection of open ultrafilters. Then the theorem below is immediate.

Theorem 2.4.4. [21],[46] Let X be a Hausdorff topological space. Then there exists a one-to-one correspondence between the balanced free open filters on X and the nonempty closed subsets of $\sigma X \setminus X$, the remainder of the Fomin H-closed extension of X.

Definition 2.4.2. An open filter $\mathcal{O}$ is called *regular* provided $O \in \mathcal{O}$ implies there exists $Q \in \mathcal{O}$ with $\overline{Q} \subseteq O$. Equivalently, $\mathcal{O} = \mathcal{S}(\mathcal{G}(\mathcal{O}))$.

Lemma 2.4.2. [21] Let $\mathcal{O}$ be an open ultrafilter. Then $\mathcal{O}$ is regular if and only if $\mathcal{O}$ is closed-generated.

Proof. Suppose that $\mathcal{O}$ is regular. By definition, for each $O \in \mathcal{O}$ there exists $Q \in \mathcal{O}$ such that $\overline{Q} \subseteq O$. Therefore there exists a closed filter $\mathcal{F}$ such that for each $Q \in \mathcal{O}$, $\overline{Q} \in \mathcal{F}$. Therefore

$$\mathcal{O} = \{O \mid O \text{ is open and there exists } \overline{Q} \in \mathcal{F} \text{ with } \overline{Q} \subseteq O\}$$

$$\mathcal{O} = \mathcal{S}(\mathcal{F})$$

and hence $\mathcal{O}$ is closed-generated.

Conversely, suppose that $\mathcal{O}$ is closed-generated. Then, by definition, $\mathcal{O} = \mathcal{S}(\mathcal{F})$ for some closed filter $\mathcal{F}$. But

$$\mathcal{S}(\mathcal{F}) = \{O \mid O \text{ is open and there exists } A \in \mathcal{F} \text{ such that } A \subseteq O\}.$$

Since $\text{int}(A)$ is open and $\text{int}(A) \subseteq A$, there exists an open filter $\mathcal{O}$ such that $\text{int}(A) \in \mathcal{O}$ for each $A \in \mathcal{F}$. Put $\text{int}(A) = Q$. Then

$$\mathcal{O} = \{O \mid O \text{ is open and for each } Q \in \mathcal{O} \text{ we have } \overline{Q} \subseteq O\}.$$

That is, for each $O \in \mathcal{O}$ there exists $Q \in \mathcal{O}$ such that $\overline{Q} \subseteq O$. Hence $\mathcal{O}$ is regular. □

The following theorem and proof is taken verbatim in [21].

Theorem 2.4.5. [21] Let X be a regular space. The following statements are equivalent:

(1) Every free open ultrafilter is regular.

(2) Every free prime open filter is a free open ultrafilter.

Proof. (1) $\Rightarrow$ (2): Suppose $\mathcal{P}$ is a free prime open filter that is not an open ultrafilter. Then there exists an open ultrafilter $\mathcal{M}$ such that $\mathcal{P} \subseteq \mathcal{M}$ and $\mathcal{P} \neq \mathcal{M}$. Now, by (1) and Lemma 2.4.2, there exists a closed ultrafilter $\mathcal{N}$ such that $\mathcal{M} = \mathcal{O}(\mathcal{N})$. Hence $\mathcal{F}(\mathcal{P}) \supsetneqq \mathcal{F}(\mathcal{M}) = \mathcal{F}(\mathcal{O}(\mathcal{N}))$, which is impossible. Thus, $\mathcal{P}$ is a free open ultrafilter.

(2) $\Rightarrow$ (1): Let $\mathcal{M}$ be a free open ultrafilter. Set $\mathcal{K} = \mathcal{F}(\mathcal{M})$. Then $\mathcal{K}$ is a minimal prime closed filter and there exists a closed ultrafilter $\mathcal{N} \supseteq \mathcal{K}$. Let $\mathcal{P} = \mathcal{O}(\mathcal{N})$. Then, $\mathcal{P} = \mathcal{O}(\mathcal{N}) \subseteq \mathcal{O}(\mathcal{K}) = \mathcal{O}(\mathcal{F}(\mathcal{M})) = \mathcal{M}$. By condition (2), $\mathcal{S}(\mathcal{N}) = \mathcal{O}(\mathcal{N}) = \mathcal{P} = \mathcal{M}$ provided $\mathcal{P}$ is free. $\mathcal{P}$ is clearly a minimal prime open filter. In order to show that $\mathcal{P}$ is free, we first note that $\text{adh}\mathcal{K} = \bigcap \mathcal{K} \subset \{\bar{M} \mid M \in \mathcal{M}\} = \emptyset$. Similarly, $\mathcal{K} \subseteq \mathcal{N}$ implies $\mathcal{N}$ is a free closed ultrafilter. Now $\mathcal{P} = \mathcal{O}(\mathcal{N}) = S(\mathcal{N})$. Let $\chi \in \bigcap \overline{\mathcal{S}}(\mathcal{N})$. Now $\chi \notin \mathcal{N}$ and so there exist $N_\chi \in \mathcal{N}$ with $\chi \notin N$. Since X is regular there exists disjoint open sets O_χ and O_N containing χ and N, respectively. Now $O_N \in \mathcal{S}(N)$ and $\chi \notin \overline{O}_N$. Thus, $\mathcal{P}$ is a free open filter. Hence, we have that $\mathcal{M} = S(\mathcal{N})$, where $\mathcal{N}$ is a free closed ultrafilter. $\mathcal{M}$ is regular by Lemma 2.4.2. □

Lemma 2.4.3. [47] Let $f : \kappa X \to Y$ be a Katětov function of an H-closed extension Y of X. Then for each open set U in X, $o_Y U \subseteq f(\mathcal{O}_U) = U \cup (\text{cl}_Y U \setminus X)$.

Definition 2.4.3. A binary relation $\sim$on a set X is said to be an equivalence relation, if and only if it is reflexive, symmetric and transitive. That is, for all a, b and c in X:

(1) $a \sim a$. (Reflexivity)

(2) $a \sim b$ if and only if $b \sim a$. (Symmetry)

(3) if $a \sim b$ and $b \sim c$, then $a \sim c$. (Transitivity)

The following theorem [46] and proved.

Theorem 2.4.6. [46] Let R be an equivalence relation on κX such that $R(x) = \{x\}$ for $x \in X$. The following are equivalent:

(i) There is an H- closed extension Y of X such that $R = R_Y$.

(ii) Let $y, z \in \kappa X \setminus X$. Then $\bigcap R(y)$ is a free open filter on X, and if $\bigcap R(y) \subseteq z$, then $z \in R(y)$.

(iii) For $y \in \kappa X,\ R(y) = a_{\kappa X}(N_{R(y)})$.

Theorem 2.4.7. [47] Let $f : \kappa X \to Y$ be a Katětov function of an H-closed extension Y of X. f factor through σX (i.e., $f : \sigma X \to Y$ is continuous) if and only if Y is regular relative to $Y \setminus X$.

Proof. Suppose f factors through σX. Let $y \in Y$ and U an open set containing y. Then, by Theorem 2.4.3 and Theorem 2.4.6, $f^{-1}(y)$ is a compact subset of σX. It follows that there is an open set V in X such that $f^{-1}(y) \subseteq \mathcal{O}_V$ and $f(\mathcal{O}_V) \subseteq U$. If $y \in X$, then $y \in V \subseteq o_Y V$. If $y \notin X$, then $V \in \bigcap f^{-1}(y)$; since

$$f^{-1}(y) = \{z \mid z \text{ is open ultrafilter and } \mathcal{O}^y \subseteq z\},$$

then by Theorem 2.4.3, $\mathrm{int}_X V \in \mathcal{O}^y$. Then, by Theorem 2.4.1.,

$$y \in (\mathrm{int}_Y \mathrm{cl}_Y o_Y V) \setminus X = o_Y(\mathrm{int}_X \mathrm{cl}_X V) \setminus X.$$

Since $f(\mathcal{O}_V) \subseteq U$, then by Theorem 2.4.1 and Lemma 2.4.3, $(o_Y V) \cup (\mathrm{cl}_Y o_Y V) \setminus X \subseteq U$. Thus, Y is regular relative to $Y \setminus X$. Conversely, suppose Y is regular relative to $Y \setminus X$ and that $f(z) = y$. Let U be an open set containing y. There is an open set V such that

$$y \in V \cup (\mathrm{int}_Y \mathrm{cl}_Y V \setminus X) \subseteq V \cup (\mathrm{cl}_Y V \setminus X) \subseteq U.$$

Since $W = V \cap X$. If $y \in X$, then $z \in W \subseteq \mathcal{O}_W$. Suppose $y \notin X$. By Theorem 2.4.1, $W \in z$ implying $z \in \mathcal{O}_W$. By Lemma 2.4.3, $f(\mathcal{O}_W) = W \cup (\mathrm{cl}_Y W \setminus X) \subseteq V \cup (\mathrm{cl}_Y V \setminus X) \subseteq U$. This shows that f factors through σX. □

Theorem 2.4.8. [47] Let Y be an extension of X. Then Y is regular if and only if Y is regular relative to $Y \setminus X$ and $\mathcal{O}^y$ is regular filter on X for each $y \in Y$.

Proof. It is easy to verify that a regular space Y satisfies the two conditions. To prove the converse, let $y \in U \subseteq Y$ where U is open in Y. Since $U \cap X \in \mathcal{O}^y$, then there is an open set W in Y that contains y and $\mathrm{cl}_X(W \cap X) \subseteq U \cap X$ and $W \subseteq U$. There is an open set V in Y such that

$$y \in V \cup (\mathrm{int}_Y \mathrm{cl}_Y V) \setminus X \subseteq V \cup (\mathrm{cl}_Y V) \setminus X \subseteq W.$$

Now, $y \in \mathrm{int}_Y \mathrm{cl}_Y V \subseteq \mathrm{cl}_Y V = ((\mathrm{cl}_Y V) \cap X) \cup (\mathrm{cl}_Y V \setminus X)$. Since $(\mathrm{cl}_Y V) \cap X = (\mathrm{cl}_Y (V \cap X)) \cap X = \mathrm{cl}_X(V \cap X) \subseteq \mathrm{cl}_X(W \cap X) \subseteq U \cap X$, then it follows that $\mathrm{cl}_Y V \subseteq U$. Thus, X is regular. $\square$

If X is a Tychonoff space, then by Theorem 2.4.3., σX is projectively larger than any compactification of X, including the Stone-Čech compactification. So, σX is the Stone-Čech compactification of X if and only if σX is compact, or equivalently, σX is regular. Since σX is regular relative to $\sigma \setminus X$ by Theorem 2.4.7, then the following theorems is an easy consequence of Theorem 2.4.8 (see [47]).

Theorem 2.4.9. [21],[46],[47] Let X be Hausdorff space. σX is the Stone-Čech compactification of X if and only if X is regular and every free open ultrafilter is regular.

Theorem 2.4.10. [21],[46],[47] Let X be Hausdorff space. σX is the Stone-Čech compactification if and only if X is regular and every free prime open filter is a free open ultrafilter.

2.5 Minimal prime co-zero filters and balanced z-filters

In this section we consider minimal cozero filters and balanced z-filters. We start by showing the relationship between minimal prime cozero filters and the z-ultrafilters. In particular, if $\mathcal{M}$ is a z-ultrafilter on a completely regular space X, then the set of the complements of members of $\mathcal{M}$ is a minimal prime co-zero ideals on X and vice-versa. We also show that there is a one-to-one correspondence between the closed sets in Stone-Čech compactification (βX) and the balanced z-ultrafilters in X.

Lemma 2.5.1. Let $\mathcal{F}$ be a closed filter and $\mathcal{G} = \{O \in \tau \mid X \smallsetminus O \notin \mathcal{F}\}$. Then:

(1) $\mathcal{F}$ is a closed ultrafilter if and only if $\mathcal{G}$ is a minimal prime open filter.

(2) $\mathcal{G}$ is an open ultrafilter if and only if $\mathcal{F}$ is a minimal prime closed filter.

Theorem 2.5.1. [20] Let $\mathcal{M}_p$ be a z-ultrafilter on a completely regular space X. Set

$$\mathcal{V}_p = \{V \in \text{co} - Z(X) \mid X \smallsetminus V \notin \mathcal{M}_p\}.$$

Then $\mathcal{V}_p$ is a minimal prime co-zero filter on X.

Proof. Easily $\emptyset \notin \mathcal{V}_p$ and if $V \in \mathcal{V}_p$ and $V \subseteq W \in \text{co} - Z(X)$ then $X \smallsetminus W \subseteq X \smallsetminus V \notin \mathcal{M}_p$ and thus $X \smallsetminus W \notin \mathcal{M}_p$. Hence $W \in \mathcal{V}_p$. Let $V_1 \in \mathcal{V}_p$ and $V_2 \in \mathcal{V}_p$. Then $X \smallsetminus V_1 \in \mathcal{M}_p$ and $X \smallsetminus V_2 \in \mathcal{M}_p$ and thus $X \smallsetminus (V_1 \cap V_2) = (X \smallsetminus V_1) \cup (X \smallsetminus V_2) \notin \mathcal{M}_p$. Therefore, $V_1 \cap V_2 \in \mathcal{V}_p$ and $\mathcal{V}_p$ is a co-zero filter. To see that $\mathcal{V}_p$ is prime let $V_1 \cup V_2 \in \mathcal{V}_p$. Then $X \smallsetminus (V_1 \cup V_2) \notin \mathcal{M}_p$. If $V_1 \notin \mathcal{V}_p$ and $V_2 \notin \mathcal{V}_p$ it follows that $X \smallsetminus V_1 \in \mathcal{M}_p$ and $X \smallsetminus V_2 \in \mathcal{M}_p$ and thus $X \smallsetminus (V_1 \cup V_2) = (X \smallsetminus V_1) \cap (X \smallsetminus V_2) \in \mathcal{M}_p$, a contradiction. Thus, $\mathcal{V}_p$ is a prime co-zero filter.

Suppose $\mathcal{W}$ is a prime co-zero filter with $\mathcal{W} \subseteq \mathcal{V}_p$. Then,

$$\mathcal{N} = \{Z \in Z(X) \mid X \smallsetminus Z \in \mathcal{W}\}$$

is a prime z-filter. Moreover, $\mathcal{M}_p \subset \mathcal{N}$ and since $\mathcal{M}_p$ is a z-ultrafilter $\mathcal{M}_p = \mathcal{N}$. Thus, $\mathcal{W} = \mathcal{V}_p$ and $\mathcal{V}_p$ is a minimal prime co-zero filter on X. □

Theorem 2.5.2. [20] Let X be a completely regular topological space and let $\mathcal{V}$ be a minimal prime co-zero filter. Set

$$\mathcal{M} = \{Z \in Z(X) \mid X \smallsetminus Z \notin \mathcal{V}\}.$$

Then, $\mathcal{M}$ is a z-ultrafilter.

Definition 2.5.1. Let $\mathcal{F}$ be a zero filter on X. $\mathcal{F}$ is called a *balanced zero filter* provided $\mathcal{F}$ is the intersection of all the z-ultrafilters that contain it. That is; $\mathcal{F} = \bigcap\{\mathcal{M}_p \mid \mathcal{F} \subseteq \mathcal{M}_p\}$.

Theorem 2.5.3. [20] Let G be a nonempty closed set in βX. Set $\mathcal{F} = \{Z \in Z(X) \mid G \subseteq \text{cl}_{\beta X} Z\}$. Then:

(1) $G = \bigcap\{\mathrm{cl}_{\beta X} Z \mid Z \in \mathcal{F}\}$.

(2) $\mathcal{F}$ is a balanced z-filter.

Proof. (1) Follows since $\{\mathrm{cl}_{\beta X} Z \mid Z \in Z(X)\}$ is a base for the closed sets in βX. (A parallel argument is that βX is a strict extension of X and the zero sets in X form a base for the closed sets in X).
(2) Easily $\emptyset \notin \mathcal{F}$ and if $Z \in \mathcal{F}$ with $Z \subseteq Z^{'}$ then $Z^{'} \in \mathcal{F}$. If $Z_1 \in \mathcal{F}$ and $Z_2 \in \mathcal{F}$ then since $\mathrm{cl}_{\beta X} Z_1 \cap \mathrm{cl}_{\beta X} Z_2 = \mathrm{cl}_{\beta X}(Z_1 \cap Z_2)$, we have that $G \subseteq \mathrm{cl}_{\beta X}(Z_1 \cap Z_2)$ and $Z_1 \cap Z_2 \in \mathcal{F}$. Thus, $\mathcal{F}$ is a z-filter.

To see that $\mathcal{F}$ is balanced, it suffices to let $Z \in \mathcal{M}_p$ for each $\mathcal{M}_p$ containing $\mathcal{F}$ and show that $Z \in \mathcal{F}$. Let $Z \in \bigcap\{\mathcal{M}_p \mid \mathcal{F} \subseteq \mathcal{M}_p\}$. Then $p \in \mathrm{cl}_{\beta X} Z$ for each $\mathcal{M}_p$ containing $\mathcal{F}$.

Let $q \in G$. Then $q \in \mathrm{cl}_{\beta X} Z$ for each $Z \in \mathcal{F}$. Hence $\mathcal{F} \subseteq \mathcal{M}_q$. Thus, if $Z \in \cap\{\mathcal{M}_p \mid \mathcal{F} \subseteq \mathcal{M}_p\}$ we have that $G \subseteq \mathrm{cl}_{\beta X} Z$. Therefore $Z \in \mathcal{F}$ and $\mathcal{F}$ is a balanced z-filter. □

The following theorem now follows immediately.

Theorem 2.5.4. [20] Let X be a completely regular topological space. Then there exists a one-to-one correspondence between the nonempty closed sets in βX and the balanced z-filter on X. The correspondence is given by: $G \longleftrightarrow \mathcal{F} = \{Z \mid G \subseteq \mathrm{cl}_{\beta X} Z\}$.

Corollary 2.5.1. [20] Let $\mathcal{O}$ be an open set in βX and $\mathcal{O} = \beta X$. Then there exists a balanced z-filter $\mathcal{F}$ such that $\mathcal{O} = \{p \in \beta X \mid \mathcal{F} \subsetneq \mathcal{M}_p\}$.

Corollary 2.5.2. [20] There exists a natural one-to-one correspondence between the nonempty closed subsets of βN and the filters on N.

Corollary 2.5.3. [20] Let X be a completely regular space. Then there exists a one-to-one correspondence between the zero sets in βX and the balanced z-filters on X with a countable base.

Proof. It is known, (6E) in [38], that each nonempty zero set in βX is the countable intersection of sets of the form $\mathrm{cl}_{\beta X} Z$, where $Z \in Z(X)$. The result now follows from Theorem 2.5.4 above. □

Theorem 2.5.5. [20] Let G be a nontrivial open-closed subset of βX. Then there exist balanced z-filters $\mathcal{F}$ and $\mathcal{G}$ such that each z-ultrafilter contains one and only one of these filters.

Lemma 2.5.2. [20] Let X be a completely regular space with F_1 and F_2 nonempty closed subsets of βX. For $i = 1, 2$; let $O_i = \beta X \smallsetminus F_i$ and $\mathcal{F}_i = \{Z \in Z(X) \mid F_i \subseteq \mathrm{cl}_{\beta X} Z\}$. The following statements are equivalent:

(1) $F_1 \subseteq F_2$.

(2) $O_2 \subseteq O_1$.

(3) $\mathcal{F}_2 \subseteq \mathcal{F}_1$.

Proof. By Theorem 2.5.4 above, $F_i = \{p \in \beta X \mid \mathcal{F}_i \subseteq \mathcal{M}_p\}$ for $i = 1, 2$. Suppose $F_1 \subseteq F_2$. Let $Z \in \mathcal{F}_2$. Then $F_2 \subseteq \mathrm{cl}_{\beta X} Z$ and thus $F_1 \subseteq \mathrm{cl}_{\beta X} Z$ and $Z \in \mathcal{F}_1$. Suppose $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Let $p \in F_1$, then $\mathcal{F}_1 \subseteq \mathcal{M}_p$ and thus $\mathcal{F}_2 \subseteq \mathcal{M}_p$ and $p \in F_2$. $\square$

Definition 2.5.2. A point p in a topological space X is called a *P-point* if for each countable collection $\{O_i\}$ of open sets containing p there exists an open set O with $p \in O \subset \bigcap O_i$.

Theorem 2.5.6. [20] Let X be a completely regular topological space and $\mathcal{M}_p$ a z-ultrafilter on X. A point p is a p-point in βX if and only if for each countable collections $\{\mathcal{F}_i\}$ of balanced z-filter F such that $\mathcal{F} \subsetneq \mathcal{M}_p$ and for each i, $\mathcal{F} \subseteq \mathcal{F}_i$.

Proof. Let p be a p-point in βX, and $\{\mathcal{F} \mid i \in N\}$ a countable collection of balanced z-filters with $\mathcal{F}_i \subsetneq \mathcal{M}_p$. Set $F_i = \bigcap \mathrm{cl}_{\beta X} \mathcal{F}_i$. Then F_i is a closed set with $p \notin F_i$. Then $O_i = \beta X \smallsetminus F_i$ is an open set containing p for each $i \in N$. Since p is a p-point in βX, there exists an open set O in βX with $p \in O \subseteq \bigcap O_i$. Let $F = \beta X \smallsetminus O$ and $\mathcal{F}$ the corresponding balanced z-filter on X. Then for each $i \in N$, $F \subseteq F_i$ and, by Lemma 2.5.2, $\mathcal{F} \subseteq \mathcal{F}_i$. Easily, $\mathcal{F} \subsetneq \mathcal{M}_p$. The proof in the other direction follows in a similar manner. $\square$

Let $\{\mathcal{F}_i\}$ be a countable collection of filters on N. Then $\bigcap \mathcal{F}_i$ is a filter on N and we have the following corollary.

Corollary 2.5.4. [20] Let N denote the natural numbers with the discrete topology. Let $\mathcal{M}$ be an ultrafilter on N. Then $\mathcal{M}$ is a p-point in βX if and only if for each countable family of filters $\{\mathcal{F}_i\}$ on N, with each $\mathcal{J}_i \subsetneq \mathcal{M}$, then $\bigcap \mathcal{F}_i \subsetneq \mathcal{M}$.

2.6 Nearness spaces

Definition 2.6.1. Let X be a set and ξ a collection of covers of X, called *uniform covers.* Then (X, μ) is a *nearness space* provided:

(1) $\mathcal{A} \in \mu$ and $\mathcal{A}$ refines $\mathcal{B}$ implies $\mathcal{B} \in \mu$.

(2) $X \in \mu$ and $\emptyset \notin \mu$.

(3) If $\mathcal{A} \in \mu$ and $\mathcal{B} \in \mu$ then $\mathcal{A} \wedge \mathcal{B} = \{A \cap B \mid A \in \mathcal{A} \text{ and } B \in \mathcal{B}\} \in \mu$.

(4) $\mathcal{A} \in \mu$ implies $\operatorname{int}(A) = \{X \mid X \searrow x, A\} \in \mu$.

For a given nearness space (X, μ) the collection of sets that are "near" is given by $\xi = \{\mathcal{A} \subseteq \mathcal{P}(X) \mid \{X \setminus A \mid A \in \mathcal{A}\} \notin \mu\}$. The micromeric collections are given by $\mathcal{A} \in \gamma$ if and only if $\{B \subseteq X \mid A \cap B \neq \emptyset$ for each $A \in \mathcal{A}\} \in \xi$. The closure operator generated by a nearness space is given by $cl_\xi A = \{\mathbf{x} \mid \{\{\mathbf{x}\}, A\} \in \xi\}$. If we are primarily using these "near" collections we will denote the nearness space by (X, ξ). The underlying topology of a nearness space is always symmetric; that is, $\mathbf{x} \in \overline{\{\mathbf{y}\}}$ implies $\mathbf{y} \in \overline{\{\mathbf{x}\}}$.

Definition 2.6.2. Let (X, ξ) be a nearness space. The nearness space is called:

(1) *topological* provided $\mathcal{A} \in \xi$ implies $\bigcap \bar{\mathcal{A}} \neq \emptyset$.

(2) *complete* provided each ξ-cluster is fixed; that is $\bigcap \bar{\mathcal{A}} \neq \emptyset$ for each maximal element $\mathcal{A} \in \xi$.

(3) *concrete* provided each near collection is contained in some ξ-cluster.

(4) *contigual* provided $\mathcal{A} \in \xi$ implies there exists a finite $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B} \notin \xi$.

(5) *totally bounded* provided $\mathcal{A} \notin \xi$ implies there exists a finite $\mathcal{B} \subseteq \mathcal{A}$ such that $\bigcap \mathcal{B} = \emptyset$.

Recall that an extension of a topological space X is a dense embedding $e : X \to Y$ of X into a topological space Y. Since for every nonempty space X there exists a proper class of essentially different extensions, it is natural to restrict attention to a narrower class of extensions, still large enough to contain the more interesting extensions usually encountered in topology. The restriction to T_2-extensions, i.e. extensions $e : X \to Y$ with Y being T_2-space, though natural, is not stringent enough, since every infinite T_2-space still has a proper class of essentially different T_2-extensions. Among all extensions of a fixed space X which induce the same filter trace (= set of trace filters of neighborhood filters of single points) on X. An extension $e : X \to Y$ such that $\{\mathrm{cl}_Y(eA) \mid A \subseteq X\}$ is a base for the closed sets in Y.

To characterize the latter, some more information concerning nearness space is needed. If $X = (X, \xi)$ is a nearness space then a non-empty collection of subsets of X is called an X*-cluster* provided it is a maximal (with respect to inclusion) X-near collection. Every X-cluster is easily seen to be a grill on X. A nearness space is called complete provided $\bigcap \mathrm{cl}_X \mathrm{G} \neq \emptyset$ for any X-cluster G.

Definition 2.6.3. Let (X, ξ) be a nearness space. Let $\mathcal{A} \subseteq \rho(X)$ and let $c(\mathcal{A})$ denote the set of all ξ- clusters that contain $\mathcal{A}$. Let $\rho(\mathcal{A})$ denote the set of all the fixed ξ- clusters that contain $\mathcal{A}$. Set $b(\mathcal{A}) = \bigcap \mathrm{c}(\mathcal{A})$. $\mathcal{A}$ is said to be a *balanced near collection* provided $\mathcal{A} \in \xi$, $\mathcal{A} = \bigcap \mathrm{c}(\mathcal{A})$ and $\mathcal{A}$ is called a *rigid near collection* provided $\mathcal{A} \in \xi$ and $\mathcal{A} = \bigcap \rho(\mathcal{A})$.

Theorem 2.6.1. Let (X, ξ) be a concrete nearness space and $\mathcal{A} \in \xi$.

(1) $\mathcal{A} \in \xi$.

(2) $b(\mathcal{A})$ is the smallest balanced near collection containing $\mathcal{A}$.

(3) Each ξ-clusters is a balanced near collection.

(4) If η is a nonempty collection of ξ - clusters then $\bigcap \eta$ is a balanced near collection.

(5) If $\mathcal{A}$ is a rigid near collection then $\mathcal{A}$ is a balanced near collection.

Theorem 2.6.2. Let Y be a T_2-strict extension of X. Set $\xi = \{\mathcal{A} \subseteq \rho(X) \mid \bigcap \mathrm{cl}_Y \mathcal{A} \neq \emptyset\}$

(1) If $\emptyset \neq F \subseteq Y$ then $\mathrm{c}(\mathcal{S}(F)) = \{\mathcal{A}_y \mid y \in \mathrm{cl}_Y F\}$ and $\mathcal{S}(F)$ is a balanced near collection.

(2) If $\mathcal{A}$ is a balanced near collection then there exists a closed set F in Y such that $\mathcal{A} = \mathcal{S}(F)$.

(3) If $\emptyset \neq F \subseteq Y$ then $\mathcal{S}(F) = \mathcal{S}(\mathrm{cl}_Y F)$.

(4) If F and E are nonempty subsets of Y then $\mathcal{S}(F \cup E) = \mathcal{S}(F) \cap \mathcal{S}(E)$.

(5) There exists a one-to-one correspondence between the nonempty closed subsets of Y and the balanced near collections.

Proof. (1) Let $\emptyset \neq F \subseteq Y$. Let $U \in \mathrm{c}(\mathcal{S}(F))$.

$$\Rightarrow U \text{ is an } \xi - \text{cluster that contain } \mathcal{S}(F).$$
$$\Rightarrow U \subseteq \rho(X) \text{ and } \{A \subseteq X \mid F \subseteq \mathrm{cl}_Y A\} \subseteq U.$$
$$\Rightarrow U \in \{A \subseteq X \mid F \subseteq \mathrm{cl}_Y A\} \text{ for some } y \in F.$$
$$\Rightarrow \mathrm{c}(\mathcal{S}(F)) \subseteq A_y \text{ for some } y \in F.$$

Conversely, let $U \in \mathcal{A}_y$.

$$\Rightarrow U \in \{A \subseteq X \mid y \in \mathrm{cl}_Y A\} \text{ for some } y \in F.$$
$$\Rightarrow U \in \{A \subseteq X \mid F \subseteq \mathrm{cl}_Y A\}.$$
$$\Rightarrow \{A \subseteq X \mid F \subseteq \mathrm{cl}_Y A\} \subseteq U.$$
$$\Rightarrow U \text{ is an } \xi - \text{cluster that contain } \mathcal{S}(F).$$
$$\Rightarrow U \in \mathrm{c}(\mathcal{S}(F)).$$

and hence $\mathrm{c}(\mathcal{S}(F)) = \{\mathcal{A}_y \mid y \in \mathrm{cl}_Y F\}$. Hence $\mathcal{S}(F)$ is a balanced near collection.

(2) Follows immediately from (1).

(3) From (1) and (2), we have $\mathcal{S}(F) = \mathcal{S}(\mathrm{cl}_Y F)$.

(4) If F and E are nonempty subsets of Y.

$$\begin{aligned}
\mathcal{S}(F \cup E) &= \{A \subseteq X \mid F \cup E \subseteq \mathrm{cl}_Y A\}.\\
&= \{A \subseteq X \mid F \subseteq \mathrm{cl}_Y A \text{ and } E \subseteq \mathrm{cl}_Y A\}.\\
&= \{A \subseteq X \mid F \subseteq \mathrm{cl}_Y A\} \cap \{A \subseteq X \mid E \subseteq \mathrm{cl}_Y A\}.\\
&= \mathcal{S}(F) \cap \mathcal{S}(E).
\end{aligned}$$

(5) From (1) and (2) it follows that there is a one-to-one correspondence between the nonempty closed subsets of Y and the balanced near collections. $\square$

Theorem 2.6.3. Let Y be a strict T_2-extension of X and ξ the nearness structure on X induced by Y. Let $\mathcal{A} \subseteq \rho(X)$.

(1) If $T = \{x \mid \mathcal{A}_x \in \rho(\mathcal{A})\}$ then $T = \bigcap \mathrm{cl}_X \mathcal{A}$.

(2) $\mathcal{A}$ is a rigid near collection if and only if there exists a nonempty closed set T in X such that $\mathcal{A} = \{A \subseteq X \mid T \subseteq \mathrm{cl}_X A\}$.

(3) $\mathcal{A}$ is a rigid near collection if and only if

  (A) $\mathcal{A}$ is a balanced near collection, and

  (B) $\mathrm{cl}_Y(\bigcap \mathrm{cl}_X \mathcal{A}) = \bigcap \mathrm{cl}_Y \mathcal{A}$.

(4) Let F be a nonempty closed subset of Y. The following statements are equivalent.

  (A) $F = \mathrm{cl}_Y(F \cap X)$.

  (B) There exists a unique closed subset of X, say G, such that $F = \mathrm{cl}_Y G$.

  (C) $\mathcal{S}(F)$ is a rigid near collection.

Proof. (1) If $T = \{x \mid \mathcal{A}_x \in \rho(\mathcal{A})\}$ then $T = \bigcap \mathrm{cl}_X \mathcal{A}$. Recall that $\rho(\mathcal{A})$ denotes the set of all the fixed ξ-clusters that contain $\mathcal{A}$, so $\mathcal{A} \subseteq \rho(X)$. $\mathcal{A}_x = \{A \subseteq X \mid x \in \mathrm{cl}_X A\} \in \rho(\mathcal{A})$. Now $\mathcal{A}_x \in \rho(\mathcal{A})$ implies that $x \in \mathrm{cl}_x A$ because $\bigcap \overline{A} \neq \emptyset$. Now

$$\begin{aligned}
\mathcal{A}_x \in \rho(\mathcal{A}) &\Rightarrow \mathcal{A} \subseteq \mathcal{A}_x. \\
&\Rightarrow x \in \mathrm{cl}_X A \Rightarrow x \in \mathrm{cl}_X \mathcal{A}. \\
&\Rightarrow x \in \bigcap \mathrm{cl}_X A \Rightarrow x \in \bigcap \mathrm{cl}_X \mathcal{A}. \\
&\Rightarrow T \subseteq \bigcap \mathrm{cl}_X \mathcal{A}.
\end{aligned}$$

Now

$$\begin{aligned}
&x \in \bigcap \mathrm{cl}_X \mathcal{A}. \\
&\Rightarrow x \in \mathrm{cl}_X A \text{ for every } A \in \mathcal{A}. \\
&\Rightarrow \mathcal{A}_x \text{ is a fixed } \xi- \text{ cluster}. \\
&\Rightarrow \mathcal{A}_x \in \rho(\mathcal{A}). \\
&\Rightarrow x \in T.
\end{aligned}$$

So $\bigcap \mathrm{cl}_X \mathcal{A} \subseteq T$ and hence equality.

(2) By definition $\mathcal{A}$ is a rigid near collection provided $\mathcal{A} \in \xi$ and $\mathcal{A} = \bigcap \rho(\mathcal{A})$. Now from (1), we have $T = \bigcap \mathrm{cl}_X \mathcal{A}$, $\mathcal{A}_x \in \rho(\mathcal{A})$. Since the arbitrary intersection of closed sets in closed, it follows T is a closed set. Also $T \subseteq \mathrm{cl}_X \mathcal{A}$ for every ξ-cluster $\mathcal{A}$. Hence $\mathcal{A} = \{A \subseteq X \mid T \subseteq \mathrm{cl}_X A\}$.

Conversely, suppose there exists a nonempty closed set T in X such that $\mathcal{A} = \{A \subseteq X \mid T \subseteq \mathrm{cl}_X A\}$. Then

$$\begin{aligned}
\mathcal{A} \text{ is a fixed } \xi - \text{cluster since } T \in \bigcap \mathrm{cl}_X A. \\
\Rightarrow \mathcal{A} \in \rho(\mathcal{A}).
\end{aligned}$$

Now from (1) if $T = \bigcap \mathrm{cl}_X A$ for every $A \in \mathcal{A}$, we have $\mathcal{A} = \bigcap \rho(\mathcal{A})$ and hence $\mathcal{A}$ is a rigid near collection.

From (2),

$$\begin{aligned}
&T = \bigcap \mathrm{cl}_X \mathcal{A}. \\
&\Rightarrow \mathrm{cl}_Y T = \mathrm{cl}_Y(\bigcap \mathrm{cl}_X \mathcal{A}). \\
&\Rightarrow \bigcap \mathrm{cl}_Y \mathcal{A} = \mathrm{cl}_Y(\bigcap \mathrm{cl}_X \mathcal{A}).
\end{aligned}$$

If A and B, then it follows immediately that $\mathcal{A}$ is a rigid near collection.

(4) $(A) \Leftrightarrow (B)$: Put $G = F \cap X$.

$(A) \Rightarrow (C)$: $\mathcal{S}(F) = \{A \subseteq X \mid F \subseteq \mathrm{cl}_Y A\}$ is a balanced near collection. That $\mathcal{S}(F) = \mathcal{A}$, where $\mathcal{A}$ is a balanced near collection. Now, from

$$\begin{aligned}
&F = \mathrm{cl}_Y(F \cap X) = \mathrm{cl}_Y(\mathrm{cl}_X F). \\
&\Rightarrow \bigcap \mathrm{cl}_Y F = \mathrm{cl}_Y(\bigcap \mathrm{cl}_X F).
\end{aligned}$$

which is statement (B) in part (3) above. Hence $\mathcal{S}(F)$ is a rigid near collection.

$(C) \Rightarrow (A)$:

$$\begin{aligned}
&\bigcap \mathrm{cl}_Y(\mathcal{S}(F)) = \mathrm{cl}_Y(\bigcap \mathrm{cl}_X \mathcal{S}(F)).\\
&\Rightarrow \mathrm{cl}_Y(\mathcal{S}(F)) = \mathrm{cl}_Y(\mathrm{cl}_X \mathcal{S}(F)).\\
&\Rightarrow \mathrm{cl}_Y(F) = \mathrm{cl}_Y(\mathrm{cl}_X F).\\
&\Rightarrow F = \mathrm{cl}_Y(F \cap X).
\end{aligned}$$

□

The following theorem has been taken as verbatim in [14].

Theorem 2.6.4. [14], [20] For any T_2-nearness space (X, ξ) the following conditions are equivalent:

(1) ξ is a nearness structure induced on X by a strict extension.

(2) The completion X^* of X is topological.

(3) ξ is concrete.

Proof. (1) $\Rightarrow$ (3): If ξ is induced by the strict extension $e : X \to Y$ and if G is a non-empty X-near collection, then there exists some $y \in \bigcap \mathrm{cl}_Y$. The collection $\mathfrak{L}_Y$ of all subsets of X whose Y-closure contains y is easily seen to be an X-cluster.

(3) $\Rightarrow$ (2): For any non-empty X^*-near collection $\mathfrak{L}$, the collection of all subsets A of X for which there exists a $B \in \mathfrak{L}$ with $B \subseteq \mathrm{cl}^*_X A$ is an X-near collection and hence is contained in an X-cluster G. If $x \in \bigcap \mathrm{cl}_X G$ then $x \in \bigcap \mathrm{cl}_{X^*} \mathfrak{L}$ and if $\bigcap \mathrm{cl}_X G = \varphi$ then $G \in \bigcap \mathrm{cl}_{X^*} \mathfrak{L}$.

(2) $\Rightarrow$ (1): If $e : X \to X^*$ is the completion then $e : X \to TX^*$ is a strict extension, inducing the nearness structure ξ on X. □

Chapter 3

Ideals and filters in pointfree topology

The dual notion of a filter in topological spaces is an ideal. Ideals, just like filters, play a vital role in topological spaces and rings. In pointfree topology Dube in [28] introduced balanced filters, closed-generated filters, open-generated ideals and stably closed-generated filters in frames. Dube and Mugochi in [34] introduced the notion of a balanced ideal in frames. We introduce the notion of stably open-generated ideals in frames and show that an ideal is regular if and only if it is stably open-generated. Dube has shown as an application that a frame is normal if and only if its closed-generated filters are precisely the stably closed-generated ones (see [28]). By dualising Dube's statement we show that a frame is extremally disconnected if and only if its open-generated ideals are precisely the stably open-generated ones. We show that there is a one-to-one correspondence between points of βL and the balanced ideals of Coz L. Furthermore we study locally finite nearness frames, Pervin nearness frames and fine nearness frames. We end the chapter with strict extensions in frames and introduce a notion of a Baire frame.

3.1 Filters and ideals

Definition 3.1.1. A subset F of L is called a *filter* if it is satisfies the following

(1) $1 \in F$,

(2) $a, b \in F \Rightarrow a \wedge b \in F$,

(3) $a \leq b$ and $a \in F \Rightarrow b \in F$.

A filter F of L is said to be a *maximal filter* if for any filter G,

$$F \subseteq G \subseteq L \Rightarrow F = G \text{ or } G = L.$$

Definition 3.1.2. A subset I of L is called an *ideal* if it satisfies the following

(1) $0 \in I$,

(2) $a, b \in I \Rightarrow a \vee b \in I$,

(3) $a \leq b$ and $b \in I \Rightarrow a \in I$.

An ideal I of L is said to be a *maximal ideal* if for any ideal J,

$$I \subseteq J \subseteq L \Rightarrow I = J \text{ or } J = L.$$

In Definition 2.1.2 we notice that α is in general the collection of subsets of a topological space X. If α is a collection of either open or closed subsets of X statements (1) and (2) cannot be expressed solely in terms of open sets and hence cannot be extended to frames. For all results which we have shown in topological spaces using statement (1) and (2) of Definition 2.1.2 we will not attempt them in frames. However we point that the statements (1) and (2) hold in the cases where α is an ideal or a filter.

Lemma 3.1.1. [28] Let $F \subseteq L$ be a filter and $0 \neq x \notin F$. Then there is a prime filter G such that $x \notin G$ and $F \subseteq G$.

Proof. The principal ideal generated by x misses F, so by Stone's Separation Lemma there is a prime ideal J that misses F and contains $\downarrow x$. Then $G = L \setminus J$ is a prime filter with the required property. $\square$

3.2 Balanced filters

Unlike in spaces, every filter in a frame is the intersection of prime filters containing it. This is captured by the following proposition which is taken as is from [28].

Proposition 3.2.1. [28] Every filter in a frame is the intersection of prime filters containing it.

Proof. Let F be a filter in a frame L and P be the collection of all prime filters containing F. Then $F \subseteq \bigcap \mathrm{P}$. On the other hand let $x \in \bigcap \mathrm{P}$. If $x \notin F$, then let $F \subseteq L$ be a filter and $0 \neq x \in F$. Then there is a prime filter G such that $x \notin G$ and $F \subseteq G$ there exists $Q \in P$ such that $x \notin Q$, which is a contradiction since $x \in \bigcap \mathrm{P}$. □

Definition 3.2.1. The *balance of a filter* F in a frame L is the filter $b(F) = \bigcap\{P \subseteq L \mid P$ is an ultrafilter containing $F\}$. We say F is balanced if $F = b(F)$.

We remark that the definition of a balance of an ideal is not the consequence of the dualisation of the above definition. The definition as introduced by Dube and Mugochi in [34] will be studied in the section.

Proposition 3.2.2. [28] A filter is an ultrafilter if and only if it is prime and balanced.

Proof. Suppose that a filter F is an ultrafilter. By Proposition 3.2.1, F is the intersection of all prime filters containing F. Furthermore F is an ultrafilter, so it is maximal in terms of containment, so $F = b(F)$. Hence F is prime and balanced.
Conversely, suppose a filter G is prime and balanced. By definition of a balanced filter, G is an ultrafilter. □

Dualisation of the preceding proposition does not hold in general.

The following proposition is found in [28] together with its proof. Here we only give the statement and leave out the proof.

Proposition 3.2.3. [28] For any filter F in a frame L we have that

(a) $b(F) = \{x \in L \mid x^{**} \in F\}$.

(b) $b(F) = \langle F \cup D(L) \rangle$.

Corollary 3.2.1. [28] A filter is balanced if and only if it contains all the dense elements.

Proof. By the definition of balanced filters which contains all the dense elements.

conversely, Let F be a filter containing all the dense elements. Since $F \subseteq b(F)$ we need to show that $b(F) \subseteq F$. If $x \notin F$, then every filter in a frame is the intersection of prime filters containing it there is a prime filter $P \supseteq F$ with $x \notin P$. So, A filter is an ultrafilter if and only if it is prime and balanced, P is an ultrafilter that does not containing x; therefore $x \notin b(F)$. □

The following definitions were also introduced by Dube in [28].

Definition 3.2.2. A filter F in a frame L is *closed-generated* if there is an ideal J in L such that $F = \{x \in L \mid x \vee y = 1 \text{ for some } y \in J\}$. In this case we shall write $F = \gamma(J)$, and call such an ideal J a witness for F.

Dually,

Definition 3.2.3. An ideal I in a frame L is *open-generated* if, for some filter F, it is of the form $\delta(F) = \{x \in L \mid x \wedge y = 0 \text{ for some } y \in F\}$ and call F a witness for the ideal I.

The following result is taken verbatim as in [28], here we include the proof for the sake of completeness. The author took the result from Gratzer's book but stated in a less general form.

Lemma 3.2.1. [28] A prime ideal J in a frame L is a minimal prime ideal if and only if it contains no dense elements.

Proof. Suppose J is a minimal prime ideal in L. Take I to be the prime ideal in L. Suppose on contrary that J contains a dense element u. Then $u^* = 0 \in J$. So we have $u, u^* \in J$. For any $a \in I$, we have $u \wedge a \in J$ and $a \neq 1$ because I is an ideal. Hence $u^* \vee (u \wedge a) = (u^* \vee u) \wedge (u^* \vee a) = (u^* \vee u) \wedge a = a \in J$. So $I \subseteq J$. But I is a prime ideal, so this contradicts that J is a minimal prime ideal. Therefore J contains no dense elements. Conversely, suppose J contains no dense elements. We want to show that J is a minimal prime ideal. We claim that the maximal $w \in L$ such that w is not dense is in J. If not, there exist $y \in L$, $y \notin J$ and y not dense $w < y$. Then $y^* \notin J$ because J as an ideal is a downset. Now $y \wedge y^* = 0 \in J$ which contradict the primeness of J. Indeed a maximal element $w \in L$ such that w is not dense is an element of J. Hence J is a minimal prime ideal. □

Corollary 3.2.2. [28] A filter F in a frame L is an ultrafilter if and only if $L \smallsetminus F$ is a minimal prime ideal.

Proof. If F is an ultrafilter then it is a prime filter (so that $L \smallsetminus F$ is a prime ideal) and it contains all the dense elements, so that $(L \smallsetminus F) \cap D(L) = \emptyset$. Therefore $L \smallsetminus F$ is a minimal prime ideal by Lemma 3.2.1. Conversely, since $L \smallsetminus F$ is a prime ideal F is a prime filter, and since $(L \smallsetminus F) \cap D(L) = \emptyset$ we have that F is balanced by Proposition 3.2.2. □

Corollary 3.2.3. Let L be a frame. Then the following statements are equivalent:

(1) Every prime filter is an ultrafilter.

(2) Every prime ideal is a minimal prime ideal.

Proof. (1) $\Rightarrow$ (2): Suppose that J is a prime ideal. Then $L \smallsetminus J$ is a prime filter and since $(L \smallsetminus J)$ is a prime filter, it is an ultrafilter by (1). Hence $L \smallsetminus (L \smallsetminus J) = J$ is a minimal prime ideal.

(2) $\Rightarrow$ (1): Suppose that F is a prime filter. Then $(L \smallsetminus F)$ is a prime ideal and so by (2) is a minimal prime ideal and by Corollary 3.2.2, $L \smallsetminus (L \smallsetminus F) = F$ is an ultrafilter. □

The following lemma is culled from [28], the author proved the first statement and indicated that the second statement can be shown in the similar fashion. Here we include proofs for both statements.

Lemma 3.2.2. [28] Let F be a filter and J be an ideal in a frame L. Then

(a) F is an ultrafilter if and only if $\delta(F) = L \smallsetminus F$.

(b) J is a maximal ideal if and only if $\gamma(J) = L \smallsetminus J$.

Proof. (a) Suppose F is an ultrafilter. Let $x \in \delta(F)$. Then $x \wedge y = 0$ for some $y \in F$ and hence $x \notin F$, that is, $x \in L \smallsetminus F$; establishing the one inclusion. Now let $w \in L \smallsetminus F$ and put $T = \{t \in L \mid t \geq w \wedge x \text{ for some } x \in F\}$. If we assume that w meets every member of F then T is a filter properly containing F, contradicting the maximality of F. Thus $w \wedge x = 0$ for some $x \in F$; proving the reverse inclusion.

Conversely, suppose $\delta(F) = L \smallsetminus F$ and let G be a filter with $G \supseteq F$. Let $g \in G$. For each $x \in F$ we have that $g \wedge x \neq 0$, and therefore $g \notin \delta(F)$, that is, $g \notin L \smallsetminus F$. This shows that $g \in F$, so that $F = G$, and therefore proves that F is an ultrafilter.

(b) Suppose J is a maximal ideal. Let $x \in \gamma(J)$. Then $x \vee y = 1$ for some $y \in J$ and hence $x \notin J$, that is, $x \in L \smallsetminus J$, establishing the one inclusion. Now let $w \in L \smallsetminus F$ and put $S = \{s \in L \mid s \leq w \vee x \text{ for some } x \in J\}$. If w joins every member of J then S is an ideal properly containing J contradicting the maximality of J. Then $w \vee x = 1$ for some $x \in J$. Proving the reverse inclusion.

Conversely, suppose $\gamma(J) = L \smallsetminus J$ and let K be an ideal with $K \supseteq J$. Let $y \in K$. For each $x \in J$ we have $x \vee y \neq 1$, and therefore $y \notin \gamma(J)$, that is $y \notin L \smallsetminus J$. This shows that $y \in J$, so that $K = J$, and therefore proves that J is a maximal ideal. □

Corollary 3.2.4. [28] A filter F is a minimal prime filter if and only if it is closed-generated witnessed by a maximal ideal.

The dual of the preceding corollary also holds as is seen below.

Corollary 3.2.5. An ideal I is a maximal prime ideal if and only if it is open-generated witnessed by a minimal filter.

Definition 3.2.4. A filter F in frame L is said to be *regular* if for each $a \in F$, there exists $x \in F$ such that $x \prec a$.

Dually,

Definition 3.2.5. An ideal I in a frame L is said to be *regular* if for each $b \in I$, there exists $x \in I$ such that $b \prec x$.

Definition 3.2.6. A filter F is *stably closed-generated* in case $F = \gamma(\delta(F))$.

The dual notion of the above definition were not defined in [28] we give the definition here.

Definition 3.2.7. An ideal I is *stably open-generated* in case $I = \delta(\gamma(I))$.

Observation 3.2.1. An ideal J is stably open-generated in case $J = \delta(\gamma(J))$. If J is any ideal, then $J \subseteq \delta(\gamma(J))$. For any $a \in J$ then $a \wedge b = 0$. If $a \in \delta(\gamma(J))$, then $a \wedge b = 0$ for some $b \in \gamma(J)$; so that for some $c \in J$, $b \vee c = 1$ whence we deduce that $a \leq c$ implying that $a \in J$.

The following result and its proof appear in [28]. Here we leave out the proof.

Proposition 3.2.4. [28] A filter is regular if and only if it is stably closed-generated.

We show below that the dualisation of the preceding proposition also holds.

Proposition 3.2.5. An ideal I is regular if and only if it is stably open-generated.

Proof. Suppose that an ideal I is regular. If I is a regular ideal and $t \in I$, then $t \prec s$ for some $s \in I$. Then $t^* \vee s = 1$ implies that $t^* \in \gamma(I)$, and $t^* \wedge t = 0$ implies that $t \in \delta(\gamma(I))$. Thus $I \subseteq \delta(\gamma(I))$. Let $a \in \delta(\gamma(I))$, then $a \wedge b = 0$ for some $c \in I$ $b \vee c = 1$, whence we deduce that $a \leq c$ implies that $a \in I$. So $\delta(\gamma(I)) \subseteq I$ and hence equality.

Conversely, suppose that an ideal I is stably open-generated. That is $I = \delta(\gamma(I))$. Then for any $x \in I$, $x \wedge y = 0$ for some $y \in \gamma(I)$. But then there exists $w \in I$ such that $w \vee y = 1$. Thus $x \prec w$ and therefore I is regular. □

Corollary 3.2.6. [28] An ultrafilter is regular if and only if it is closed-generated

Dually,

Corollary 3.2.7. A minimal prime ideal is regular if and only if it is open-generated.

Proof. The necessity part follows immediately from Proposition 3.2.5.

Conversely, for the sufficient part, suppose $J = \delta(F)$ for some (proper) filter F. Let $x \in F$. Then $x \notin J$, otherwise $y \wedge x = 0$ for some $y \in F$ implying that $0 \in F$, a contradiction. Thus $F \subseteq L \smallsetminus J$, and hence $J = \delta(F) \subseteq \delta(L \smallsetminus J) = \delta(\gamma(J))$ by Lemma 3.2.2 since is a prime ideal. So $J = \delta(\gamma(J))$ and the result follows by Proposition 3.2.5. □

Recall that for any filter F in L, $\mathrm{sec}(F) = \{a \in L \mid a \wedge b \neq 0, \text{ for each } b \in F\}$.

Definition 3.2.8. A filter F in a frame L is said to be

(a) *Convergent* if $F \cap S \neq \emptyset$ for any cover S of L.

(b) *Clustered* if $\mathrm{sec}(F) \cap S \neq \emptyset$ for any cover S of L.

Definition 3.2.9. A filter F in a frame L is said to be *free* if for any cover S of L such that $\mathrm{sec}(F) \bigcap S = \emptyset$.

Proposition 3.2.6. [28] Let F be a filter in a frame L. Then:

(a) $\sec(F) = \bigcup\{U \mid U \text{ is an ultrafilter and } U \supseteq F\}$.

(b) $b(F) = \sec^2(F)$.

Proof. (a) Let U be the collection of all ultrafilters that contain F. Then for any $Q \in U$ we have $Q = \sec(Q) \subseteq \sec(F)$, which shows that $\bigcup U \subseteq \sec(F)$. On the other hand let $x \notin \bigcup U$. Therefore, if $Q \in U$ then $x \notin Q$; so that $x^* \in Q$ as Q is an ultrafilter. Thus $x^* \in \bigcap U = b(F)$. Thus, by the (a) part of Proposition 3.2.3, we have that $x^{***} = x^* \in F$. Since $x \wedge x^* = 0$, $x \notin \sec(F)$. Therefore $\sec(F) \subseteq \bigcup U$, proving the result.

(b) Let $x \in b(F)$. Then $x^{**} \in F$. If t is an arbitrary element of $\sec(F)$ then $t \wedge x^{**} \neq 0$ which implies that $t \wedge x \neq 0$. So x meets every member of $\sec(F)$ and is therefore in $\sec^2(F)$, establishing the one inclusion. Now let $a \in \sec^2(F)$. Then $a^* \notin \sec(F) = \sec(F) = \bigcup U$. As argued above that $a \in Q$ for each $Q \in U$. So $a \in b(F)$, and the reverse inclusion follows. □

Lemma 3.2.3. Let F and Q be filters in a frame L such that $F \subseteq Q$. If F is free, then so is Q.

Proof. Suppose that $F \subseteq Q$ and F is a free filter in L and Q is a maximal. Since F is free there is a cover S of L such that $\sec(F) \cap S = \emptyset$. Since Q is a maximal filter, it follows that $Q = \sec(Q)$ in [39]. But $\sec(F) \subseteq \sec(Q)$. Since $\sec(F)$ is maximal filter it follows that $\sec(F) = \sec(Q)$. Hence $\sec(Q) \cap S = \emptyset$ showing that Q is also free. □

The following propositions is the frame analogue of Theorem 2.4.5, it has been taken verbatim in [28].

Proposition 3.2.7. [28] The following are equivalent for a regular frame L.

(a) Every free ultrafilter in L is regular.

(b) Every free prime filter in L is an ultrafilter.

Proof. (a) $\Rightarrow$ (b): Let F be a free prime filter and Q be an ultrafilter with $F \subseteq Q$. We want to show that $F = Q$. Since F is free and $Q \supseteq F$, Q is also free. So Q is a free ultrafilter and therefore, by hypothesis, Q is regular. Thus by Proposition 3.2.5 we have that $Q = \gamma(\delta(Q))$. Since Q is an ultrafilter, $\delta(Q) = L \smallsetminus Q$ by Lemma 3.2.2. Thus, $Q = \gamma(L \smallsetminus Q)$. Now let $x \in Q$. Then there is $y \in L \smallsetminus Q$ such that $x \vee y = 1$. So $x \vee y$ is an element of the prime filter F, hence $x \in F$ or $y \in F$. But $y \notin F$ because $y \notin Q$ and $F \subseteq Q$. So $x \in F$, and therefore $F = Q$ and we are done.

(b) $\Rightarrow$ (a): Let F be a free ultrafilter. We must show that F is regular. Put $J = L \smallsetminus F$. Then J is an ideal. Find a maximal ideal I with $I \supseteq J$ and put $P = L \smallsetminus I$. Notice that P is a prime filter and, by Lemma 3.2.2,

$$P = \gamma(I) = L \smallsetminus I \subseteq L \smallsetminus J = F.$$

If we can show that P is free, then (P being prime) we will have that P is an ultrafilter by hypothesis. Thus we will have that $P = F$, and so F will be a closed-generated ultrafilter and will therefore be regular by Corollary 3.2.5. Because F is a free ultrafilter, there is a cover C of L such that $C \cap F = \emptyset$. We claim that $\mathrm{sec}(P) \cap C = \emptyset$. If not, let t be in the intersection and pick $c \in C$ such that $t \prec c$. Then $t^* \vee c = 1$, so that $t^* \in P$ or $c \in P$ because P is prime. Since $t \in \mathrm{sec}(P)$, we cannot have $t^* \in P$. Thus $c \in P$; which is a contradiction because $P \subseteq F$ and F misses C. It follows therefore that P is free, and we are done. $\square$

Definition 3.2.10. A frame L is normal if for any elements $a, b \in L$ such that $a \vee b = 1$, there are elements $c, d \in L$ such that $c \wedge d = 0$ and $a \vee c = 1 = b \vee d$.

Definition 3.2.11. A frame L is said to be extremally disconnected if $a^* \vee a^{**} = 1$ for every $a \in L$.

The equivalent definition is that the frame L is said to be extremally disconnected if for $a, b \in L$ such that $a \wedge b = 0$ there exist $u, v \in L$ such that $u \vee v = 1$ and $u \wedge a = 0 = b \wedge v$.

One can see clearly from the equivalent statement of the preceding definition that extremally disconnectedness is a dual notion of normality.

Proposition 3.2.8. [28] A frame is normal if and only if its closed-generated filters are precisely the stably closed-generated ones.

The dual statement also holds as seen below.

Proposition 3.2.9. A frame is extremally disconnected if and only if its open-generated ideals are precisely the stably open-generated ones.

Proof. Let L be an extremally disconnected frame and J be an open-generated ideal in L. We must show that J is stably open-generated. Say $J = \delta(F)$ for some filter F. In view of the earlier Observation 3.2.1, it suffices to show that $J \subseteq \delta(\gamma(J))$. Let $x \in J$ and pick $y \in F$ such that $x \wedge y = 0$. By extremal disconnectedness find $u, v \in L$ such that $u \vee v = 1$ and $u \wedge v = 0 = y \wedge v$. Now $y \wedge v = 0$ implies $v \in J$ since $y \in F$, whence $u \vee v = 1$ implies that $u \in \gamma(J)$. Thus $x \wedge u = 0$ implies that $x \in \delta(\gamma(J))$, as required.

Conversely, let L be a frame with the stated property and suppose $a \wedge b = 0$. To prove that L is extremally disconnected we may assume, without loss of generality, that $b \neq 0$ and $a \neq 1$. Since $a \wedge b = 0$, we have, by hypothesis, that

$$a \in \delta(\uparrow b) = \delta(\gamma(\delta(\uparrow b))).$$

Now take $p \in \gamma(\delta(\uparrow b)),\ q \in \delta(\uparrow b)$ and $r \in \uparrow b$ such that $p \wedge a = 0,\ p \vee q = 1,\ q \wedge r = 0$. Since $r \geq b$, it follows that $q \wedge b = 0$, and therefore p and q are the elements required to show extremal disconnectedness. $\square$

3.3 Balanced ideals

In this section we give the definition of a sparce ideal in frames and show that an ideal is sparce if for any nonzero element in the frame meets a nonzero element in the ideal. We use the definition of a balanced ideal introduced by Dube and Mugochi in [34] to show that there is a one-to-one correspondence between the points of βL and the balanced ideals of Coz L. Although ideals are dual notions of filters in topological spaces, the notion of a balanced ideal in a frame is not obtained from dualisation of a balanced filter and extend it to frames. Here we show that there is a one-to-one correspondence between the points of βL and the balanced ideals of Coz L.

Definition 3.3.1. Let J be an ideal in a frame L.

(a) $O(J) = \{a \in L \mid \text{there exists } b \in J \text{ such that } a \vee b = 1\}$.

(b) $\sec_*(J) = \{a \in L \mid a \vee b \neq 1, \text{ for each } b \in J\}$.

(c) J is nontrivial provided $J \neq \{0\}$.

(d) J is a sparce ideal if for each nontrivial ideal I there exists a minimal prime ideal K such that $K \nleq I$ and $K \nleq J$.

(e) A collection of ideals $\{J_\alpha \mid \alpha \in I\}$ is called uniformly sparce if for each nontrivial ideal H there exists a minimal prime ideal K such that $K \nleq H$ and $K \nleq J_\alpha$ for each $\alpha \in I$.

The above definition is the frame analogue of the Definition 2.1 in [18]. Here the reader must not confuse the definition of $\sec_*(J)$ of an ideal J with that of $\sec(F)$ for any subset F of L. We used $\sec_*(J)$ of an ideal J just to follow Carlson's terminology in [18]. To avoid any confusion we will indicate with subscript $(*)$ for $\sec(J)$ of an ideal J by $\sec_*(J)$.

Observation 3.3.1. For a closed filter F in a topological X, we have $\sec^2(F) = \sec(\sec(F))$. But for an ideal J in a frame L, it is not necessarily true that $\sec_*^2(J) = \sec_*(\sec_*(J))$. Indeed if $a \vee b \neq 1$ and $b \vee c \neq 1$, one cannot conclude that $a \vee b \vee c \neq 1$ or $a \vee c \neq 1$.

Lemma 3.3.1. Let J_1 and J_2 be an ideals on a frame L. Then $J_1 \leq J_2$ if and only if $\mathcal{G}(J_1) \leq \mathcal{G}(J_2)$.

Proof. Let $a \in \mathcal{G}(J_1)$. Then $a^* \in J_1$, so $a^* \in J_2$ and hence $a \in \mathcal{G}(J_2)$.

Conversely, suppose that $\mathcal{G}(J_1) \leq \mathcal{G}(J_2)$. Let $a^* \in J_1$. Then $a \in \mathcal{G}(J_1)$, and hence $a \in \mathcal{G}(J_2)$, so $a^* \in J_2$. □

Proposition 3.3.1. Let J be an ideal on a frame L. Then the following are equivalent:

(1) J is a sparce ideal.

(2) For $0 \neq a \in L$ there exists $b \in J$ such that $a \wedge b \neq 0$.

Proof. (1) $\Rightarrow$ (2): Let $0 \neq a \in L$ and J be a sparce ideal. Then $\downarrow a$ is a nontrivial ideal. So by hypothesis there exists a minimal prime ideal K such that $K \nleq \downarrow a$ and $K \nleq J$. Take $b = \bigvee J$. We claim that $a \vee b \notin K$. Otherwise either $a \in K$ or $b \in K$ because K is a prime ideal. If $a \in K$, then $\downarrow a$ is a prime ideal contained in K, a contradiction because

K is a minimal prime ideal. Similarly, if $b \in K$, then $\downarrow b$ is a prime ideal contained in K, contradicting the minimality of K. Now $a \notin K \Rightarrow a \in L \setminus K$ where $L \setminus K$ is a filter. Similarly $b \in L \setminus K$. So $a, b \in L \setminus K$ and $L \setminus K$ is filter, so $a \wedge b \in L \setminus K$ and hence $a \wedge b \neq 0$.

(2) $\Rightarrow$ (1): Let J be an ideal such that for each $0 \neq a \in L$ there exists $b \in J$ such that $a \wedge b \neq 0$. Let $c = a^* \vee b^*$. Then $\downarrow c$ is a minimal prime ideal containing both a^* and b^*. We claim that $\downarrow c$ is a minimal prime ideal containing both a^* and b^*. If it is, then we are done. If not, there is a minimal prime ideal K containing a^* and b^*. Now $\downarrow a$ is a nontrivial ideal. So indeed K is such that $K \nleq \downarrow a$ and $K \nleq J$. Hence J is a sparce ideal.

□

To show that there is a one-to-one correspondence between the points of βL and the balanced ideals of Coz L, we first extend the result of Carlson [20]. But before embarking on this journey we first extend to frames two important results of Carlson in [20] repeated in this dissertation as Theorem 2.5.1 and Theorem 2.5.2. We start with a lemma.

Lemma 3.3.2. If $a, b \in \text{Coz } L$, then $a \wedge b \in \text{Coz } L$.

Proof. A cozero element of L is an element of the form coz φ for some $\varphi \in \mathcal{R}L$. Now $a = \text{coz } \gamma$ for some $\gamma \in \mathcal{R}L$ and $b = \text{coz } \delta$ for some $\delta \in \mathcal{R}L$. Now

$$a \wedge b = \text{coz } \gamma \wedge \text{coz } \delta = \text{coz } \gamma\delta,\ \gamma\delta \in \mathcal{R}L.$$

Hence $a \wedge b \in \text{Coz } L$. □

Theorem 3.3.1. Let L be a completely regular frame and I_p be a minimal prime ideal in Coz L. Set

$$V_p = \{a \in \text{Coz } L \mid a \text{ misses } I_p\}.$$

Then V_p is a minimal prime filter in Coz L.

Proof. Clearly from the definition of V_p, $0 \notin V_p$ since $0 \in I_p$ because I_p is an ideal. Let $a, b \in V_p$. Then we need to show that $a \wedge b \in V_p$. Then $a \in \text{Coz } L$, a misses I_p and $b \in \text{Coz } L$, b misses I_p. Now $a \wedge b \leq a$ and a misses I_p, so $a \wedge b$ misses I_p. That $a \wedge b \in \text{Coz } L$ follows from Lemma 3.3.2. Hence $a \wedge b \in V_p$.

Let $a, b \in \text{Coz } L,\ a \leq b$ with $a \in V_p$. We need to show that $b \in V_p$. Suppose on contrary that $b \notin V_p$. Then b is a cozero element which does not miss I_p. Then $b \in I_p$ and so $a \in I_p$ since I_p is an ideal is a down-set. But this contradicts the hypothesis that $a \in V_p$. Hence $b \in V_p$ and hence V_p is a filter in Coz L.

It remains to show that V_p is a prime filter. To this end, let $a, b \in \text{Coz } L$ such that $a \vee b \in V_p$. Then if $a \vee b$ misses I_p, then both a and b misses I_p. That is both a and b are elements of V_p, so V_p is a prime filter. Suppose W is a prime filter in Coz L which misses I_p with $W \subseteq V_p$. Take any $b \in V_p$, then $b \in \text{Coz } L$ and b misses I_p, so that $b \in W$. Hence $W = V_p$ so that V_p is a minimal prime filter in Coz L. □

Theorem 3.3.2. Let L be a completely regular frame and let V be a minimal prime filter in Coz L. Set

$$I = \{a \in \text{Coz } L \mid a \notin V\}.$$

Then, I is a minimal ideal in Coz L.

Proof. Clearly, the bottom element $0 \in I$ because $0 \notin V$. Take $a, b \in I$, then $a, b \in \text{Coz } L$ and $a, b \notin V$. We want to show that $a \vee b \in I$. Suppose on contrary that $a \vee b \in V$. Then by primeness of V either $a \in V$ or $b \in V$. Which means either $a \notin I$ or $b \notin I$ but both a and b are in I. A contradiction, so $a \vee b \notin V$, so that $a \vee b \in I$. Let $a, b \in \text{Coz } L,\ a \leq b$ and $b \in I$. Then $b \notin V$ and clearly $a \notin V$ because V is a filter. So indeed $a \in I$ and hence I is an ideal. Suppose J is an ideal in Coz L which does not contain elements of V such that $J \subseteq I$. Pick $u \in I$, then $u \in \text{Coz } L$ and $u \notin V$, so $u \in J$. Therefore $I = J$ and hence I is a minimal ideal of Coz L. □

We now give the definition of a balanced ideal which plays significant role on this dissertation.

Definition 3.3.2. An ideal I in a frame L is said to be *balanced* if, for any $a \in L,\ a^{**} \in I$ whenever $a \in I$.

Theorem 3.3.3. Let L be a completely regular frame. For any $I \in \beta L$ with $I < 1_{\beta L}$, the set

$$J = \{c \in \text{Coz } L \mid r_L(c) \leq I\}$$

is a balanced ideal of Coz L.

Proof. Clearly $0 \in \text{Coz } L$ and $r_L(0) = 0 \in I$, $r_L(0) \leq I$, so $0 \in J$. Take $a, b \in J$. Then $a, b \in \text{Coz } L$ with $r_L(a) \leq I$ and $r_L(b) \leq I$. Then $r_L(a \vee b) = r_L(a) \vee r_L(b) \leq I \vee I = I$. Hence $r_L(a \vee b) \leq I$ and so $a \vee b \in J$. Furthermore, let $a \leq b$ and $b \in J$. Then $b \in \text{Coz } L$ with $r_L(b) \leq I$. Now $a \leq b \Rightarrow r_L(a) \leq r_L(b) \leq I$. Since I is an ideal, it follows that $r_L(a) \leq I$ and hence $a \in J$.

Let $a \in J$. Then $a \in \text{Coz } L$ and $r_L(a) \leq J$. Since I is a regular ideal in βL, it follows that there is $r_L(b) \in I$ such that $r_L(a) \prec r_L(b)$. Now $(r_L(a))^* \vee r_L(b) = 1_{\beta L}$. Since $r_L(a)^{**} \wedge r_L(a)^* = 0_{\beta L}$, it follows that $r_L(a)^{**} \prec r_L(b)$. Thus $r_L(a^{**}) \leq (r_L(a))^{**} \prec r_L(b)$. Thus $r_L(a^{**}) \prec r_L(b)$. Hence $r_L(a^{**}) \prec I$. Therefore $r_L(a^{**}) = r_L(a)^{**} \prec I$. It remains to show that $a^{**} \in \text{Coz } L$. Furthermore, I is a regular ideal, so for each $r_L(a^{**}) \in I$ there is $r_L(a) \in I$ such that $r_L(a) \prec r_L(a^{**})$. Now $r_L(a^{**}) \vee r_L(a^*) = 1_{\beta L}$. The join map $j : \beta L \to L$, takes

$$j(1_{\beta L}) = j(r_L(a^{**}) \vee r_L(a^*)) = j(r_L(a^{**})) \vee j(r_L(a^*)) = a^{**} \vee a^* = 1_L.$$

That is, a^{**} is complemented and hence $a^{**} \in \text{Coz } L$. So indeed $a^{**} \in J$ and thus J is a balanced ideal of $\text{Coz } L$. $\square$

From the preceding Theorem, the following theorem is apparent.

Theorem 3.3.4. The map $I \to J$, where J is as above, is a bijection between $\beta L \setminus \{1_{\beta L}\}$ and the balanced ideals of Coz L.

Corollary 3.3.1. The map $I \to J$, is a bijection between $\beta\mathbb{N} \setminus \{1_{\beta\mathbb{N}}\}$ and the ideals of $\mathbb{N}$.

Corollary 3.3.2. The map $I \to J$, where J is as above, is a bijection between ideals of Coz (βL) and the balanced ideals of Coz L.

Theorem 3.3.5. Let $0_{\beta L} \neq I \neq 1_{\beta L}$ be a complemented element in βL. Then there exist balanced ideals of Coz L, J and K such that each minimal ideal H of Coz L is contained in either J or K but not both, that is $H \leq J$ or $H \leq K$.

Proof. Let I be a complemented element in βL such that $0_{\beta L} \neq I \neq 1_{\beta L}$. Let I^* be the complement of I in βL. Then $I \wedge I^* = 0_{\beta L}$ and $I \vee I^* = 1_{\beta L}$. By Theorem 3.3.3

$$J = \{u \in \text{Coz } L \mid r_L(u) \leq I\} \text{ and } K = \{v \in \text{Coz } L \mid r_L(v) \leq I^*\}$$

are balanced ideals of Coz L and they don't meet. If H is a minimal ideal of Coz L, then H is either contained in J or K and cannot be contained in both because J and K do not meet. $\square$

Proposition 3.3.2. For a completely regular frame L, For $i = 1, 2;\ I_i < 1_{\beta L}$, let

$$J_i = \{a \in \text{Coz } L \mid r_L(a) \leq I_i\}.$$

Then the following are equivalent:

(1) $I_2 \leq I_1$,

(2) $J_2 \leq J_1$.

Proof. (1) $\Rightarrow$ (2): Let $a \in J_2$. Then $a \in \text{Coz } L$ with $r_L(a) \leq I_2$, then $r_L(a) \leq I_1$ since $I_2 \leq I_1$ by hypothesis. So $a \in J_1$. Hence $J_2 \leq J_1$.

(2) $\Rightarrow$ (1): Let $w \in I_2$. Then $w \leq I_2$. Therefore $w = r_L(a)$ for some $a \in J_2$, then $w = r_L(a)$ for some $a \in J_1,\ w \in I_1$ and we are done. □

3.4 Nearness frames

By a cover A of a frame L we mean a subset of L such that $\bigvee A = 1$. We write $\text{Cov}(L)$ for the set of all covers of the frame L. The frame L is compact if for any $A \in \text{Cov}(L)$, there is a finite $F \subseteq A$ in $\text{Cov}(L)$.

Let L be a completely regular frame. Then by a compactification of L we mean a dense onto frame homomorphism $h : M \to L$ with M being a compact regular frame. The Stone-Čech compactification homomorphism of L is normally denoted by $\beta L \to L$.

Next recall the background of nearness frames in section 1.3.8, Given a collection $\mu \subseteq \text{Cov}(L)$, we say $x \in L$ is $\mu-$*strongly below* $y \in L$, written $x \lhd_\mu y$ (or simply $x \lhd y$) if there is a cover $A \in \mu$ such that $Ax \leq y$.

Lemma 3.4.1. [6], [7],[43] A frame L has a nearness if and only if L is a regular frame.

Proof. Suppose that the frame L has a nearness μ say. Then each $x \in L$ has the representation

$$x = \bigvee\{y \in L \mid y \lhd_\mu x\}$$

by the admissibility of μ. If $Ay \leq x$ for some $A \in \mu$. Then for $z = \bigvee\{t \in A \mid t \wedge y = 0\}$, we have $z \leq y^*$. So, $z \wedge y \leq y^* \wedge y = 0$. Thus $z \vee x = 1$. So, we have $y \wedge z = 0$ and $z \vee x = 1$. Thus $y \prec x$ and so, $x = \bigvee\{y \in L \mid y \prec x\}$. Hence L is regular.

Conversely, suppose that L is regular. Let μ be the filter in L generated by all finite covers and let $x \in L$. By regularity, $x = \bigvee\{y \in L \mid y \prec x\}$. If $y \prec x$, then $A = \{x, y^*\}$ is a μ-cover since it is clear that $y^* \vee x = 1$. Also, $Ay = \bigvee\{a \in A \mid a \wedge y \neq 0\} = x$. Thus $y \lhd_\mu x$, so $x = \bigvee\{y \in L \mid y \lhd_\mu x\}$. Hence, μ is admissible and thus a nearness on L. □

Lemma 3.4.2. [43],[48],[49]

(1) If L is a compact regular frame, then $\mathrm{Cov}(L)$ is the unique nearness on L, and, in fact, $\mathrm{Cov}(L)$ is a uniformity on L.

(2) A frame has a uniformity if and only if it is completely regular.

(3) If L is a regular frame, then $\mathrm{Cov}(L)$ is a uniformity if and only if the frame L is paracompact.

Remark 3.4.1. For any regular frame L, any filter $\mu \subseteq \mathrm{Cov}(L)$ containing all finite covers is thus admissible and so is a nearness by Lemma 3.4.1. Thus, $\mathrm{Cov}(L)$ itself is a nearness, which we call the fine nearness.

A frame homomorphism $h : L \to M$ between nearness frames (L, μ_L) and (M, μ_M) is called a *uniform* or *nearness* homomorphism if $h(A) \in \mu_M$ for each $A \in \mu_L$. Thus we have the category **NFrm** of nearness frames and uniform homomorphisms. Also, **NFrm** $\subseteq$ **RegFrm** is a subcategory of **RegFrm**.

We thus adopt the convention that all frames considered hereafter in this section are assumed to be regular.

Let L be a regular frame. Put

(1) $\mu_T = \{A \in \mathrm{Cov}(L) \mid \text{there exists } B \in \mathrm{Cov}(L) \text{ such that } B \leq A\}$.

(2) $\mu_P = \{A \in \mathrm{Cov}(L) \mid \text{there exists finite } B \in \mathrm{Cov}(L) \text{ such that } B \leq A\}$.

(3) $\mu_L = \{A \in \mathrm{Cov}(L) \mid \text{there exists countable } B \in \mathrm{Cov}(L) \text{ such that } B \leq A\}$.

(4) $\mu_{LF} = \{A \in \mathrm{Cov}(L) \mid \text{there exists locally finite } B \in \mathrm{Cov}(L) \text{ such that } B \leq A\}$.

Theorem 3.4.1. [43] $\mu_T, \mu_P, \mu_L, \mu_{LF}$ are nearness structures on the frame L.

Proof. μ_T is clearly the fine nearness, i.e. $\mu_T = \text{Cov}(L)$. We shall show that

(1) μ_P

(2) μ_L

(3) μ_{LF}

are all nearnesses.

(1) If $A, B \in \mu_P$, then there exists finite covers $A^{'}$ and $B^{'}$ of L such that $A^{'} \leq A$ and $B^{'} \leq B$. Then $A^{'} \wedge B^{'}$ is a finite cover and $A^{'} \wedge B^{'} \leq A \wedge B$. So, $A \wedge B \in \mu_P$. Also, if $A \leq C$ then there exists $A^{'}$ finite such that $A^{'} \leq A \leq C \Rightarrow C \in \mu_P$. Hence, μ_P is a filter on L. But, μ_P clearly contains all finite covers, so by Remark 3.4.1, μ_P is a nearness.

(2) Let $A, B \in \mu_L$ be any. Then there exists countable covers $A^{'}$ and $B^{'}$ of L such that $A^{'} \leq A$ and $B^{'} \leq B$. Then $A^{'} \wedge B^{'}$ is a countable refinement of $A \wedge B$, so $A \wedge B \in \mu_L$. Also if $A \leq C$, then $A^{'} \leq A \leq C \Rightarrow A^{'} \leq C$. Thus $A^{'}$ is a countable refinement of C and so $C \in \mu_L$. So, μ_L is a filter. Also, as every finite cover is countable, μ_L contains all finite covers and thus by Remark 3.4.1, μ_L is admissible. Hence μ_L is a nearness on L.

(3) Let $A, B \in \mu_{LF}$ be arbitrary. Then there exists locally finite covers $A^{'}$ and $B^{'}$ of L such that $A^{'} \leq A$ and $B^{'} \leq B$. Then $A^{'} \leq B^{'}$ is a locally finite refinement of $A \wedge B$. Thus $A \wedge B \in \mu_{LF}$. Again, if $A \leq C$, then $A^{'}$ is a locally finite refinement of C. So, μ_{LF} is a filter. Again, as μ_{LF} contains all finite covers, by Remark 3.4.1, μ_{LF} is admissible and hence a nearness on L. □

Remark 3.4.2. [43] Using the notation in [18] and [19], we call μ_P the Pervin nearness structure and μ_L the Lindelöf nearness structure on the regular frame L.

3.5 Locally finite nearness frames

We refer to [15], [18], [19] and [22] and attempt to find generalisations to the concepts contained therein for nearness frames.

In a frame L, a subset A is locally finite provided that there exists $B \subseteq \text{Cov}(L)$ such that each $b \in B$ meets only finitely many elements of A. The frame L is *paracompact* provided that each cover of L has a locally finite refinement, and L is *countably paracompact* provided that each countable cover of L has a locally finite refinement. A nearness $\mu \subseteq \mathrm{Cov}(L)$ is called locally finite provided that for each uniform cover A there exists a locally finite cover $B \in \mu$ such that $B \leq A$. Then (L, μ) is called a locally finite nearness frame.

Theorem 3.5.1. [43]:

(1) μ_{LF} is a locally finite nearness on L.

(2) If v is any locally finite nearness on L, then $v \subseteq \mu_{LF}$.

Proof. (1) Proved in Theorem 3.4.1.

(2) If v is any locally finite nearness on L and $A \in v$, then there exists locally finite $B \in v$ such that $B \leq A$. Then obviously, $B \leq B$ and B locally finite $\Rightarrow B \in \mu_{LF}$. Since μ_{LF} is a filter and $\mu_{LF} \ni B \leq A \Rightarrow A \in \mu_{LF}$. Thus $v \subseteq \mu_{LF}$.

□

Remark 3.5.1. [43] We call μ_{LF} the locally finite nearness on L.

Definition 3.5.1. The nearness frame (L, μ) is called

(a) *paracompact* provided that each uniform cover has a locally finite uniform refinement i.e. for all $A \in \mu$ there exists $B \in \mu$ such that B is locally finite and $B \leq A$.

(b) *countably paracompact* provided that each countable uniform cover has a uniform locally finite refinement.

Remark 3.5.2. [43] It is quite vacuous that a regular frame L is paracompact provided that $(L, \mathrm{Cov}(L))$ is a paracompact nearness. Paracompact nearness spaces have been studied in [15].

3.6 Pervin nearness frame

Recall the definition of a cover of L from section 1.3.8. For two covers A and B of L, we have

$$A \wedge B = \{a \wedge b \mid a \in A, b \in B\}.$$

Note that $A \wedge B$ is a common refinement of A and B, maximal in the preorder $\leq$ of covers. Furthermore, if A, B are two covers we will write

$$AB = \{Ab \mid b \in B\}.$$

Facts.

(1) For any cover $A, x \leq Ax$,

(2) $A \leq B$ and $x \leq y \Rightarrow Ax \leq By$,

(3) $A(Bx) \leq (AB)x = A(B(Ax))$, and

(4) $(A_1 \wedge .... \wedge A_n)(B_1 \wedge ... \wedge B_n) \leq (A_1B_1) \wedge ... \wedge (A_nB_n)$.

Definition 3.6.1. A *uniformity on a frame* L is a nonempty admissible system of covers $\mathcal{A}$ such that

(U1) $A \in \mathcal{A}$ and $A \leq B \Rightarrow B \in \mathcal{A}$,

(U2) $A, B \in \mathcal{A} \Rightarrow A \wedge B \in \mathcal{A}$,

(U3) for every $A \in \mathcal{A}$ there is a $B \in \mathcal{A}$ such that $BB \leq A$.

A cover B such that $BB \leq A$ is often called a *star-refinement* of A; thus, $(U3)$ is often expressed by saying that each $A \in \mathcal{A}$ has a star-refinement in $\mathcal{A}$.

The collection of all covers of a frame generates the Pervin quasi-uniform structure for that frame. All covers refined by some finite cover forms a nearness structure for a frame. This nearness structure will be called the Pervin nearness structure.

The Pervin nearness structure plays an interesting role in the family of all compatible nearness structure on a frame. It is the smallest totally bounded structure, the smallest contigual structure, and the largest ultrafilter generated structure.

The completion of the contigual reflection for T_2-nearness frame is the Wallman compactification. But the contigual reflection of a nearness frame is the Pervin nearness frame. A prime extension is one for which each trace filter is a prime filter. Since the Pervin nearness structure is ultrafilter generated it follows that the Stone-Čech compactification, if the frame is normal, is a prime extension.

We can construct the Stone-Čech compactification for a normal frame, using the strict extension on the family of all minimal prime filters.

Definition 3.6.2. Let (L, η_L) be a nearness frame. Let $\mathsf{S} \subseteq L$. Then S is called a *sparce near collection* if $\mathcal{S} \in \eta_L$ and for each $\mathcal{B} \in \eta_L$ such that $\mathcal{B}$ is not contained in each η_L-cluster, then there exists an η_L-cluster $\mathcal{A}$ such that $\mathcal{B} \nleq \mathcal{A}$ and $\mathsf{S} \nleq \mathcal{A}$.

Let $\{\mathsf{S}_\alpha \mid \alpha \in I\} \subset \eta_L$. Then $\{\mathsf{S}_\alpha \mid \alpha \in I\}$ is called *a uniformly sparce* family if for each $\mathcal{B} \in \eta_L$ such that $\mathcal{B}$ is not contained in each η_L-cluster, there exists an η_L-cluster $\mathcal{A}$ such that $\mathcal{B} \nleq \mathcal{A}$ and $\mathsf{S}_\alpha \nleq \mathcal{A}$ for each $\alpha \in I$.

Clearly each member of a uniformly sparce family is itself a sparce near collection.

Definition 3.6.3. Let L be a regular frame. A nearness frame (L, μ) is said to be

(i) *totally bounded* provided that for each $A \in \mu$ there exists $B \in \mathrm{Cov}(L)$ such that $B \leq A$ and B is finite.

(ii) *contigual* if for each μ-cover A there exists a μ-cover B such that B is a finite refinement of A.

Theorem 3.6.1. Let (L, μ) be a nearness frame. Then

(1) μ_P is contigual.

(2) μ_P is the smallest compatible contigual nearness structure on L.

(3) $\mu_P = \bigcap\{\mu \mid \mu$ is a compatible contigual nearness structure on $L\}$.

(4) $\mu_P = \bigcap\{\mu \mid \mu$ is a compatible totally bounded nearness structure on $L\}$.

(5) μ_P is the smallest compatible totally bounded nearness structure on L.

(6) The Pervin nearness structure is contained in each compatible totally bounded nearness structure on L.

Definition 3.6.4. A cover B is said to *corefines* A if for each $b \in B$ there exists $a \in A$ such that $a \leq b$.

Theorem 3.6.2. Let (L, μ_P) be a nearness frame.

(1) $\mu_P = \{A \in \text{Cov}(L) \mid G(J) \leq A$ for some minimal prime ideal $J\}$.

(2) A is minimal in μ_P if and only if there exists a minimal prime ideal J such that $G(J) = A$.

(3) $\gamma_P = \{A \in \text{Cov}(L) \mid$ there exists a minimal prime filter B that corefines $A\}$.

Lemma 3.6.1. [18] Every contigual nearness structure in a frame L is concrete.

Proof. Let $S \in \mu$. The nearness structure μ is contigual, so there exists $A \in \mu$ such that A is a finite refinement of S. Also, $A \in \mu$ so there is a $B \in \mu$ such that B is also a finite refinement of A. Now the union of all the finite sets in μ is also in μ and contain S. Thus S is contained in the μ-cluster. Hence μ is concrete. □

Corollary 3.6.1. Every Pervin nearness frame is concrete.

Proof. Every Pervin nearness structure is contigual and by the preceding Lemma 3.6.1 the result follows immediately. □

The following theorem compares the Locally finite nearness, the Pervin nearness, the Lindelöf nearness and the fine nearness.

Theorem 3.6.3. [43] For any regular frame L

(1) $\mu_P \subseteq \mu_{LF} \subseteq \mu_T$.

(2) $\mu_{LF} = \mu_T \Leftrightarrow L$ is a paracompact frame.

(3) $\mu_L \subseteq \mu_{LF} \Leftrightarrow L$ is a countably paracompact frame.

(4) $\mu_{LF} \subseteq \mu_L \Leftrightarrow$ every locally finite cover of L has a countable subcover.

Proof. (1): If $A \in \mu_P$, then there exists finite $B \in \text{Cov}(L)$ such that $B \leq A$. As every finite cover is locally finite, $B \in \mu_{LF}$. Since $B \leq A$ and μ_{LF} is a filter $\Rightarrow A \in \mu_{LF}$. Thus $\mu_P \subseteq \mu_{LF}$. As $\mu_T = \text{Cov}(L)$, clearly $\mu_{LF} \subseteq \mu_T$.

(2): Suppose that $\mu_{LF} = \mu_T$. Let $A \in \text{Cov}(L)$ be any. Then $\text{Cov}(L) = \mu_T \Rightarrow A \in \mu_T = \mu_{LF}$. Then there exists locally finite $B \in \text{Cov}(L)$ such that $B \leq A$. Thus each cover of L has a locally finite refinement $\Rightarrow$ L is paracompact.

Conversely, if L is paracompact, then by (1) it suffices to show that $\mu_T \subseteq \mu_{LF}$. So, let $A \in \mu_T$ be arbitrary. Then $A \in \text{Cov}(L)$ and $A \leq A$. Since L is paracompact and $A \in \text{Cov}(L) \Rightarrow$ there exists $B \in \text{Cov}(L), B$ locally finite such that $B \leq A$. Then B is a locally finite refinement of A. Thus $A \in \mu_{LF}$. So, $\mu_T \subseteq \mu_{LF}$.

(3): Suppose that $\mu_L \subseteq \mu_{LF}$. Let A be any countable cover of L. Then clearly, $A \in \mu_L$. Thus $A \in \mu$. Thus $A \in \mu_L$. So, there exists a locally finite $B \in \text{Cov}(L)$ such that $B \leq A$. Thus A has a locally finite refinement and clearly L is countably paracompact. For the converse, if $A \in \mu_L$ then there exists a countable cover B such that $B \leq A$. By countably paracompactness there exists a locally finite refinement C of B and hence of A i.e., $A \in \mu_{LF}$. So, $\mu_L \subseteq \mu_{LF}$.

(4): Suppose that $\mu_{LF} \subseteq \mu_L$. Let A be a locally finite cover of L. Then $A \in \mu_{LF} \Rightarrow A \in \mu_L$. Thus there exists a countable $B \in \text{Cov}(L)$ such that $B \leq A$. Then for each $b \in B$ there exists $a_b \in A$ such that $b \leq a_b$. As B is countable cover, $A_B = \{a_b \in A \mid b \in B\} \subseteq A$ is a countable subcover of A.

Conversely, suppose that every locally finite cover of L has a countable subcover. Let $A \in \mu_{LF}$. Then there exists a countable subcover A' of A. Since $A' \subseteq A$ we have $A \in \mu_L$, whence $\mu_{LF} \subseteq \mu_L$. □

Theorem 3.6.4. [43] (L, μ_{LF}) is a locally fine nearness frame.

Proof. Let $A \in \mu_{LF}$ and $\{B_a \mid a \in A\}$ be a family of μ_{LF}-covers. We require that $\{a \wedge b \mid a \in A \text{ and } b \in B_a\} \in \mu_{LF}$. Since $A \in \mu_{LF}$, there exists a locally finite $S \in \text{Cov}(L)$ such that $S \leq A$. Then for each $s \in S$, there exists $a_s \in A$ such that $s \leq a_s$. Then $\mathcal{B} = \{B_a \mid s \in S\} \subseteq \{B_a \mid a \in A\}$. Thus $\mathcal{B} \subseteq \mu_{LF}$. Thus for each $s \in S$, $B_{a_s} \in \mu_{LF} \Rightarrow$ there exists T_s locally finite such that $T_s \leq B_{a_s}$. Since S and T_s are locally finite for each $s \in S$ we have $S \wedge T_s$ is locally finite and for each $x \in S \wedge T_s$, $x = u \wedge v$ for some $u \in S$ and $v \in T_s$. Then there exists $a_s \in A$ such that $u \leq a_u$ as $S \leq A$. Also, there exists $b \in B_{a_s}$ such that $v \leq b$ as $T_s \leq B_{a_s}$. Thus $x = u \wedge v \leq a_u \wedge b$. So, $S \wedge T_s \leq \{a \wedge b \mid a \in A \text{ and } b \in B_a\}$. Since μ_{LF} is a filter we have $\{a \wedge b \mid a \in A \text{ and } b \in B_a\} \in \mu_{LF}$. □

Theorem 3.6.5. [43] For a regular frame L, the following are equivalent:

(1) $\mu_P = \mu_{LF}$.

(2) Every locally finite cover of L has a finite subcover.

(3) μ_{LF} is totally bounded.

(4) μ_{LF} is contigual.

Proof. (1) $\Rightarrow$ (2): Suppose that $\mu_P = \mu_{LF}$. Let A be any locally finite cover of L. Then $A \in \mu_{LF} \Rightarrow A \in \mu_P$. Thus there exists a finite $B \in \mathrm{Cov}(L)$ such that $B \leq A$. Then for each $b \in B,$ there exists $a_b \in A$ such that $b \leq a_b$. Then $\{a_b \in A \mid b \in B\}$ is a finite subcover of A.

(2) $\Rightarrow$ (3): Suppose (2). For any $A \in \mu_{LF}$, there exists a locally finite $B \in \mathrm{Cov}(L)$ such that $B \leq A$. Then B has a finite subcover C. Then C is a finite refinement of A. Thus μ_{LF} is totally bounded.

(3) $\Rightarrow$ (4): Suppose that μ_{LF} is totally bounded. Let A be a μ_{LF} - cover of L. Then by total boundedness there exists a finite $B \in cov(L)$ such that $B \leq A$. Since every finite cover is locally finite, clearly B is a μ_{LF} - cover. Thus every μ_{LF} - cover has a finite μ_{LF} - refinement $\Rightarrow \mu_{LF}$ is contigual.

(4) $\Rightarrow$ (1): Suppose that μ_{LF} is contigual. By the previous Theorem 3.6.3, it suffices to show that $\mu_{LF} \subseteq \mu_P$. Let $A \in \mu_{LF}$ be any. As μ_{LF} is contigual, A has a finite refinement $B \in \mu_{LF}$. So, clearly $A \in \mu_P$. Thus $\mu_{LF} \subseteq \mu_P$. □

We call the nearness frame (L, v) a locally fine nearness frame if whenever $A \in v$ and $\{B_a \in v \mid a \in A\} \subseteq v$ is any v-subcollection, then $\{a \wedge b \mid a \in A \text{ and } b \in B_a\}$ is a v-cover.

3.7 Strict Extensions in Frames

The type of frame homomorphism under investigation here is described as follows:

Definition 3.7.1. A frame homomorphism $h : M \to L$ is called

(1) *strict* if M is generated by the image of the right adjoint $h_* : L \to M$, and

(2) *a strict extension* if it is strict, dense, and onto.

It is worth adding that, for regular M, any dense $h : M \to L$ is strict because $x \prec a$ in M implies $x \leq h_*h(x) \leq a$ since $h_*h(x) \wedge x^* = 0$ (act the dense h).

Note that, for topological spaces, an extension $X \subseteq Y$ (meaning: X is a dense subspace of Y) is called strict whenever the corresponding homomorphism $\mathfrak{D}Y \to \mathfrak{D}X$ between the frames of open sets of Y and X is a strict extension in the present sense, and there is a description of all strict extension of a given space by means of the open filters on that space (Banaschewski [3]). In the next section, we shall study the analogue of the latter for frames; here we present a few generalities concerning strict extensions of frames.

The basic example in this context is given by the frame $\mathfrak{D}L$ of all non-void downsets of a frame L, that is, the $U \subseteq L$ such that $0 \in U$ and $a \in U$ implies $b \in U$, for all $b \leq a$, with union as join and intersection as meet: the map $\bigvee : \mathfrak{D}L \to L$ taking each U to its join $\bigvee U$ is a frame homomorphism, with right adjoint $\downarrow : L \to \mathfrak{D}L$ where

$$\downarrow a = \{x \in L \mid x \leq a\},$$

and hence indeed a strict extension.

The map $\downarrow : L \to \mathfrak{D}L$ is obviously a $(\wedge, 0, 1)$-homomorphism, and in fact universally so since any arbitrary homomorphism $\varphi : L \to N$ of this kind into a frame N determines a (necessarily unique) frame homomorphism $\overline{\varphi} : \mathfrak{D}L \to N$ such that $\overline{\varphi}(\downarrow a) = \varphi(a)$:

$$\overline{\varphi}(U) = \bigvee \varphi[U].$$

It follows from this fact that the strict extension $\bigvee : \mathfrak{D}L \to L$, in turn, is universal in the sense that, for any other such $h : M \to L$, there is a factorization

$$\bigvee : \mathfrak{D}L \xrightarrow{\overline{h}} M \xrightarrow{h} L.$$

This results from the fact that $h_* : L \to M$ is a $(\wedge, 0, e)$-homomorphism and hence determines $\overline{h} = \overline{h_*}$ such that

$$h\overline{h}(\downarrow a) = hh_*(a) = a = \bigvee \downarrow a$$

for any $a \in L$, and therefore $h\overline{h} = \bigvee$. Also, $\overline{h}$ is onto since h is strict extension $h : M \to L$.

Lemma 3.7.1. [10] If $h = fg$ for onto $g : M \to N$ and arbitrary $f : N \to L$ then $f_* = gh_*$ and f is a strict extension.

Proof. Since h is a strict extension, it follows that it is onto so that f is also onto. Also f is dense since h is dense and g is onto. Further, $f(x) \leq y$ implies $fgg_*(x) \leq y$, hence $g_*(x) \leq h_*(y)$ and consequently $x \leq gh_*(y)$ since g is onto; conversely, the latter implies $f(x) \leq hh_*(y) \leq y$ by the properties of h. Hence $f_* = gh_*$, as claimed, and since g is onto this shows f is strict. $\square$

Together with the previous observation concerning $\mathfrak{D}L$, this immediately leads to the following characterization:

Proposition 3.7.1. [10] The strict extensions $M \to L$ are exactly the homomorphisms obtained by factoring $\mathfrak{D}L \to L = \mathfrak{D}L \to M \to L$ with onto $\mathfrak{D}L \to M$.

We note that this is the precise counterpart for frames of a result of Banashewski [3] on strict extensions of T_0 spaces, where one has a universal such extension for any given space, provided by the space of all proper filters in the corresponding frame of open sets. The connection between the two results lies in the fact that, for any frame L, the proper filters in L correspond exactly to the completely prime filters in $\mathfrak{D}L$, and the original result of [3] is in fact a consequence of Proposition 3.7.1.

We now turn to a topologically motivated construction of particular strict extensions of a frame L due to Hong [39].

Given a frame L and a set X of (proper) filters F in L, the latter viewed as a space with its usual topology $\mathfrak{D}X$ generated by the sets

$$X_a = \{F \in X \mid a \in F\},$$

we have a $(\wedge, 0, 1)$-homomorphism $L \to \mathfrak{D}X$ given by $a \longmapsto X_a$ and hence homomorphism $\mathfrak{D}L \to \mathfrak{D}X$, taking $\downarrow a$ to X_a and consequently each $U \in \mathfrak{D}L$ to

$$X_U = \bigcup\{X_a \mid a \in U\} = \{F \in X \mid F \cap U \neq \emptyset\}$$

(*note* $X_{\downarrow a} = X_a$). Together with $\bigvee : \mathfrak{D}L \to L$, this then determines the homomorphism

$$\mathfrak{D}L \to L \times \mathfrak{D}X,\ U \longmapsto (\bigvee U, X_U),$$

and we let $\tau_X L$ be the corresponding images frame. Thus, we have the decomposition

$$\bigvee : \mathfrak{D}L \to \tau_X L \to L$$

where the second map is given by the first projection $L \times \mathfrak{D}X \to L$. It follows from Lemma 3.7.1 that $\tau_X L \to L$ is a strict extension, with right adjoint taking $a \in L$ to (a, X_a).

Definition 3.7.2. $\tau_X L \to L$ is called the *strict extension* of L determinant by X.

Note that, for any U and W in $\mathfrak{D}L$,

$$(\bigvee U, X_U) \leq (\bigvee W, X_W) \text{ if and only if } s \leq \bigvee W \text{ and } X_s \subseteq X_W$$

for each $s \in U$, and hence

$$\overline{W} = \{s \in L \mid s \leq \bigvee W,\ X_s \subseteq X_W\}$$

is the largest $U \in \mathfrak{D}L$ mapped to $(\bigvee W, X_W)$ by the above homomorphism $\mathfrak{D}L \to \tau_X L$. Consequently, the right adjoint of the latter takes each $(\bigvee W, X_W)$ to $\overline{W}$ so that the nucleus induced by $\mathfrak{D}L \to \tau_X L$ takes W to $\overline{W}$. Thus, $\tau_X L$ can also be described by means of this nucleus on $\mathfrak{D}L$. As an application of this, we have the following:

Theorem 3.7.1. Let (L, μ) be a nearness frame. The following are equivalent:

(1) μ is a nearness structure on L induced by strict extension.

(2) (L, μ) is concrete.

Theorem 3.7.2. Let (L, μ_P) be the Pervin nearness frame. Then the completion $h : (M, v) \to (L, \mu)$ is the Wallman compactification (wL) of L.

In a normal frame the Wallman compactification coincides with the Stone-Čech compactification βL of L.

Corollary 3.7.1. Let L be a normal frame. Let (L, μ_P) be the Pervin nearness frame. The completion $h : (M, v) \to (L, \mu_P)$ is the Stone-Čech compactification (βL) of L.

Corollary 3.7.2. Let L be a frame. Then the Wallman compactification is a prime extension of L. If L is normal, then βL is a prime extension of L.

Corollary 3.7.3. Let L be a regular frame. Then the Pervin nearness structure on L is induced by the Wallman compactification of L. If L is normal, the Pervin nearness structure on L is induced by the Stone-Čech compactification βL of L.

Definition 3.7.3. Let (L, μ) be a nearness frame. Let $S \in \mathrm{Cov}(L)$. Then S is a *sparce near collection* if $S \in \mu$ and for each $B \in \mu$ such that B is not contained in each μ-cluster, then there exists μ-cluster A such that $B \nleq A$ and $S \nleq A$.

Definition 3.7.4. Let (L, μ) be a nearness frame. A collection $v \in \mathrm{Cov}(L)$ is said to be a *μ-cluster* provided v is maximal.

Let $S_\alpha \in \mu$. Then $S_{\alpha \in I}$ is called a *uniformly sparce* family if for each $B \in \mu$ such that B is not contained in each μ-cluster, then there exists a μ-cluster A such that $B \nsubseteq A$ and $S_\alpha \nsubseteq A$ for each $\alpha \in I$.

We observe that each member of a uniformly sparce family is itself a sparce near collection.

Recall from [52, p. 185], that a topological space X is a *Baire space* if and only if the intersection of each countable family of dense open sets in X is dense. A set $A \subseteq X$ is nowhere dense in X if and only $\mathrm{int}_X \mathrm{cl}_X A = \emptyset$. A set $A \subseteq X$ is of *first category* in X if and only if $A = \bigcup_{n=1}^{\infty} A_n$, where each A_n is nowhere dense in X. All other subsets of X are of *second category* in X. Also, recall that $a \in L$ is dense if $a^* = 0$.

Definition 3.7.5. Let $\{\mathcal{U}_\alpha\}$ be a countable family of elements in a frame L. Then L is a *Baire frame* if and only if $(\bigwedge u_i)^* = 0$ for all dense $u_i \in \mathcal{U}_\alpha$ and each $\mathcal{U}_\alpha$ in L.

Recall from Dube [29] that a quotient $h : L \to M$ of L is *nowhere dense* in L if for any nonzero $x \in L$ there is nonzero $y \leq x$ in L such that $h(y) = 0$.

Definition 3.7.6. A quotient $h : L \to M$ of L is of *first category* in L if and only if for every $a \in M$, $a = h(\bigvee(h_*(a_n)))$, where each $h_*(a_n)$ is nowhere dense in L. All other quotients of L are of *second category* in L.

Theorem 3.7.3. A quotient $h : L \to L$ of L is of second category in L if and only if $(\bigwedge u_i) \neq 0$ for all dense $u_i \in \mathcal{U}_\alpha$ for each countable family $\mathcal{U}_\alpha$ in L.

Theorem 3.7.4. Let (L, μ) be a nearness frame. The following statements are equivalent:

(1) μ is a nearness induced on L by a second category strict extension.

(2) (L, μ) is concrete and for each countable $\{s_i \mid i \in \mathbb{N}\}$ of sparce near collections there exists a μ-cluster A such that $s_i \vee A \notin \mu$ for each $i \in \mathbb{N}$.

Definition 3.7.7. Let (L, μ) be a nearness frame. A nearness frame is said to be *concrete* if each $S \in \mu$ is contained in some μ-cluster.

We end the chapter by providing the pointfree version of Theorem 4.10, Theorem 4.11, Theorem 4.12, Theorem 4.13 and Corollary 4.14 in [18]. Here we only mention the results in the context of frames without giving the proofs. Although they seem to be slightly out of the research topic, they are worthy to be noted.

Proposition 3.7.2. Let (L, μ) be a nearness frame. The following are equivalent:

(1) μ is a nearness structure induced on L by strict Baire extension.

(2) (L, μ) is concrete and each countable family of sparce near collections is uniformly sparce.

Proposition 3.7.3. Let (L, μ_P) be the Pervin nearness frame. Let $A \in \mu_P$.

(1) v is not contained in each μ-cluster if and only if an ideal $L \smallsetminus v$ is not a trivial ideal.

(2) v is a sparce near collection if and only if $L \smallsetminus v$ is a sparce ideal.

Proposition 3.7.4. Let L be a regular frame. The following statements are equivalent:

(1) The Wallman compactification of L is of second category.

(2) For each countable collection $\{J_i \mid i \in \mathbb{N}\}$ of sparce ideals there exists a minimal prime sparce ideal K such that $K \nleq J_i$ for each $i \in \mathbb{N}$.

Theorem 3.7.5. Let L be a regular frame. The Wallman compactification of L is a Baire frame if and only if each countable family of sparce ideals is uniformly sparce.

Corollary 3.7.4. Let L be a normal frame. Then each countable family of sparce ideals is uniformly sparce.

Chapter 4

Miscellaneous

In this chapter we study remote points which was studied by Dube in [31] and the remote points in perfect extensions studied by Dube and Mugochi in [34]. It is shown that if $h : M \to L$ is a perfect extension of L and p is a point in M, then p is a remote point if and only if I_p is a balanced ideal of L, where

$$I_p = \{a \in L \mid h_*(a) \leq p\}.$$

We also study the roundness of the quotient $h : \beta L \to M$ of βL as defined in [31]. It is also shown that for any $I \in \beta L$, the closed quotient $h : \beta L \to {\uparrow} I$ is round if and only if there is only one ideal J of Coz L such that $I = \bigvee\{r(x) \mid x \in J\}$. We end this chapter (dissertation) with allusions to some future work by studying z_∞-ideals and a nicely balanced ideals of $C_\infty(X)$ defined by Ghosh [37]. Our ultimate goal is to put this in frame perspective.

4.1 Remote points

Let X be a topological space and $Y \supseteq X$ be an extension of X. A point $p \in Y \smallsetminus X$ is said to be remote from X if for any nowhere dense set $D \in X$, $p \notin \mathrm{cl}_Y D$. Remote point were first defined by Fine and Gillman and subsequently studied by several authors (see [35], [41], [42] and [45]). Remote points in pointfree topology were first considered by Dube in [31]. We recall from [29] that a quotient $h : L \to M$ of L is said to be nowhere dense if for every nonzero $x \in L$ there exists a nonzero $y \leq x$ in L such that $h(y) = 0$. It is also shown in [29] that a closed quotient $L \to {\uparrow} a$ is nowhere dense if and only if a is dense.

Recall that a point p of βX, where X is a Tychonoff space, is called a *remote point* if for any nowhere dense $D \subseteqq X$, $p \notin \mathrm{cl}_{\beta X} D$.

In frame perspective.

Definition 4.1.1. A point I of βL is *remote* if for each nowhere dense quotient $L \xrightarrow{h} M$, $I \vee r(h_*(0)) = 1_{\beta L}$.

Lemma 4.1.1. [31] A quotient $L \xrightarrow{h} M$ is nowhere dense if and only if $h_*(0)$ is dense.

Proposition 4.1.1. [31] (cf.[[27], 5.1 and 14.2]). The following are equivalent for a point I of βL:

(1) I is remote.

(2) $I \vee r(a) = 1_{\beta L}$ for each dense $a \in L$.

(3) For any $a \in L$, $r(a) \leqq I$ implies $a \in I$.

(4) For any $a \in L$, $r(a^*) \leqq I$ implies $r(a) \vee I = 1_{\beta L}$.

(5) For any $J \in \beta L$, $J^* \leqq I$ implies $r(\bigvee J) \vee I = 1_{\beta L}$.

(6) The set $F = \{a \in L \mid r(a) \vee I = 1_{\beta L}\}$ is an ultrafilter in L.

Proof. (1) $\Rightarrow$ (2): Let a be a dense element of L. Then the closed quotient $L \xrightarrow{h} \uparrow a$ is nowhere dense. Therefore, by hypothesis, $I \vee r(h_*(0_{\uparrow a})) = 1_{\beta L}$. But $h_*(0_{\uparrow a}) = a$, so $I \vee r(a) = 1_{\beta L}$.

(3) $\Rightarrow$ (4): Suppose $r(a^*) \leqq I$. Since $r(a \vee a^*) = r(a) \vee r(a^*)$, $r(a) \nleqq I$, lest we have that $r(a \vee a^*) \leqq I$, implying, by hypothesis, that I contains the dense element $a \vee a^*$. It follows, therefore that $r(a) \vee I = 1_{\beta L}$ since I is a point.

(4) $\Rightarrow$ (5): This is so in light of the fact that $J^* = r((\bigvee J)^*)$ for each $J \in \beta L$.

(5) $\Rightarrow$ (6): We check first that F is a filter. Clearly, F is not empty (as $1 \in F$) and is an upset. Suppose a, $b \in F$. Then $I \vee r(a \wedge b) = I \vee (r(a) \wedge r(b)) = (I \vee r(a)) \wedge (I \vee r(b)) = 1_{\beta L}$, and so $a \wedge b \in F$. In order to show that F is an ultrafilter, let $z \in L$, and suppose $z^* \notin F$. We must show that $z \in F$. Since $z^* \notin F$, $r(z^*) \vee I \neq 1_{\beta L}$, and therefore $r(z)^* = r(z^*) \leqq I$. So, by hypothesis, $r(\bigvee r(z)) \vee I = 1_{\beta L}$, that is, $r(z) \vee I = 1_{\beta L}$, and therefore $z \in F$.

(6) $\Rightarrow$ (1): Let $L \xrightarrow{h} M$ be a nowhere dense quotient of L. Then, by Lemma 4.1.1, $h_*(0)$ is dense, and so, $h_*(0) \in F$ since an ultrafilter in a frame contains all dense elements of the frame. So, by hypothesis, $I \vee r(h_*(0)) = 1_{\beta L}$, and therefore I is remote. □

Corollary 4.1.1. [31] A frame is Boolean if and only if every point of its Stone-Čech compactification is remote.

Definition 4.1.2. A frame L is *δ-normally separated* in case for every $a \in L$ and $c \in \mathrm{Coz}\, L$, such that $a \vee c = 1$, there exists $d \in \mathrm{Coz}\, L$ such that $d \leqq a$ and $d \vee c = 1$. It is apparent that L is δ-normally separated if and only if for each $a \in L$ and $c \in \mathrm{Coz}\, L$, $a \vee c = 1$ implies $r(a) \vee r(c) = 1_{\beta L}$.

Proposition 4.1.2. [31] Let L be δ-normally separated. A point I of βL is remote if and only if I is a maximal ideal in L.

Proof. Let a be an element of L such that $a \vee x \neq 1$ for each $x \in I$. We must show that $a \in I$, which will prove that I is a maximal ideal in L. Now $r(a) \vee I \neq 1_{\beta L}$, for otherwise $a \vee u = 1$ for some $u \in I$. Therefore $r(a) \leqq I$, and hence $a \in I$ by the previous proposition.

Conversely, let a be a dense element of L. Then $a \notin I$, and therefore, by maximality, $a \vee v = 1$ for some $v \in I$. Since I is a completely regular ideal, there exists a cozero element c such that $v \leqq c \in I$. So $a \vee c = 1$, and therefore, by δ-normal separation, $r(a) \vee r(c) = 1_{\beta L}$, and hence $r(a) \vee I = 1_{\beta L}$. Therefore I is a remote point. □

Following [13], we say an extension $M \xrightarrow{h} L$ is perfect if $h_*(a \vee a^*) = h_*(a) \vee h_*(a^*)$ for every $a \in L$. This is equivalent to saying $h_*(a \vee b) = h_*(a) \vee h_*(b)$ for all disjoint a and b in L. The extensions $\beta L \to L$ and $\kappa L \to L$ are perfect. For perfect extensions there are more equivalent conditions for a point to be remote. As in [23], we say a filter F in a frame L is *disjoint-prime* if, for any $a \in L$, $a \vee a^* \in F$ implies $a \in F$ or $a^* \in F$. Because a filter is an ultrafilter if and only if, for every $a \in L$, either $a \in F$ or $a^* \in F$, it follows easily that a filter is an ultrafilter if and only if it is saturated and disjoint-prime. Observe that if $M \xrightarrow{h} L$ is a perfect extension, then $\mathcal{U}^p$ is saturated for every $p \in Pt(M)$, where

$$\mathcal{U}^p = \{a \in L \mid h_*(a) \nleq p\}.$$

Recall Definition 3.3.2 that an ideal I in a frame L is *balanced* if, for any $a \in L$, $a^{**} \in I$ whenever $a \in I$. Minimal prime ideals are balanced because they do not contain dense elements, so that if one such contains a^{**}, then it does not contain a^*, and hence it must contain a by primeness. For an extension $M \xrightarrow{h} L$ and $p \in Pt(M)$, we set

$$I_p = \{a \in L \mid h_*(a) \leq p\}$$

so that $I_p = L \smallsetminus \mathcal{U}^p$.

Proposition 4.1.3. [34] Let $M \xrightarrow{h} L$ be a perfect extension of L. The following statements about a point $p \in Pt(M)$ are equivalent:

(1) p is a remote point.

(2) For any dense $a \in L$, $h_*(a) \leq p$.

(3) For any $a \in L$, $h_*(a) \leq p$ implies $h_*(a^*) \nleq p$.

(4) For any $a \in L$, $h_*(a^*) \leq p$ implies $h_*(a) \nleq p$.

(5) For any $b \in M$, $b^* \leq p$ implies $h_*h(b) \nleq p$.

(6) $\mathcal{U}^p$ is an ultrafilter.

(7) I_p is a minimal prime ideal of L.

(8) I_p is a balanced ideal of L.

Proof. (2) $\Rightarrow$ (3): Let $a \in L$ be such that $h_*(a) \leq p$. Since $a \vee a^*$ is dense, (2) implies $h_*(a \vee a^*) \nleq p$. Since $h_*(a \vee a^*) = h_*(a) \vee h_*(a^*)$ and $h_*(a) \leq p$, it follows that $h_*(a^*) \nleq p$.

(3) $\Rightarrow$ (4): Clearly the denial of (4) contradicts (3).

(4) $\Rightarrow$ (5): For any $b \in M$, $b^* = h_*h(b^*)$, so $b^* \leq p$ implies $h_*h(b^*) \leq p$, that is, $h_*(h(b)^*) \leq p$, so that, $h_*h(b) \nleq p$ by (4).

(5) $\Rightarrow$ (6): Let $a \in L$ be such that $a^* \notin \mathcal{U}^p$. Then $h_*(a^*) \leq p$, that is, $h_*(a)^* \leq p$. So, by (5), $h_*hh_*(a) \nleq p$, that is, $h_*(a) \nleq p$, so that $a \in \mathcal{U}^p$. Therefore $\mathcal{U}^p$ is an ultrafilter.

(6) $\Rightarrow$ (7): Since $I_p = L \smallsetminus \mathcal{U}^p$, it follows from [28, Corollary 3], which states that a filter is an ultrafilter if and only if its set-theoretic complement is a minimal prime ideal, that I_p is a minimal prime ideal in L.

(7) $\Rightarrow$ (8): Minimal prime ideals are balanced.

(8) $\Rightarrow$ (1): If $L \xrightarrow{\eta} N$ is a nowhere dense quotient of L, then $\eta_*(0)$ is dense, and is therefore not in I_p, otherwise $1 = \eta_*(0)^{**} \in I_p$ because I_p is balanced. Thus, $h_*\eta_*(0) \nleq p$, hence p is remote from L. $\square$

For each $I \in \beta L$ the ideals $\mathbf{M}^I$ and $\mathbf{O}^I$ of $\mathcal{R}L$ are defined by

$$\mathbf{M}^I = \{\varphi \in \mathcal{R}L \mid r(\text{coz } \varphi) \leqq I\} \text{ and } \mathbf{O}^I = \{\varphi \in \mathcal{R}L \mid r(\text{coz } \varphi) \prec I\}.$$

Clearly, $\mathbf{O}^I \subseteq \mathbf{M}^I$. Since, for any $I \in \beta L$ and $a \in L$, $r(a) \prec I$, and only if $a \in I$, it follows that

$$\mathbf{O}^I = \{\varphi \in \mathcal{R}L \mid \text{coz } \varphi \in I\}.$$

These are of course frame counterparts of the ideals $\mathbf{O}^p$ and $\mathbf{M}^p$ of $C(X)$ (see [38]). It is shown in [30] that:

(1) A subset Q of $\mathcal{R}L$ is a maximal ideal if and only if there is a unique point I of βL such that $Q = \mathbf{M}^I$.

(2) If P is a prime ideal, then there is a unique point I of βL such that $\mathbf{O}^I \subseteqq P \subseteqq \mathbf{M}^I$.

(3) For each point I of βL, $\mathbf{M}^I$ is the unique maximal ideal containing $\mathbf{O}^I$.

(4) For any point I of βL and $\varphi \in \mathcal{R}L$, $\varphi \in \mathbf{O}^I$ if and only if $\gamma\varphi = 0$ for some $\gamma \notin \mathbf{M}^I$.

Definition 4.1.3. The *annihilator* of a set $S \subseteq A$ is the ideal

$$Ann(S) = \{a \in A \mid as = 0 \text{ for every } s \in S\}.$$

The following proposition extends some characterizations of remote points of $\mathcal{R}$ obtained by Mandelker [41].

Proposition 4.1.4. [31] Let L be perfectly normal. Then the following are equivalent for a point I of βL:

(1) I is remote.

(2) $\mathbf{M}^I = \mathbf{O}^I$.

(3) For each $\varphi \in \mathbf{M}^I$, coz φ is not dense.

(4) $\mathbf{M}^I$ is a minimal prime ideal of $\mathcal{R}L$.

Proof. (1) $\Rightarrow$ (2): Let I be remote and $\varphi \in \mathbf{M}^I$. Then $r(\mathrm{coz}\ \varphi) \leqq I$, and therefore, by hypothesis, $\mathrm{coz}\ \varphi \in I$, implying that $\varphi \in \mathbf{O}^I$. Therefore $\mathbf{M}^I \subseteqq \mathbf{O}^I$, and hence $\mathbf{M}^I = \mathbf{O}^I$ as the other inclusion always holds.

(2) $\Rightarrow$ (3): If $\varphi \in \mathbf{M}^I$, then $\mathrm{coz}\ \varphi$ cannot be dense since $\mathrm{coz}\ \varphi \in I$, by hypothesis.

(3) $\Rightarrow$ (1): Suppose, for contradiction, that I is not remote. Let $L \xrightarrow{h} M$ be a nowhere dense quotient of L such that $I \vee r(h_*(0)) \neq 1_{\beta L}$. This implies that $r(h_*(0)) \leqq I$. Take $\gamma \in \mathcal{R}L$ such that $\mathrm{coz}\ \gamma = h_*(0)$. Then $\gamma \in \mathbf{M}^I$. But $\mathrm{coz}\ \gamma$ is dense by Lemma 4.1.1, so we have a contradiction.

(2) $\Rightarrow$ (4): Recall that in any ring a prime ideal is minimal prime if and only if each element in the annihilated by an element outside it. Let $\alpha \in \mathbf{M}^I$. Then, by hypothesis, $\alpha \in \mathbf{O}^I$. By the last of the results cited from [30], α is annihilated by an element outside $\mathbf{M}^I$. Since $\mathbf{M}^I$ is a prime ideal as it is a maximal ideal, it follows that it is a minimal prime ideal.

(4) $\Rightarrow$ (2): Let $\varphi \in \mathbf{M}^I$. Then, in virtue of $\mathbf{M}^I$ being a minimal prime ideal as hypothesized, φ is annihilated by an element outside $\mathbf{M}^I$. So, $\varphi \in \mathbf{O}^I$. This shows that $\mathbf{M}^I \subseteqq \mathbf{O}^I$, and hence $\mathbf{M}^I = \mathbf{O}^I$ as the other inclusion always holds. $\square$

Definition 4.1.4. A quotient $\beta L \xrightarrow{h} M$ of βL is *round* if for each $c \in \mathrm{Coz}\ L$, $h(r(c)) = 0$ implies $h(r(c^*)) = 1$. It is strongly round if for each $a \in L$, $h(r(a)) = 0$ implies $h(r(a^*)) = 1$.

Proposition 4.1.5. [31] A quotient $\beta L \xrightarrow{h} M$ of βL is round if and only if $h(r(c^*)) = h(r(c))^*$ for each $c \in \mathrm{Coz}\ L$.

The example is taken from [31] and the definition of nearly open map was considered in [5].

Example 4.1.1.

(a) Recall that a homomorphism $h : L \to M$ is said to be *nearly open* if $h(a^*) = h(a)^*$ for all $a \in L$. Every nearly open quotient is strongly round. Hence, every open quotient is strongly round, and every dense quotient is strongly round. Thus, non-spatial frames that admit round quotients abound. Indeed, every quotient of a Boolean frame is open, so that every quotient of a Boolean frame is strongly round. Conversely, if every quotient of L is strongly round, then L is Boolean. To see this, let $a \in L$ and consider the quotient $\beta L \xrightarrow{\varphi} \uparrow a$ given by $J \mapsto a \vee \bigvee J$. Since $\varphi(r(a)) = 0_{\uparrow a}$, $\varphi(r(a^*)) = 1$, that is, $a \vee a^* = 1$.

(b) If in the composite $\beta L \xrightarrow{h} M \xrightarrow{g} N$ of quotients h is round (resp. strongly round) and g is dense, then $\beta L \xrightarrow{gh} N$ is round (resp. strongly round). For, let $c \in \mathrm{Coz}\, L$ such that $gh(r(c)) = 0$. Then, by denseness, $h(r(c)) = 0$, so that $h(r(c^*)) = 1$, implying that $gh(r(c^*)) = 1$. Hence, for any quotient $\beta L \xrightarrow{h} M$, if the quotient $\beta L \xrightarrow{\bar{h}} h_*(0)$ is round (resp. strongly round) then $\beta L \xrightarrow{h} M$ is also round (resp. strongly round).

(c) A quotient map $h : L \to M$ is nearly open if and only if the composite $\beta L \xrightarrow{h\beta_L} M$ is strongly round.

Having observed that open quotients are round, we provide necessary and sufficient conditions for closed quotient to be round. Given $I \in \beta L$, let

$$I^\bullet = \{c \in \mathrm{Coz}\, L \mid r(c) \leqq I\} \text{ and } I_\bullet = I \cap \mathrm{Coz}\, L.$$

Then of course $I^\bullet$ and $I_\bullet$ are ideals of $\mathrm{Coz}\, L$, and, in fact, $I_\bullet = \{\mathrm{coz}\, \varphi \mid \varphi \in \mathbf{O}^I\}$ and $I^\bullet = \{\mathrm{coz}\, \varphi \mid \varphi \in \mathbf{M}^I\}$. Furthermore,

$$I_\bullet \subseteqq I^\bullet$$

and

$$(\ddagger) \qquad I = \bigvee_{c \in I_\bullet} r(c) = \bigvee_{c \in I^\bullet} r(c).$$

Lemma 4.1.2. [31] Let $I \in \beta L$. For any ideal J of $\mathrm{Coz}\, L$, $I = \bigvee\{r(x) \mid x \in J\}$ if and only if $I_\bullet \subseteqq J \subseteqq I^\bullet$.

Proof. Let J be an ideal of $\mathrm{Coz}\, L$ and suppose $I = \bigvee\{r(x) \mid x \in J\}$. Then, in fact, $I = \bigcup\{r(x) \mid x \in J\}$ since the join is directed. So, if $c \in I_\bullet$ then $c \in I$, and hence $c \in r(d)$ for some $d \in J$, implying that $c \in J$. Therefore $I_\bullet \subseteqq J$. On the other hand, if $x \in J$ then

$r(x) \leqq I$, and so $x \in I^\bullet$. Thus, $I_\bullet \subseteqq J \subseteqq I^\bullet$. Conversely, if $I_\bullet \subseteqq J \subseteqq I^\bullet$, then it follows from (‡) above that $I = \bigvee\{r(x) \mid x \in J\}$. □

The desired characterizations of round closed quotients are:

Proposition 4.1.6. [31] The following are equivalent for any $I \in \beta L$:

(1) The closed quotient $\beta L \xrightarrow{h} \uparrow I$ is round.

(2) $I_\bullet = I^\bullet$.

(3) There is only one ideal J of Coz L such that $I = \bigvee\{r(x) \mid x \in J\}$.

Proof. (1) ⇒ (2): If $\beta L \xrightarrow{h} \uparrow I$ is round, then, for any $\varphi \in \mathbf{M}^I$, $r(\mathrm{coz}\ \varphi) \leqq I$, and therefore coz $\varphi \in I$, implying that $\varphi \in \mathbf{O}^I$. Thus, $\mathbf{M}^I \subseteqq \mathbf{O}^I$, and hence $\mathbf{M}^I = \mathbf{O}^I$. Therefore $I_\bullet = I^\bullet$.

(2) ⇒ (3): This follows from the lemma. In fact, the ideal in question is $I_\bullet$.

(3) ⇒ (1): The current hypothesis implies that $I_\bullet = I^\bullet$. Let $\varphi \in \mathbf{M}^I$. Then coz $\varphi \in I^\bullet = I_\bullet$. So there exists $\varphi \in \mathbf{O}^I$ such that coz φ = coz $\psi \in I$. But this implies that $\varphi \in \mathbf{O}^I$. Consequently, $\mathbf{M}^I \subseteqq \mathbf{O}^I$, and hence $\mathbf{M}^I = \mathbf{O}^I$. Therefore, for any $c \in$ Coz L (say, c = coz φ), if $I \vee r(c) = I$, then $r(\mathrm{coz}\ \varphi) \leqq I$, and hence $\varphi \in \mathbf{M}^I = \mathbf{O}^I$, so that $r(c^*) \prec I$, implying $I \vee r(c^*) = 1_{\beta L}$. Therefore $\beta L \xrightarrow{h} \uparrow I$ is round. □

4.2 $\mathcal{R}_K(L)$ and $\mathcal{R}_\infty(L)$

We end the dissertation by giving the results of Ghosh [37] on the note of ideals of $C_\infty(X)$ of a topological space X. The author defined z_∞-ideal and nicely balanced ideals of $C_\infty(X)$ and showed that if I is a z_∞-ideal of $C_\infty(X)$ then I is the intersection of all free maximal ideals of $A(X)$ for some $A(X) \in \Sigma(X)$ if and only if I is nicely balanced. For the background on all free maximal ideals of $A(X)$ for some $A(X) \in \Sigma(X)$ we refer the reader to [17], [26], [37] and [50]. We want to put this result in frame perspective.

In this dissertation we just lay the foundation in terms of frames and extend all the required results in pointfree setting. For a given frame L, let

$$\mathcal{R}_\infty(L) = \{\varphi \in \mathcal{R}L \mid \uparrow\varphi(\frac{-1}{n}, \frac{1}{n}) \text{ is compact for each } n \in \mathbb{N}\}.$$

This is the frame analogue of the subset of $C(X)$ consisting of functions that vanish at infinity (*see*[33], [38]). Again set

$$\mathcal{R}_K(L) = \{\varphi \in \mathcal{R}L \mid \uparrow(\text{coz } \varphi)^* \text{ is compact}\}$$

an ideal that was introduced by Dube in [33].

Observation 4.2.1. Observe that $\mathcal{R}_K(L) \subseteq \mathcal{R}_\infty(L)$. It is shown in [33] that $\mathcal{R}_K(L) \subseteq \mathcal{R}_s(L)$, where

$$\mathcal{R}_s(L) = \{\varphi \in \mathcal{R}(L) \mid \text{ coz } \varphi \text{ is small}\},$$

and that $\mathcal{R}_s(L) \subseteq \mathcal{R}_\infty(L)$.

Definition 4.2.1. An ideal I of $\mathcal{R}(L)$ or $\mathcal{R}^*(L)$ is said to be *fixed* if $\bigvee\limits_{\varphi \in I} \text{ coz } \varphi < 1$.

Definition 4.2.2. An ideal I of $\mathcal{R}(L)$ or $\mathcal{R}^*(L)$ is said to be *free* if $\bigvee\limits_{\varphi \in I} \text{coz } \varphi = 1$.

Let $\Sigma(L) = \{\text{family of all subrings of } \mathcal{R}(L) \text{ contaning } \mathcal{R}^*(L)\}$. If $A(L) \in \Sigma(L)$, then the intersection of all free maximal ideals of $A(L)$ is the set

$$A_\infty(L) = \{\alpha \in A(L) \mid \alpha\varphi \in \mathcal{R}_\infty(L) \text{ for all } \varphi \in A(L)\}.$$

Proposition 4.2.1. If $A(L) \in \Sigma(L)$, the intersection of all free maximal ideals of $A(L)$ is an ideal of $\mathcal{R}_\infty(L)$ containing $\mathcal{R}_K(L)$.

Proposition 4.2.2.

(a) $\mathcal{R}_\infty(L) = \bigcap\{Q \in \mathcal{R}^*L \mid Q \text{ is a free maximal ideal}\}$.

(b) If L is realcompact, then $\mathcal{R}_K(L) = \bigcap\{Q \in \mathcal{R}L \mid Q \text{ is a free maximal ideal}\}$.

(c) If L is realcompact, then $\mathcal{R}_K(L) = \mathcal{R}_s(L)$.

Definition 4.2.3. A proper ideal J of $\mathcal{R}_\infty(L)$ is called a z_∞-*ideal* if $\alpha \in J$, $\varphi \in \mathcal{R}_\infty(L)$ and coz $(\alpha) =$ coz (φ) imply that $\varphi \in J$.

Definition 4.2.4. A proper ideal J of $\mathcal{R}_\infty(L)$ is called a *nicely balanced* if $\alpha \in \mathcal{R}_\infty(L) \smallsetminus J$ implies that there is $\varphi \in \mathcal{R}L$ such that $\varphi\gamma \in \mathcal{R}_\infty(L)$ for some $\gamma \in J$ but $\alpha\varphi \notin \mathcal{R}_\infty(L)$.

Proposition 4.2.3. Let J be a z_∞-ideal of $\mathcal{R}_\infty(L)$. Then $J = A_\infty(L)$ for some $A(L) \in \Sigma L$ if and only if J is nicely balanced.

Proposition 4.2.4. [33] $\mathcal{R}_s(L) = \bigcap\{M \subseteq \mathcal{R}(L) \mid M$ is a free maximal ideal$\}$.

Proof. Let $\varphi \in \mathcal{R}_s(L)$ and I be a point of βL with $\bigvee I = 1$. We must show that $r(\text{coz } \varphi) \leqq I$. If not, then $r(\text{coz } \varphi) \vee I = 1_{\beta L}$, and therefore there is a cozero element c in I such that $c \vee \text{coz } \varphi = 1$. Thus, $\uparrow c$ is compact since $\varphi \in \mathcal{R}_s(L)$. But now the set $I' = I \cap \text{Coz } L$ is a proper ideal of Coz L such that $c \in I'$ and $\bigvee I' = 1$. This violates the lemma, and hence establishes the inclusion $\subseteqq$. On the other hand, let φ be in the stated intersection. Suppose, for contradiction, that $\varphi \notin \mathcal{R}_s(L)$. Then there is a cozero element c such that $c \vee \text{coz } \varphi = 1$ but $\uparrow c$ is not compact. By the lemma, select a proper ideal J of Coz L such that $c \in J$ and $\bigvee J = 1$. Put $Q = \{\alpha \in \mathcal{R}L \mid \text{coz } \alpha \in J\}$. Clearly Q is a free proper ideal of $\mathcal{R}L$, and so is contained in some free maximal ideal M. Take $\gamma \in \mathcal{R}L$ such that $c = \text{coz } \gamma$. Then M contains both γ and φ, and hence the invertible element $\gamma^2 + \varphi^2$, which is impossible. Therefore the reverse inclusion also holds. □

Corollary 4.2.1. [33] $\mathcal{R}_K(L) = \bigcap\{Q \subseteq \mathcal{R}(L) \mid Q$ is a free ideal$\}$.

Proof. By the Proposition, $\mathcal{R}_K(L)$ contains the stated intersection. For the reverse inclusion, let $\varphi \in \mathcal{R}_K(L)$ and let Q be a free ideal of $\mathcal{R}L$. As shown in [33, Proposition 3.4], if we set $J = \bigvee\{r(\text{coz } \alpha) \mid \alpha \in Q\}$, then $\mathbf{O}^J = mQ \subseteqq Q$. Notice that

$$\bigvee J = \bigvee\{\text{coz } \alpha \mid \alpha \in Q\} = 1$$

since Q is free. Thus, $\{(\text{coz } \varphi)^* \vee x \mid x \in J\}$ is a cover of the compact frame $\uparrow(\text{coz } \varphi)^*$. Since J is an ideal of L, compactness therefore yields an element u of J such that $(\text{coz } \varphi)^* \vee u = 1$. Therefore $\text{coz } \varphi \leqq u$, and hence $\text{coz } \varphi \in J$, so that $\varphi \in \mathbf{O}^J \subseteqq Q$. Thus, $\mathcal{R}_K(L) \subseteqq Q$, which establishes the outstanding inclusion. □

Corollary 4.2.2. Suppose $\mathcal{R}_K \neq \mathcal{R}_\infty(L)$. Then $\mathcal{R}_K(L)$ is nicely balanced ideal of $\mathcal{R}_\infty(L)$ if and only if $\mathcal{R}_K(L) = \mathcal{R}_s(L)$.

Lemma 4.2.1. Let $A(L),\ B(L) \in \Sigma(L)$ with $A(L) \leq B(L)$. Then $B_\infty(L) \leq A_\infty(L)$.

Proposition 4.2.5. Suppose $\mathcal{R}_K(L) \neq \mathcal{R}_\infty$. Then $\mathcal{R}(L)$ is a nicely balanced ideal of $\mathcal{R}_\infty(L)$ if and only if $\mathcal{R}_K(L) = \bigcap\{Q \in \mathcal{R}(L) \mid$ is a free maximal ideal$\}$.

Definition 4.2.5. For any sublattice A of a frame L, an ideal $J \subseteq A$ is called

(a) *σ-proper* if $\bigvee S \neq 1$ for any countable $S \subseteq J$ and

(b) *completely proper* if $\bigvee J \neq 1$, the join understood in L.

Definition 4.2.6. A frame L is called *realcompact* if any σ-proper maximal ideal in Coz L is completely proper.

Corollary 4.2.3. If L is realcompact such that $\mathcal{R}_K(L) \neq \mathcal{R}_\infty(L)$. Then $\mathcal{R}_K(L)$ is a nicely balanced ideal of $\mathcal{R}_\infty(L)$.

For any frame L, $\varphi : \mathcal{L}(\mathbb{R}) \to L$ is called bounded if $\varphi(p,q) = 1$ for some $p, q \in \mathbb{Q}$, and L is called *pseudocompact* if all $\varphi : \mathcal{L}(\mathbb{R}) \to L$ are bounded.

Corollary 4.2.4. If L is pseudocompact such that $\mathcal{R}_K(L) \neq \mathcal{R}_\infty(L)$ then $\mathcal{R}_K(L)$ is not a nicely balanced ideal of $\mathcal{R}_\infty(L)$.

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