AN EXPLORATION OF ACE TEACHING CYCLE IN IMPROVING GRADE 12 LEARNERS' UNDERSTANDING OF QUADRATIC FUNCTIONS

By

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DEDICATION

I dedicate this dissertation to my wife, Natasha, and my children, Lwandle and Langavi. To my father Willie (deceased) and Mother Maria, who were my first teachers.

DECLARATION

I declare that **An exploration of the ACE teaching cycle in improving Grade 12 learners' understanding of quadratic functions** is my work and that all the sources I have quoted or used have been acknowledged and indicated through a complete reference list. This work has not been submitted before for any other degree at any other institution.

27 March 2023

Hlangwani, W (Mr) Date

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ABSTRACT

The knowledge of quadratic functions is essential in improving learners' conceptual understanding of algebraic and geometric concepts. Hence, such knowledge is attained when learners understand quadratic functions' multiple representations. Therefore, the focus of the study was to explore the role of the Activities, Classroom discussions, and Exercises [ACE] teaching cycle in improving Grade 12 learners' conceptual understanding of quadratic functions. I adopted the APOS theory as a lens to improve learners' conceptual understanding of quadratic functions. Additionally, I used Merriam's case study design which incorporated the ACE teaching cycle as an instructional style for data collection with 30 criterion sampled learners. Subsequently, learners posit various conceptual obstacles after implementing the ACE cycle. Firstly, they seem to grapple at the action level relating to the knowledge of the properties of quadratic functions. Secondly, they posed conceptual obstacles to quadratics while interacting with quadratic functions. Thirdly, learners posed difficulties transitioning from one form to another, posing pitfalls to conceptual understanding quadratic functions. Lastly, they faced difficulty making connections between the forms of quadratic functions due to a deficiency in solving techniques. However, through the intervention of the exercise, learners' conceptual obstacles seem to be remedied. Yet, some conceptual obstacles appeared to be persistent: failure to correctly translate quadratics, confusion about the y-intercept of the function and the y-coordinate of the vertex, and difficulty linking the connection between the range and vertex of a function. Therefore, I recommend that future studies be broadened on learners understanding of the vertex and the y-intercept of quadratic functions.

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CHAPTER ONE: INTRODUCTION AND BACKGROUND

1.1. INTRODUCTION

In this chapter, I present a synopsis of the study. The overview contains the background and motivation of the study, the statement of the research problem, the purpose of the study, and the research question and methodology. Then, the chapter ends with the significance and a chapter summary.

1.2. BACKGROUND AND MOTIVATION

Literature reveals that conceptual understanding of the concept function remains a challenge to most learners. Consequently, the study of functions continues to receive significant input from different researchers (de Sousa & Alves, 2022; Santia & Sutawidjadja, 2019; Sönnerhed, 2021; Wilkie, 2021; Zaslavsky, 1997). However, questions about learning functions in mathematical understanding remain unanswered. Therefore, the focus of this study limits itself to quadratic functions when addressing the unanswered questions. I paid attention to the following:

(1) The emergence of quadratic functions from algebra.
(2) Philosophical underpinnings towards quadratic functions.
(3) Mathematical understanding of quadratic functions.
(4) The concept of a quadratic function.
(5) Conceptual obstacles that inhibit learners' understanding of quadratic functions; and
(6) mitigating learners' conceptual obstacles of quadratic functions.

1.2.1. The emergence of quadratic functions from algebra

Quadratic functions emerge from the understanding of arithmetic and algebraic concepts. Algebra is the study of mathematical symbols and procedures for engaging with those symbols (Ko et al., 2021). Conceptually, quadratic functions draw from the knowledge of arithmetic and algebraic concepts: arithmetic studies, properties of their operations and roots (Rahayu et al., 2021). However, learners encounter difficulties in transitioning from arithmetic to algebraic concepts. These difficulties include failing to make connections between algebra and arithmetic concepts because learners treat algebra as abstract. O’Connor and Norton (2016) found that lacking both the action and process of algebra inhibits the development of an object and schema level of understanding quadratics. O'Connor and Norton also noted that the absence of these levels of understanding hindered comprehension of quadratic function concepts.

Algebraic concepts are structural and operational; structural algebra entails objects, and operational algebra involves computational processes (Sfard, 1991). Sfard’s view of the term object concurs with Arnon et al.’s (2014) view, as they both narrow it to the ability to construct a static structure. The static structures are mental constructions the learner has reflected upon various times, i.e., the action and process levels of understanding. The learners' reflection on the action and process develops an object level for understanding (Bansilal et al., 2017). Thus, the ability to see a quadratic function as a process and object is indispensable for conceptual understanding.

Algebra precedes quadratic functions; thus, learning quadratic functions is done after learners have learned algebraic concepts (Ko et al., 2021). Learning quadratic function is embedded in the skills developed through algebraic reasoning. Algebraic reasoning entails forming generalisations from prior knowledge of computations, constructing these ideas into meaningful symbolic systems, and exploring the concept of functions (Barana, 2021). Algebraic reasoning permeates all mathematical concepts, meaning that without understanding the concepts, learners will struggle with conceptual comprehension of quadratic functions. In high school mathematics, learners are engaged in algebraic reasoning when they can be able to make connections between the concepts of the algebraic and quadratic functions.

Algebraic reasoning allows the learner to operate in the indeterminate form of algebra when dealing with quadratic functions. However, algebra is not easily understood due to its complexity since it is indeterminate (Pinto & Cañadas, 2021). Usiskin (1998) explains the indeterminacy of algebra using five tenets, i.e., an equation to solve, a formula, a property, an identity, and a function with a direct variation. I adapted the last conception, i.e., a function with a direct variation to study quadratic functions. Treating algebra as an equation of a function enhances algebraic reasoning. Algebraic reasoning provides affordances for understanding the development of the quadratic function concept.

1.2.2. Philosophical underpinnings toward quadratic functions

Learning mathematics is inseparable from mathematics concepts. In the process of learning mathematics, the concept of a quadratic function is a vital aspect to be understood by the learner since its understanding lies in the transition from arithmetic and algebra. The transition has been viewed from the traditional perspective since it is assumed that learners would have mastered the necessary skills in arithmetic before interacting with algebra. These learned arithmetic skills permit the learner to treat the quadratic function as not a new concept (Wagner & Kieran, 2018). To demonstrate this transition, for the example given that $f(x) = x^2 + 1$. From the function, the learner must first appreciate that this is a quadratic function. Thus, from it, we can determine the x-intercepts, which is a skill embedded in algebra, and determine the value of $f(2)$, which is a skill embedded in arithmetic. Therefore, the learner engaging in this activity will demonstrate their understanding of prior knowledge of arithmetic and algebra.

Understanding prior knowledge allows learners to interact with any quadratic function task since functions are the basis of mathematics (Cahyani & Rahaju, 2019). Learners are not expected to demonstrate an understanding of the quadratic function but to show how the concept is developed. The development of mathematical concepts is the basis for learning mathematics. In developing mathematical concepts, learners are engaged in abstraction since concepts are abstract. The concept development process emanates from abstraction, which is a conscious activity of making connections of mathematical structures (Narhasanah et al., 2017). The process of abstraction is considered the last resort and the highest form in the concept development process. Arnon et al. (2014) assert that abstraction is the process of encapsulating processes. Therefore, this abstraction is organised into four levels: action, process, object, and schema.

In context, the abstraction of quadratic functions looks at the concept in four tenets, i.e., action, process, object, and schema. At an action level, the quadratic function implies that the learner can now use the Table method using arithmetic skills to determine the vertex, find the vertex using the formula, and describe the shape of a quadratic function. Secondly, at the process level, the learner can now define the quadratic function, sketch the function using the Table method, and understand the meaning of the vertex of the quadratic. Thirdly, the object level allows the learner to understand the word problems of quadratic function concepts and can interact with the transformation of the graph. Lastly, the schema level is the highest form of abstraction since it is an organised and logical framework of all the tenets above.

The process of abstraction involving quadratic function is in line with Monaghan and Ozmantar (2006), which is an activity of vertically recognising prior mathematical knowledge into a new mathematical structure. This type of abstraction requires the learner to reflect on their constructed schema. The reflection process is called reflective abstraction. Cahyani and Rahaju (2019) assert that reflective abstraction, which is mathematics highest form of thinking, is the ability to construct new understanding through the mechanism of linking certain mathematical constructs (Dubinsky, 2002). In context, learners' reflective abstraction is the process of acknowledging the fact that quadratic functions are constructed from the transition of arithmetic to algebra, as articulated above. Moreover, reflective abstraction can develop new knowledge of learners by identifying problems and then seeking solutions using different appropriate procedures from the action construct to the developed schema.

The mechanism applied to access this abstraction can be promoted by using a mental structure involving the action, process, object, and schema (APOS) (Dubinsky, 2002). Reflective abstraction is intertwined with the mathematical understanding of quadratic function. Understanding is explained in the next section. However, blending understanding with the APOS constructs results in the development of the genetic decomposition of the quadratic function concept. Genetic decomposition is a mental structure a learner might demonstrate to understand a mathematical concept (Dubinsky, 2002). Moreover, genetic decomposition is a tool that shows how the concept of a quadratic function is organised in the learner's mind

(Jojo, 2019). The development of the genetic decomposition is guided by the researcher’s knowledge of understanding the quadratic function concepts.

The genetic decomposition allows the researcher to offer affordances to learning quadratic functions using the four tenets of the APOS. The tenet action is the cognitive structure of using prior knowledge as an external part in dealing with operations. During the action level of understanding, the learner working with $f(x) = (x + 1)^2$ engaged with the quadratic function concept like a vertex. This learner will need a vertex formula to locate it and then move to the process stage. The process is the cognitive structure that uses the learned actions; but during the process, learners conduct these operations as part of the internal process. The process level of understanding produces learners who no longer require a formula to determine the vertex. Instead, the learner can now determine any vertex of a quadratic function at this stage and link it to the axis of symmetry. The action and process are encapsulated into an object level of understanding.

The object is the cognitive structure that allows learners to acknowledge the learned actions and processes as totality transformations and constructs the knowledge explicitly (Luneta & Makonye, 2010). The learner operating at this level of understanding can make a linkage between quadratic function concepts. Schema is the collection of the action, process, and objects into a skeletal framework to help learners solve any mathematical problem presented. The learner operating at this stage of understanding can simply make connections of the concepts of a quadratic function, and the knowledge can be applied in real-life instances.

The four tenets laid above are coined into APOS theory by Dubinsky (2002). APOS theory is the framework of how mathematical concepts are developed and learned (Arnon et al., 2014). Moreover, the effort of the APOS theory alludes to what is going on in the learner's mind when trying to learn mathematics concepts. Learners engaged in the process of understanding are said to be involved in construction. The movement through the four tenets of understanding alluded to above strengthens constructivism, which is the reference for the emergence of new understanding through constructive action, process, and objects. Thus, my philosophical underpinning is rooted in constructivism, which is a philosophy explaining how mathematical knowledge occurs in a learner's mind, which conforms to APOS theory.

1.2.3. Mathematical understanding of quadratic functions

The concept of understanding in mathematics is fluid and poses a challenge in defining it. Thus, understanding is a covert behaviour and remains the internal proponent of the learner. Understanding, simply put, is a process that goes beyond knowing more than routine procedures (Perkins & Blythe, 1994). Furthermore, understanding is fully immersed in the content to explain, find evidence and examples, and represent the topic in a new way (Raheem & Jawad, 2019). This process of understanding emphasises not only the correctness of the eventual answer but has shifted to emphasising the process, context, and comprehension. The learning of mathematics should advance understanding of the concepts being taught.

The concept of mathematical understanding was also researched by Skemp (1976), who identified two types of understanding, i.e., instrumental, and relational understanding. Instrumental understanding is perceived as rules without reasons, while relational understanding is the process of knowing what to do and why it is done in that context. For example, if a learner is given the following function $f(x) = x^2 - 2x + 1$, and it is required to determine the value of $(0; p)$. This question nurtures instrumental understanding, requiring learners to substitute zero into the function. Therefore, the question will not be difficult for learners as it is routine work.

Nickerson (1985) asserts that understanding entails the ability to see the concept in a deeper context, make connections of the concept, and envision the concept using mental structures. This view coincides with Star's (2005) view of deep procedural understanding, which is the ability to comprehend a mathematical concept flexibly while taking all the critical aspects into a thorough judgment. Moreover, understanding is the ability to coherently build context or cognitive structures (Hiebert & Carpenter, 1992). As a result, understanding is seen as an action or an outcome of actions. Sierpinska (1994) clarified this by proposing three tenets that guide understanding, i.e., an act of understanding, understanding and the process of understanding. These three tenets are intertwined and interwoven in that one cannot operate in isolation from the other. Sierpinska's view of understanding is that the process is a cognitive task achieved after a lengthy period.

Duffin and Simpson (2000) extend Sierpinska's (1994) view of understanding. They hold that the tenets proposed by Sierpinska are the building blocks for conceptual understanding. The concept of understanding is fluid and extraordinarily complex, lacking research grounding. Therefore, I adopted Sierpinska's view of understanding. Concisely, I view conceptual understanding as the ability of the learner to make connections between mental representations of a mathematical concept. Subsequently, this view is like what Rittle-Johnson (2017) termed procedural flexibility, which nurtures both types of knowledge in an iterative bidirectional view. In addition, understanding is the outcome of representations linked to mathematical concepts. This definition of understanding is in line with the APOS theory. The theory outlines four tenets that depict learners' ability to understand mathematical concepts. The movement through these tenets permits the construction of the highest form of understanding.

Sierpinska's view of understanding permits me to assess learners' comprehension of mathematical concepts. Knowing that, one can argue to say what informs the assessment of conceptual understanding. In the present study, assessment of conceptual understanding was informed by the APOS theory, which allowed me to grade learners' understanding irrespective of the correctness or incorrectness of the answer. I used the APOS theory with its instructional method, i.e., the ACE teaching cycle to categorise the learners' understanding at various levels based on their procedural flexibility. This theory is in line with the view by Hiebert and Carpenter (1992) that understanding cannot be inferred from one activity of a single response. This means that for one to say they understand the quadratic function concept, they can now interiorise the concept.

In acknowledging that understanding is a complex and challenging task, this means that if I am to assess it, I need to access various connection networks of concepts that a learner has. Therefore, when I assess learners' conceptual understanding of quadratic functions, I look for traits of comprehension of the concepts. Thus, the ability of the learner to conduct a mathematical task means that some glimpses of understanding are there, and not to the extent. Therefore, assessing understanding looks for the learners' conceptual obstacles that inhibit connections between the mathematical concepts, the procedures, and the connections made between the mathematical representations (Hiebert & Carpenter, 1992). Learners' understanding viewed from the APOS theory demonstrates learners' potential at various levels. Therefore, this form of understanding can be assessed through the different tenets of the theory.

1.2.4. The concept of a quadratic function

Parent (2015) asserts that a function is a one-to-one mapping from one set, its domain, to another, its range. A function is a relation in which the first coordinate is never repeated. Every input has only one output. For example, if there is a coordinate $(1;2)$ on a function, it cannot be repeated. However, functions are limited to quadratic functions in the present study. The quadratic function is one of the forms $f(x) = ax^2 + bx + c$ where the variables a, b and c are integers and both (determinate and indeterminate) and the value of $a \neq 0$ (Nielsen, 2015). The quadratic function can be expressed in three different forms, i.e., standard form, i.e., $f(x) = ax^2 + bx + c$, factored form, i.e., $f(x) = a(x - x_1)(x - x_2)$, and vertex form, i.e., $f(x) = a(x - p)^2 + q$ (Ousby et al., 2008). In high school mathematics, learners interact with the critical components of quadratic function, i.e., the intercepts, turning point, axis of symmetry, domain, range, and the effects of determinate and indeterminate parameters (Parent, 2015). Due to its indeterminate and abstract nature, the comprehension of quadratic function also hinges on the strength of connections and interrelatedness of its concepts, as mentioned above (Mutambara et al., 2019).

The strength of connections of the tenets of quadratic function revolves around the integration of algebraic reasoning. This implies that the objective of quadratic function content shifts from the abstract nature to the concrete nature. The connection is seen as follows, given $f(x) = x(x + 6)$, the learner must use their skills in arithmetic and algebra to determine the x-intercepts, the vertex, the axis of symmetry, its domain and range. Parent (2015) found that learners struggled with making the connections as they failed to note that the $f(x)$ is quadratic. However, suppose learners could interact with the question; in this case, it should have allowed them to solve equations to determine the x-intercepts, demonstrate the relationship of the parent function to other related functions, and sketch the quadratic function using the vertex form, standard form, and factored form.

The integration of algebraic reasoning yields learners to transit from one representation to another by demonstrating the interrelatedness of the concept. The transition from one representation to another is depicted when learners are required to sketch the graph of the quadratic function. Given $g(x) = -2x^2$ it is challenging for learners to visualise this abstract concept but to see, they need a representation concretely. The strength of the connection is the view that learners' understanding is rooted in prior knowledge of functions. Learners must connect prior and present knowledge (Pirie, 1988). This view indicates that conceptual understanding is an ongoing process and is not achieved in isolation. Therefore, when learners develop their conceptual understanding of quadratic function concepts, they make various levels of understanding, as Dubinsky (2002) outlined in action, process, object, and schema. Consequently, these learners are guiding themselves to make these connections since understanding is flexible and a covert behaviour (Hiebert & Carpenter, 1992). However, during the process learners can hit a snag which poses a conceptual obstacle in comprehending the concept.

1.2.5. Learners' conceptual obstacles of quadratic functions

Learning quadratic functions is crucial in mathematics since a sound understanding of it is essential to excel in a variety of mathematics content (Hiebert & Carpenter, 1992). Therefore, improving their conceptual understanding of quadratic functions is paramount to learners' courses as they progress working with other polynomials (Didiş et al., 2011). Learners' development of conceptual understanding of quadratic functions implies that they can: solve quadratic functions using different strategies to find the critical points of the function; and use their understanding of quadratic functions to analyse, represent and create graphs.

Literature asserts that conceptual understanding quadratic functions is vital; however, learners grapple to grasp the concept, which posed conceptual obstacles (Astuti & Hidayat, 2020; Didiş et al., 2011; Kotsopoulos, 2007; Mutambara et al., 2019; Parent, 2015). Conceptual obstacles lie in the process of accommodation and assimilation that learners make when they interact with mathematical concepts. The former is the process where learners make integration of prior knowledge, while the latter is the change in learners' schema triggered by new knowledge (Wagner, 2010). During the accommodation process, learners encounter conceptual obstacles due to the difficulty of changing the existing schema to link to the present. Therefore, conceptual obstacles are cognitive difficulties that can be elaborated by mathematical concepts that underpin learners’ prior learning.

Conceptual obstacles inhibit learners’ comprehension of the quadratic function concept. The conceptual obstacles include, amongst others, i.e., misinterpretation of graphical information; failing to note the relation between quadratic equation and quadratic function; inability to comprehend the analogy of quadratic function and linear functions; failure to transit from one form of a quadratic function to a standard form, vertex form or factored form; and emphasising one unique coordinate at the expense of others (Zaslavsky, 1997).

Moreover, Nielsen (2015) noted that learners had a conceptual difficulty of the concept variable x in $f(x) = (x - 3)(x - 5)$. In the study, the learner assumed that the variable x is not consistent in the product. These conceptual obstacles are inherited from fragmented knowledge of algebra. In a related study, Zaslavsky (1997) found that learners could not use the implicit content linked to the concept of the axis of symmetry. Thus, conceptual obstacles were not only noted when learners solved quadratics, even when they were required to sketch the graphs.

If conceptual obstacles are not addressed, this will result in fragmented knowledge of quadratics functions (Ozaltun-Celik & Bukova-Guzel, 2017). Fragmented knowledge of quadratic functions results from unaddressed conceptual obstacles. Consequently, a fragmented understanding of quadratic functions results from teaching strategies. Hiebert et al. (2007), cited in Parent (2015), concur by stating that teaching strategies in high school promote a fragmented understanding of mathematical concepts. This posed difficulties in conceptual understanding quadratic functions.

1.2.6. Mitigating learners' conceptual obstacles of quadratic functions

Therefore, mitigating learners' conceptual obstacles of quadratic functions requires the teacher to conduct frequent diagnosis and to provide remedies to the conceptual obstacles. Hence, the remedy for learners' conceptual obstacles of quadratic functions requires learning for understanding. Fong (1982) explained the two terms diagnosis and remedy in a mathematical context. A diagnosis is a process to determine the nature of the conceptual obstacle, and a remedy is a method afforded to the learner to mitigate the hindrances (Fong, 1982). Mathematical understanding is a fluid concept in the subject area; however, the study is limited to conceptual understanding when dealing with the concept. Conceptual understanding mathematics means the capability to act and think flexibly with a concept.

In addition, conceptual understanding is a process that goes beyond knowing, and it is more than the collection of mathematics concepts; it is not the idea of knowing the procedure only. Conceptual understanding is the learners' skills to justify why a mathematical notion is true and makes sense. This justification process requires reflective thought, where learners can connect prior knowledge to new knowledge while modifying their present schemas (Widada et al., 2020). This process can happen either in two ways: assimilation and accommodation. As explained above, it is noted that if the assimilation process is not in cooperation with accommodation, it yields conceptual difficulties.

Therefore, the learning activities responsibility is to provide affordances for learners' understanding of quadratic functions. The affordances are achieved when the assimilation does not contradict the accommodation process. Confronted by these reflective thoughts, this challenged me to introspect on what nurtures an understanding of quadratic functions and how the growth of learners' conceptual understanding of quadratic functions can be enhanced. As a result, activities, classroom discussions, and exercises [ACE] teaching cycle became a quest for me to undertake the present study to improve learners' understanding of quadratic functions.

The ACE teaching cycle is an instructional approach guided by the constructivist theory (Arnon et al., 2014). Glasersfeld (1991) asserts that constructivism is a theory of learning that holds that mathematical concepts are developed in an active process of creating knowledge. Four principles guide constructivism, i.e., learners possess prior knowledge of mathematical concepts; mathematical knowledge is constructed uniquely and individually; the learning process is active and reflective; and learning is developed in nature (Glasersfeld, 1991). These four principles are embedded within the APOS theory. The first principle considers that the learner has specific knowledge of quadratic functions at the action level of understanding, lacking development. The second principle is in line with the process level of understanding as this is a covert activity of the learner to reflect on the actions developed to move to the second level of understanding.

The third principle requires the learner to practically reflect on the developed actions and processes to operate at the object level of understanding quadratic functions. The last principle is the last resort of understanding such that the learners' schema is developed, and it permeates them to reflect on all three levels of understanding. Therefore, mathematical understanding is a covert behaviour that occurs in an individual's mind (Bodner & Elmas, 2020). The learner must develop their understanding since the knowledge is built from their unique web of prior concepts. Therefore, this is stimulated by the teacher's activities to nurture learners' mathematical understanding (Akilli & Genç, 2017). The instructional approach that can be used to implement constructivism in the classroom is the ACE teaching cycle (Arnon et al., 2014).

The ACE teaching cycle is grounded in the APOS theory; as such, the nature of instruction adheres to the perspective of the theory on what it means to understand mathematics (Santos, 2019). The first construct of the cycle activities deals with the learners working cooperatively on mathematics concepts designed with mental constructions. Secondly, the classroom discussions involve interaction with the developed schema in the activities, and lastly, the exercises provide affordances for learners to explore the concept further.

1.3. RESEARCH PROBLEM

Knowledge of quadratic functions is vital in improving learners' conceptual understanding. Such knowledge is attained when one comprehends the multiple representations of functions and analyses them graphically (Mutambara et al., 2019). Learning representations are not limited to fluency in algebraic and geometrical processes but also hinge on the strength of connections and interrelatedness of quadratic function concepts (Santia & Sutawidjadja, 2019). This complex web of quadratic functions concept, which is determinate and indeterminate, is relational (Ubah & Bansilal, 2018). For example, studies report that learners struggle with basic processes and procedures of quadratic functions, such as simplification and finding intercepts (Ozaltun-Celik & Bukova-Guzel, 2017; Parent, 2015). An inability to comprehend quadratic functions may indicate poorly focused knowledge on how to get the intercepts, turning point, the axis of symmetry, domain, range, and the effects of parameters in quadratic functions. Poor attention to these critical aspects of quadratic functions may explain why learners possess an undeveloped, underdeveloped, and fragmented understanding of mathematics concepts (Ubah & Bansilal, 2018). Unfocused attention to essential aspects of quadratic functions was also observed in Didiş et al.'s (2011) study, which asserted that learners preferred factorisation but struggled with simplifying to find x-intercepts. In a related study by Eraslan (2005), learners applied factorisation to determine the x-intercepts. These learners promoted instrumental understanding and did not read the graph for the same purpose because they failed to represent the x-intercepts on a graph. Despite studies investigating misconceptions (Eraslan, 2008; Fonger et al., 2020; Kotsopoulos, 2007; Makonye & Shingirayi, 2014; Ruli et al., 2018), using different teaching methods (Astuti & Hidayat, 2020; Benning & Agyei, 2016) and learners thinking (Nielsen, 2015; Parent, 2015) of quadratic functions, hardly any attention was paid to improving the understanding of such functions. Hence there is limited research focusing on improving learners' understanding of quadratic concepts. Therefore, the present study explored how learning for mathematical understanding could enrich learners' conceptual understanding of quadratic functions through the ACE teaching cycle.

1.4. PURPOSE OF THE STUDY AND RESEARCH QUESTIONS

The study aimed to explore the role of the ACE teaching cycle in improving Grade 12 learners' conceptual understanding of quadratic functions. To pursue the purpose of the study, the study answered one research question:

- How does the ACE teaching cycle improve learners' conceptual understanding of quadratic functions?

1.5. RESEARCH METHODOLOGY

I adopted an interpretive approach to qualitative research. Qualitative research explores and understands the meanings of the individual (Bogna et al., 2020). In this study, I explored the learners' conceptual understandings as they interacted with mathematics tasks in the classroom. I employed Merriam's (1998) view of the case study design. Merriam asserts that a case study is the complete portrayal and investigation of a bounded phenomenon. As suggested by Merriam, the boundaries of the study were the Grade 12 learners' conceptual understanding of quadratic functions concepts, i.e., the axis of symmetry, vertex, the location of x-intercepts, whether the graph opens up or down, the maximum or minimum point of the graph, the y-intercept, and the transformations through the ACE teaching cycle. Therefore, the study adopted purposive sampling of 30 Grade 12 mathematics learners who participated in the study. I collected qualitative data using the tenets of the ACE teaching cycle.

In the activities and exercise phases, I have used Task 0, a learning task, and a test that conforms to documents. Additionally, I have used classroom discussions that conform to unstructured interviews (Merriam, 1998). The study used content analysis to analyse the qualitative data in a deductive way. Moreover, I used the APOS theory as a lens to analyse the data I collected in the activities, classroom discussions and exercises. I ensured rigor in the analysis by attending to conformability, credibility, dependability, and transferability, as recommended by Guba (1981).

1.6. SIGNIFICANCE OF THE STUDY

Potentially the study adds knowledge to the literature on the learners' conceptual understanding of quadratic functions. Moreover, the study will guide researchers and policy makers on the conceptual obstacles that learners exhibit when interacting with quadratic function tasks.

1.7. RESEARCH SETTING

The participants in this study were Grade 12 learners at a school in Namakgale Circuit under Mopani District in Limpopo Province. The sample was chosen based on performance in mathematics marks from the previous Grade. The school has Xitsonga and Sepedi speaking learners from nearby villages and around the school's location, and offers science, commercial, and general subjects. The school has one class offering Physical Science because most learners study Mathematical Literacy.

1.8. STRUCTURE OF THE STUDY

The study aimed to explore the role of the ACE teaching cycle in improving Grade 12 learners' understanding of quadratic functions. To pursue the purpose, the study answered the following research question: How does the ACE teaching cycle improve learners' conceptual understanding of quadratic functions?

I presented five chapters in this dissertation. Chapter one of outlined the introduction, background to the study, problem statement, the purpose of the study and research questions, overview of the research methodology, significance of the study, overview of the study, and lastly, the chapter's summary. In the second chapter, I presented the theoretical framework, which established the boundaries of the study. Additionally, I presented the literature review, which is divided into the following sections: quadratic function concept, mathematical understanding, and representation of quadratic functions, assessing learners' understanding, understanding of quadratic functions, conceptual obstacles about quadratic functions, and contextual teaching of quadratic functions.

Chapter three discusses the research methodology that guided the study. In the chapter, I present a rationale from the qualitative research paradigm, research design, the sampling method employed, the data collection method used, data analysis procedures, quality criteria, and ethical considerations. In the fourth chapter, I simultaneously analysed and discussed the research findings. I further divided the chapter into data analysis and discussions, synthesis of the research findings. I offered a summary of the chapter as the last section. Lastly, I presented the recommendations and conclusion of the study. This chapter is divided into the introduction, research design and method, interpretation of the research findings, recommendations, limitations, and the conclusion of the study.

1.9. SUMMARY OF THE CHAPTER

In this chapter, I presented the background and motivation of the study, the problem statement, the purpose of the study and research questions, the research setting, an overview of the research methodology, the significance of the study, and an overview of the study. The chapter concludes by providing an overview of the dissertation.

CHAPTER TWO: LITERATURE REVIEW

2.1. INTRODUCTION

The knowledge of functions is one of the essential concepts in all mathematics content (Eisenberg & Dreyfus, 1994). The study of functions generally receives much input from different researchers (Hartter, 2009; Hight, 1968; Wilson, 1994; Zaslavsky, 1997). Wilson (1994) noted that the emphasis on functions is the glue to mathematical concepts and a representation of actual instances. Although different scholars have researched the concept of functions and their studies, the question of "mathematical understanding of functions remains unanswered." In search of how the mathematical understanding of quadratic function can be nurtured, I have detailed this chapter into seven sections.

In the first section, I present the theoretical framework of the study, i.e., APOS theory, coupled with a synopsis of recent research that utilised the APOS theory. In the second section, I present the breakdown of the quadratic function concept. Thirdly, I present an extensive review of mathematical understanding and representation of quadratic functions. The fourth section deals with assessing learners' conceptual understanding. Moreover, the fifth section focuses on learners' conceptual understanding of quadratic functions. The sixth section discusses the research on learners' conceptual obstacles about quadratic functions. The seventh section looks at how quadratic functions can be taught for conceptual understanding, presenting the contextual teaching of quadratic functions. Lastly, I gave the summary of the chapter.

2.2. ROLE OF THEORY

2.2.1. My philosophical underpinnings of mathematics

The mathematics pedagogy rests on a philosophy of mathematics (Thom & Howson, 1973). The philosophy of mathematics is the aim or rationale behind the practice of teaching mathematics (Ernest, 1994). There exist two philosophies of mathematics: absolutist and fallibilist. The absolutist view of mathematics as an absolute, objective, and non-correctable body of knowledge that rests upon deductive logic.

Absolutism views mathematical understanding as timeless. Although we may discover new theories and truths to add, it is superhuman. The absolutists view mathematics as rigid, fixed, logical, inhuman, and abstract.

The absolutist view may be communicated to the learners in the mathematics classroom if the teacher has adopted this kind of worldview (Ernest 1994). As such, the transfer of this kind involves giving learners unrelated routine mathematical tasks which involve the application of learned procedures and emphasise that every answer has a static method and a solution coupled with criticisms of any failure to achieve this answer using the memorandum method. For example, a teacher teaching quadratic functions will only teach one form of the function, i.e., $f(x) = ax^2 + bx + c$, since they find this form easier to work with, unlike the other. The teacher will strictly tell the learner to use the Table method to draw the graph instead of using various methods. The absolutist philosophy of mathematics is teacher-fixed, meaning that the teacher is the source of knowledge in the mathematics classroom and cannot make mistakes.

However, unlike the absolutist view, the fallibilist view of mathematics offers a different approach to mathematics education. The fallibilist considers mathematics human, correctable and changing (Ernest, 1994). Fallibilism views mathematics as the outcome of social processes. Thus, mathematics is open for revision regarding its proof and concepts. The fallibilist philosophy of mathematics is learner-fixed, meaning that it allows the learner to construct the mathematical understanding of certain concepts. For example, a teacher adopting this type of philosophy in the mathematics classroom would enable learners to interact with all the forms of quadratic functions that are the standard form, i.e., $f(x) = ax^2 + bx + c$, vertex form $f(x) = a(x-p)^2 + q$ and the factored form $f(x) = a(x - x_1)(x - x_2)$. In this classroom, the teacher allows classroom discussions to influence the teaching and learning of the concepts of the quadratic functions and acknowledge that learners have some understanding of the concept.

The constructivist theory of mathematics underpins the fallibilist view. Constructivism is a mathematical theory that views learners as constructors of understanding a particular concept through self-modification of cognitive structures

(Glasersfeld, 1991). This process of self-modification is unconscious but goal-directed by which the learner reacts to mental disturbance by changing how they view the concept. Essentially, this means that when a learner encounters a problem, they respond by thinking about it until it makes sense. Constructivism is rooted in the idea that learning occurs through teaching practices that construct knowledge over time. The constructivist approach permits the learner to develop a deeper conceptual understanding of the concept.

The constructivism approach encourages the active and participatory role of the learner in their learning process as they engage in the activities. To assist learners in the learning process, the teacher must help them by scaffolding them to make connections of their prior knowledge. Constructivism is a cycle of conceptual levels that a mathematics learner progresses through when building a set of mathematical understanding by linking the current concept to their prior knowledge (Glasersfeld, 1991). The conceptual levels are categorised into four tenets: the action level, process level, object level, and schema level of understanding (Arnon et al., 2014). These four tenets are conceptualised by Dubinsky (2002) in the APOS theory. Therefore, I adopted the fallibilist view of mathematics as my philosophy of mathematics.

2.2.2. The theoretical framework of the study

I adopted the APOS theory proposed by Dubinsky (2002) as a lens to explore learners' understanding of quadratic functions. The theory has been operationalised in various studies as follows: learners' and teachers' understanding of quadratic functions (Cahyani & Rahaju, 2019; Listiawati & Juniati, 2021; Mutambara et al., 2019); understanding of the vertex (Burns-Childers & Vidakovic, 2018); and learners' obstacles of quadratic functions (Kabar, 2018; Ruli et al., 2018). The APOS theory affirms that understanding comes about when an individual modifies current mental structures (Bansilal et al., 2017). Even though learners can work on the same concept, their cognitive networks will vary depending on their general descriptions and mental constructions (Arnon et al., 2014). The general descriptions and mental structures are achieved through reflective abstraction. Reflective abstraction is the capacity to develop modern understanding through the instrument of connecting certain mathematical constructs (Dubinsky & Wilson, 2013). Therefore, the APOS consists of the following tenets, action, process, object, and schema (Dubinsky & Wilson, 2013).

An action conception is a modification of the external mental structures received in a logical manner (Arnon et al., 2014). This modification is first conceived as an action conception when it is a reaction to stimuli that a learner perceives as external. Thus, the change requires specific instructions and performing each stage of the transformation (Bansilal et al., 2017). Furthermore, an action conception may be a single step or multiple steps response, but it is characterised by each step being prompted by prior knowledge. When learners reflect on the action conception, they internalise it to a new coordinated mental structure; if they can reverse it, this is the process conception (Borji et al., 2018).

They summarise the action and process conception as a coherent cognitive entity resulting in an object conception (Arnon et al., 2014). The schema conception is a network of concepts built using a variety of action, process, and object conceptions to represent a more significant mathematical notion (Dubinsky et al., 2013). Mathematical schemas are logically constructed to give learners abilities to use in mathematical activities (Arnon et al., 2014). They are built and rebuilt through personal learning processes. The APOS theory emphasises the critical role that an individual mental schema plays in creating latest information based on existing knowledge (Kazunga & Bansilal, 2020). Genetic decomposition is one of the major strategies employed in APOS studies. The genetic decomposition conceptualises how learners see how learning occurs in encapsulating new concepts (Arnon et al., 2014).

2.2.3. Genetic decomposition

Genetic decomposition is a constructional schema that a learner might undergo when a mathematical concept is taught (Mutambara et al., 2019). The genetic decomposition shows how the learner might formulate understanding at the various levels of understanding. Figure 2.1 presents the proposed genetic decomposition of the quadratic function concept based on the APOS theory adapted from Mutambara et al. (2019). Ndlovu and Brijlall (2015) hold that if the genetic decomposition cannot explain learners’ understanding of the concept, it needs revision. The revision emanates from the learners' responses to the learning tasks; as such, it cannot be immediate. The genetic decomposition needs to be assessed first since it is a hypothetical model of the learners' minds (Şefik et al., 2021).

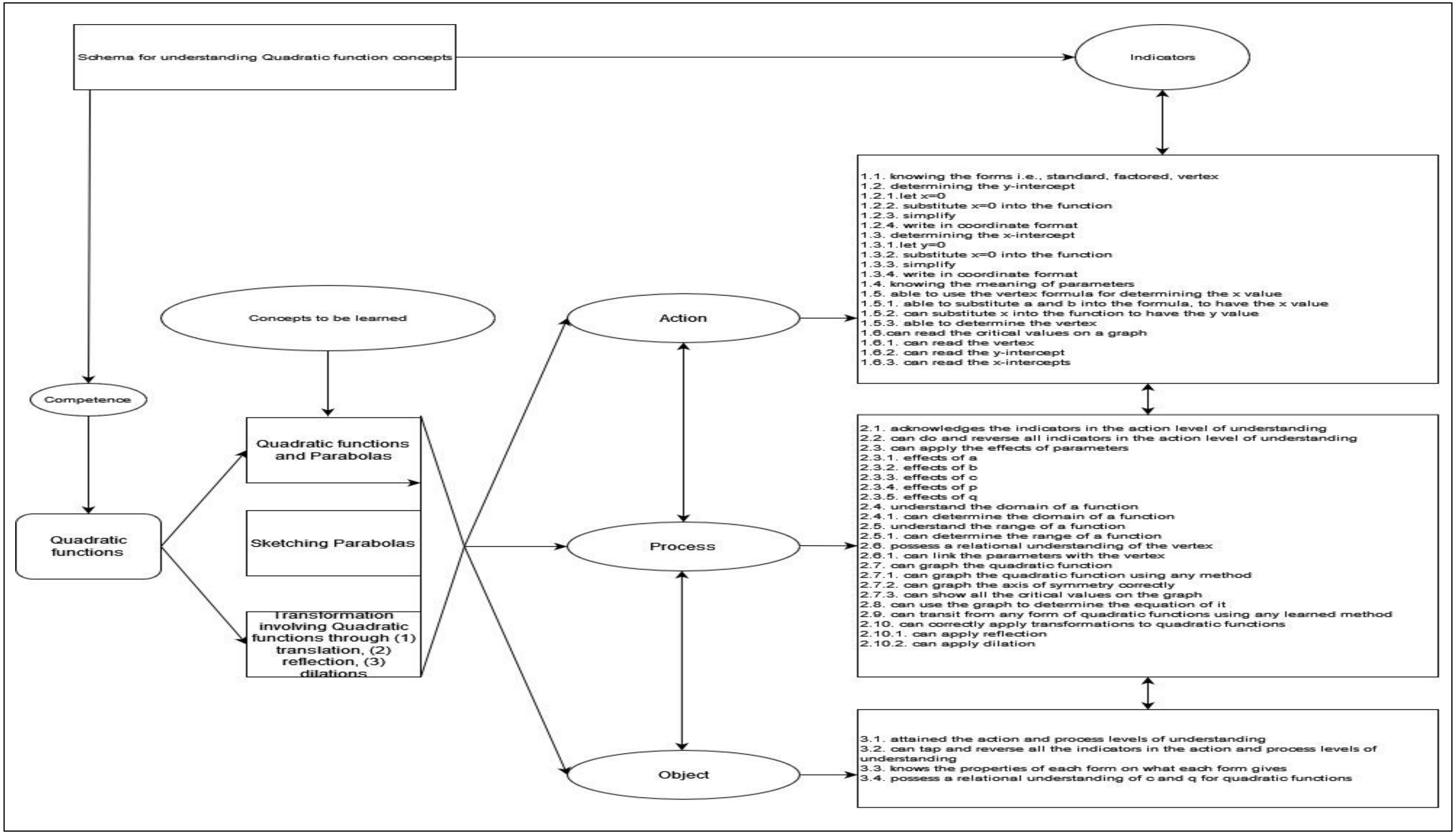


Figure 2.1: The genetic decomposition adapted from Mutambara et al. (2019)

2.2.4. Studies that used APOS theory

Studies that use APOS theory are based on classroom interaction between the learners and the teachers on a one-to-one session or with more than one learner. These studies generally employ the ACE teaching cycle and use various research designs, predominately the case study design. In most cases, mathematics activities are used as tools for data collection. Hence, learners are given activities to work on through the ACE teaching cycle in which they interact. Therefore, in this section, I briefly present how the APOS theory was used in different studies concerning the improvement of conceptual understanding of mathematics concepts (Arnon et al., 2014; Asiala et al., 1996; Carlson, 1998; Childers & Vidakovic, 2014; Dubinsky, 2002).

Childers and Vidakovic (2014) report on learners' meaning and interpretation of the vertex of a quadratic function concerning their understanding of quadratic function in two different representations, word problems and algebraic forms. Data from the study revealed several meanings of the vertex. The collected data were analysed using APOS theory (Asiala et al., 1996). The findings of the study revealed that it is essential to investigate learners' meaning of the vertex to assist them in overcoming conceptual obstacles. From this, I learned how the data analysing process is done through the APOS theory.

A study by Arnon et al. (2014) investigates the development of mathematical concepts. They combine the APOS theory, Nesher's theory, and Yerushalmy's ideas of multiple representations in their work. The finding revealed that learners whose concept was introduced as concrete actions performed better than those who held the concept as tangible objects. They later conclude their study that the concrete action operates similarly to the action in the APOS theory. Furthermore, the findings showed that the action tenet of the APOS theory is a vital tenet to begin with the development of conceptual understanding of mathematical concepts.

What I found meaningful and insightful in this study was how the authors explained the development of the action construct of the APOS theory. This supports the genetic decomposition adopted in this study. Asiala et al. (1996) report on a framework for research and curriculum development. The authors gave details of three components and provided examples of their application. The research adopted qualitative methods. The study presented a theoretical analysis of the APOS theory and later described the ACE teaching cycle. In this study, I learned how data was collected and analysed through the use of intervention phases.

In another study by Asiala et al. (1992), the authors report on how abstract algebra learners might come to understand permutations of finite sets and symmetries of a regular polygon. In the study, they first present what it could mean to understand what is expressed in APOS theory. Their results showed that teaching in context is effective in helping learners to develop strong conceptions of mathematics concepts. In their study, what I found more insightful was the presentation of what it means to understand mathematics concepts through the constructs of the APOS theory.

Carlson (1998) reports on learners' development of the function concept as they progress through their mathematics careers. The study used an exam measuring understanding of functions concept after the interviews were conducted with a sample of 5. Data analysis was done using the APOS theory to classify learners' conceptual views of function. The author concurs with Breidenbach's et al. (1992) findings that learners' understanding of functions was developed using the construction of activities with learners. In this study, I found that to improve learners' understanding of quadratic function, I have to be mindful of the activities I prepare as they are essential for learners' development of conceptual understanding.

Learners' understanding of functions requires an understanding of variables to determine the x-intercepts for plotting the function. Trigueros et al. (1995) examined learners' understanding of variables; specifically, learners' ability to use and interpret the variable as unknowns through equations or functions. Their results are all based on a large sample of 164. The results revealed the persistence of conceptual obstacles and approaches characteristic of beginning algebra learners in school mathematics (Trigueros et al., 1995). The data collected suggest that most learners operate on the action level in which the solutions are anchored on the signs presented in the expressions of the functions (quadratic function).

Moreover, learners rely on procedural understanding instead of conceptual understanding (Trigueros & Ursini, 1999). Finally, most learners possess difficulties moving into the object level of variables. From this study, I noted that to understand quadratic function, one of my activities should be in word problems to make specific conclusions in terms of whether or not the author's views are correct.

In conclusion, several issues that arose in the above reviewed studies became helpful in my study. Firstly, how conceptual understanding is outlined through the frame by Asiala et al. (1996) and Trigueros et al. (1995) enlightened my understanding of how APOS theory could be used in analysing the collected data to code the essential aspects of the data. Lastly, the genetic decomposition used by Mutambara et al. (2016) has influenced my research since it will be used as a tool to check if understanding was developed or not.

2.2.5. The role of APOS in the data analysis

The APOS theory can be used as a data analysis tool in a study (Şefik et al., 2021). When used as a data analysis tool, the theory looks for the mental structures formed in the participants' minds after applying the ACE teaching cycle. Different studies have used the APOS theory as a data analysis tool (Brijlall & Maharaj, 2015; Chimhande et al., 2017). The APOS theory in analysis guides the researcher in detailing how the learners view the concept. Therefore, to achieve this in this study, I have incorporated the genetic decomposition developed while doing the analysis. The genetic decomposition gave me the picture of what informs me that the learners operate at different levels of understanding. The APOS theory simplifies analysis because of the proposed hypothetical genetic decomposition developed. The theory allowed me to categorise the learners' understanding based on their levels of understanding.

2.3. UNDERSTANDING THE QUADRATIC FUNCTION CONCEPT

The concept of function has evolved. The evolution of the concept emanates from the competition between two notions, i.e., the geometric (expressed in the form of a curve) and the algebraic (expressed as a formula) (Kleiner, 1989). Hence, the evolution in mathematics has changed the concept of function from a curve described by motion to an expression representing the relation between two or more variables with its graph (Denbel, 2015). The concept of function has different formal and informal definitions. Therefore, I focused on the concept of a function expressed in algebra and not in a geometric. More formally, a function from x to y is defined as any subset of the Cartesian product of x and y, such that for every value of $x \in R$, there is exactly one value of $y \in R$ such that $(x; y) \in f$ (Denbel, 2015; Hasanah et al., 2021).

Parent (2015) defined the concept function as any mathematical expression containing variable x that has a definite value when a number is substituted for x. Moreover, a function is an expression that describes the relationship between two or more variables, where the input variable has exactly one output variable. In essence, this means that for every x value, there can be only one y value. In this study, the function definition is conceptualised to mean a relationship between two sets, i.e., domain, and range, where every domain is mapped to a specific range. The function is denoted by the letters f, g, h, or k, which makes them different from equations (Wijayanti & Abadi, 2019).

The development of the concept of function in mathematics prevailed in how functions were introduced and taught in high school mathematics. Today, functions are part of every high school mathematics curriculum. The expectation is that after Grade 12, learners will know the function concept in general and be familiar with specific types of functions, i.e., linear, quadratic, general polynomial, reciprocal, step, exponential, trigonometric, logarithmic, and piece-wise functions in different representations (Mpofu & Pournara, 2018). However, numerous studies show that learners have difficulties learning the concept of function in general.

Moreover, functions are defined in learners' books in a modern sense. However, learners tend to hold a restricted image of the function concept. For example, Mpofu and Pournara found that some learners found it difficult to relate the symbolic to graphical representation. In addition to the problems arising from learners' restricted images is the asymptote concept of a function, as there is clear evidence that different representations of functions are treated in isolation without stressing the connections among them. The problem arises from the notion that the asymptote is either parallel or coincides with the axes. A study by Flesher (2003) revealed that learners posed conceptual obstacles with the concept of asymptote. For example, learners viewed the concept as a number and not linear.

Conversely, there is some vague understanding among curriculum developers, textbook authors, and teachers regarding the purpose of application and modelling of function concepts (Maf'ulah et al., 2019). Therefore, all this results in an identity crisis concerning learning the concept of function in mathematics classrooms. Consequently, if the identity crisis is not defined well, it births conceptual obstacles regarding the concept of function. Although the concept of function can be more complex, this study limits functions to quadratic functions.

The quadratic function is one of the forms $f(x) = ax^2 + bx + c$, i.e., the standard form, where a, b and c are integers with $a \neq 0$ (Nielsen,2015). Moreover, the quadratic function can also be expressed in a factored form $f(x) = a(x - x_1)(x - x_2)$, and vertex form $f(x) = a(x - p)^2 + q$ (Ubah & Bansilal, 2018; Ndlovu, 2017; Ousby et al., 2018). The graph of a quadratic function is called a parabola, which is recognised for its U-shaped in Figure 2.2 (Pender et al., 2011). Hence, each form demonstrates some graphical information related to the location of critical points on the graph. The standard form reveals the location of the y-intercept $(0; c)$, the vertex form indicates the turning point of the graph $(p; q)$, and the factored form gives the roots of the function $(x_1; 0)$ and $(x_2; 0)$ (Hattikudur et al., 2012; Zaslavsky, 1997). However, Parent (2015) found that learners over-rely on one form.

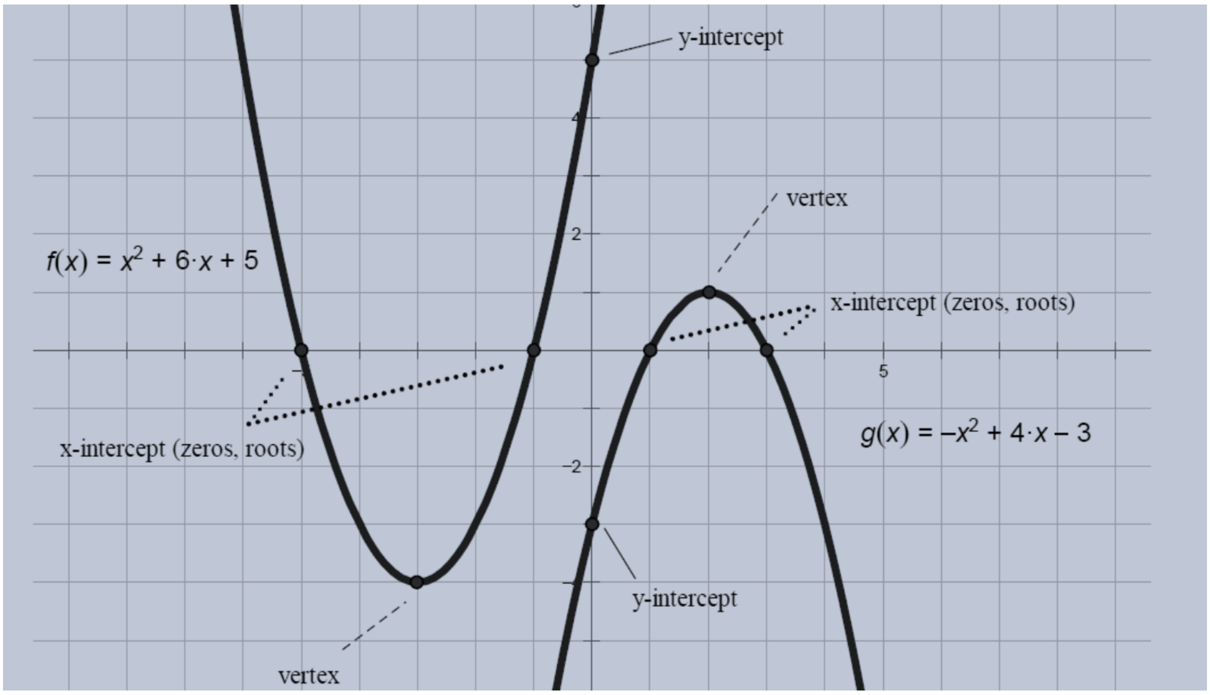


Figure 2.2: The Quadratic function

The concept of a quadratic function is one of high school mathematics' most important mathematical ideas (Benning & Agyei, 2016). This critical concept is essential in understanding mathematical concepts such as algebra and geometry (Clement, 2001). Learners' understanding of the knowledge of quadratic function is crucial for them to excel in much of the mathematics content. However, understanding quadratic functions posed conceptual obstacles of the need to connect different concept constructs (Didiş et al., 2011).

The conceptual obstacles emanate from the failure to grasp the interrelated constructs of the quadratic functions, namely, the domain and range, intercepts, turning point (minima and maxima), asymptotes, intervals in which the function is increasing or decreasing, and the discrete or continuous nature of the graph (Parent, 2015). The understanding of these constructs above results from comprehending the function's parameters. Studying the effects of the parameters of the quadratic function is vital for improving learners' understanding of the concept. Therefore, given $f(x) = ax^2$ and altering the value of a results in a vertical shift of the graph (Mutambara et al., 2019). The bigger the value of a, the thinner the graph becomes, and the smaller the value of a, the fatter the graph (Figure 2.3). The value of a also gives the shape of the graph. If $a > 0$, that is $a \in [1; 2; 3; 4]$, its concavity upward, and if $a < 0$, that is $a \in [-1; -2; -3; -4]$, its concavity downward. Figure 2.3 demonstrates the effects of the parameter a as positive or negative.

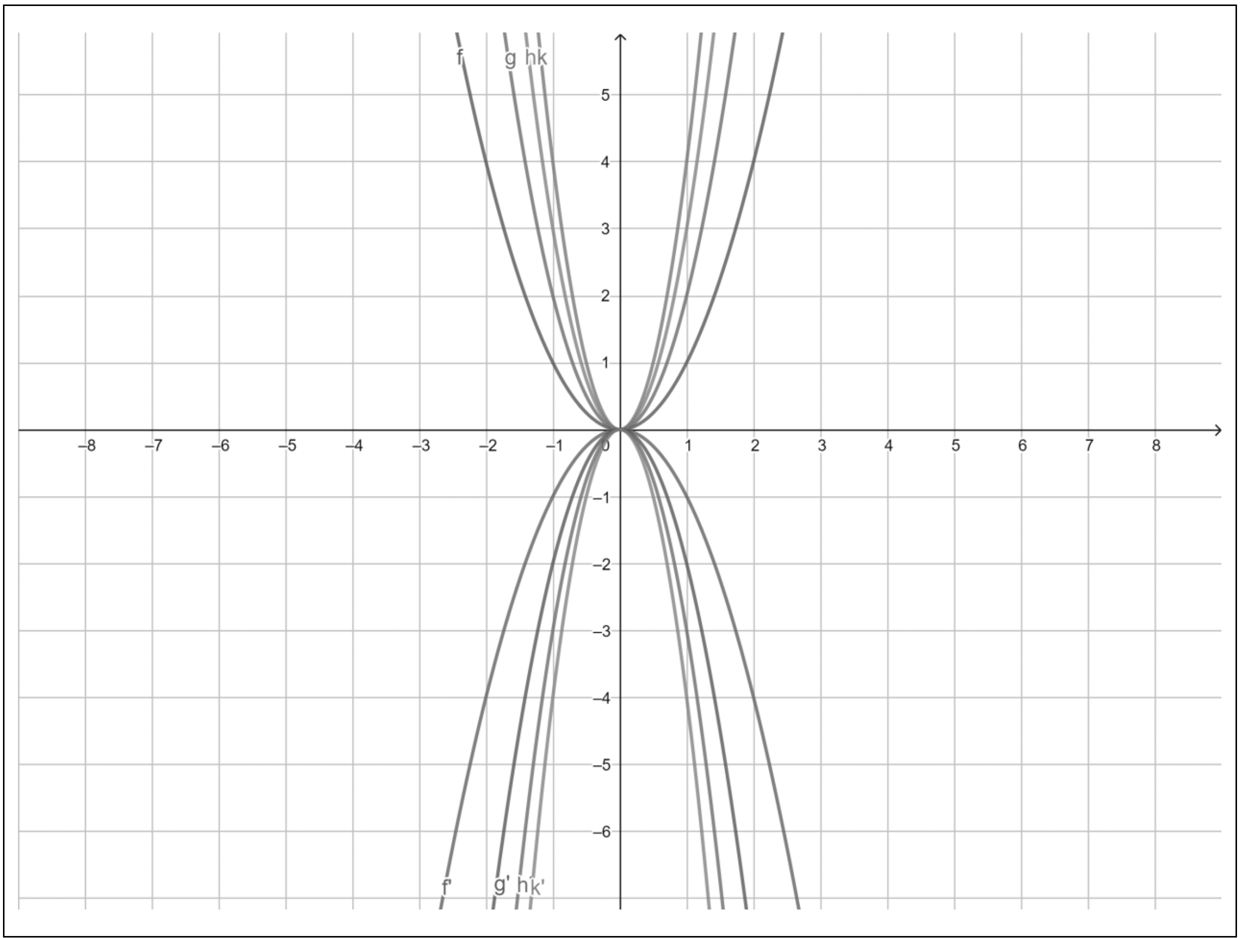


Figure 2.3: Effects of changing the value of a

However, learners possess a conceptual obstacle regarding the value of a, as it is noted by Zaslavsky (1997) that learners confuse the significance of a in a linear function with the one in a quadratic function. This is so because of the nature of writing the standard forms of the two graphs as $f(x) = ax^2 + bx + c$ and $g(x) = ax + c$. As such, from the standard forms, most learners assumed that a gives the slope of the quadratic function. The interference of concepts learned in linear functions made learners determine the slope in quadratic functions (Ellis & Grinstead, 2008).

Eraslan (2008) described that given the function $f(x) = x(x + 2) - 3$, a learner graphed the parabola so that its concavity is upward but found the vertex at $(2; -3)$, which was an incorrect vertex of the graph. Moreover, when the learner was asked to express the graph of $g(x) = (x+1)^2 + 4$ in the standard form, the response was given as $g(x) = -x^2 - 1x + 4$. These conceptual obstacles are consistent with conflating the vertex and standard form. Ellis and Grinstead (2008) studied the standard form of a quadratic function and acknowledged that the a parameter is interpreted as influencing the shape of the graph. However, learners assumed that changing the a parameter in the vertex form does not alter the vertex's location.

The quadratic function is given in the standard form $f(x) = ax^2 + bx + c$, changing the parameter b while leaving a and c constant, results in translations for the range of values implemented. The quadratic function maintains its shape and direction Figure 2.4. The parameter c from the standard form deals with the vertical shifts of the graph Figure 2.5. Changing the value of c moves the position of the vertex along the line $x = -\frac{b}{2a}$ (Owens, 1992). Learners may not recognise exceptional cases with the quadratic function assuming that c does not exist when given $f(x) = 2x^2$. When interacting with the standard form, interpreting the value of c parameter is easier.

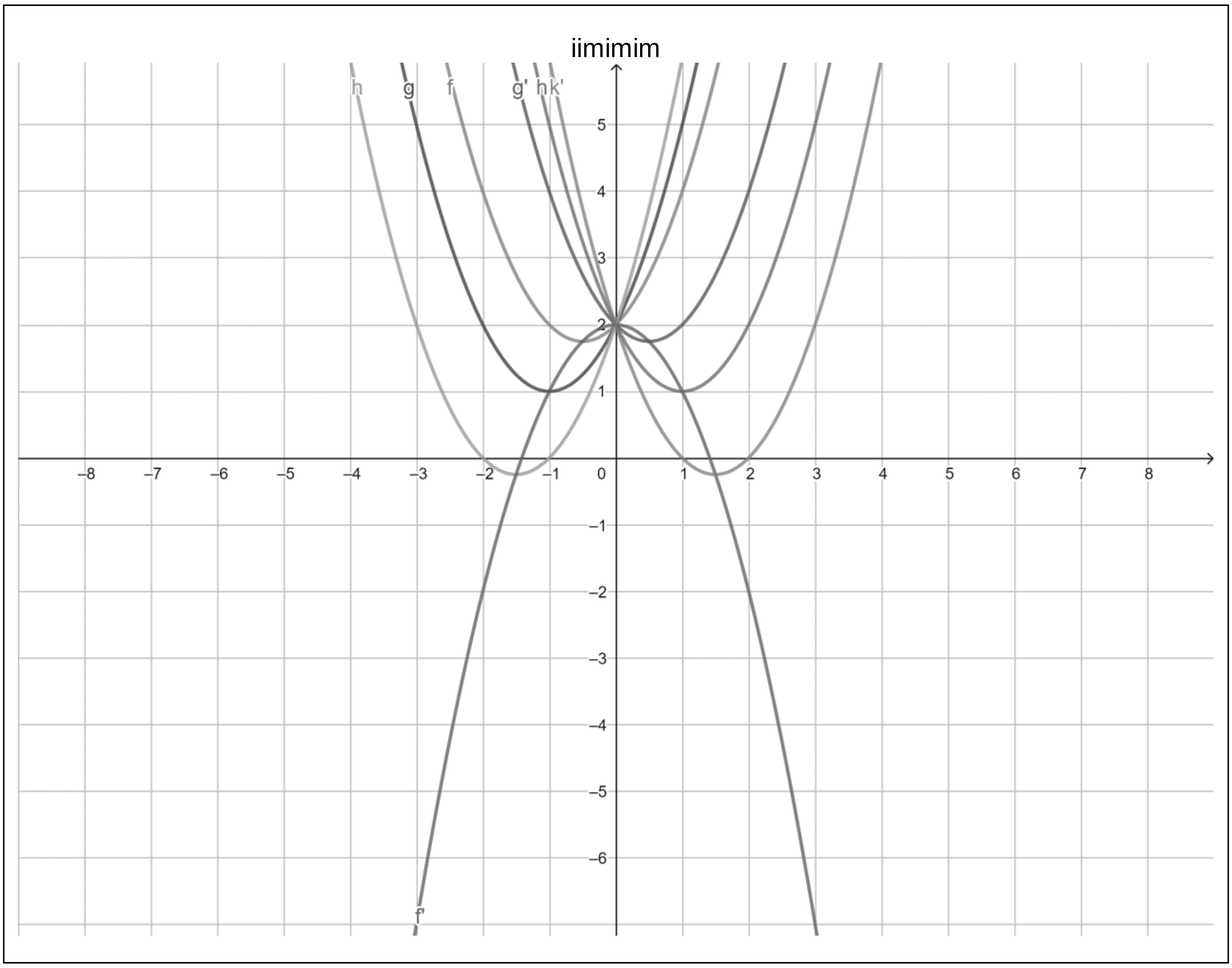


Figure 2.4: The effect of varying b

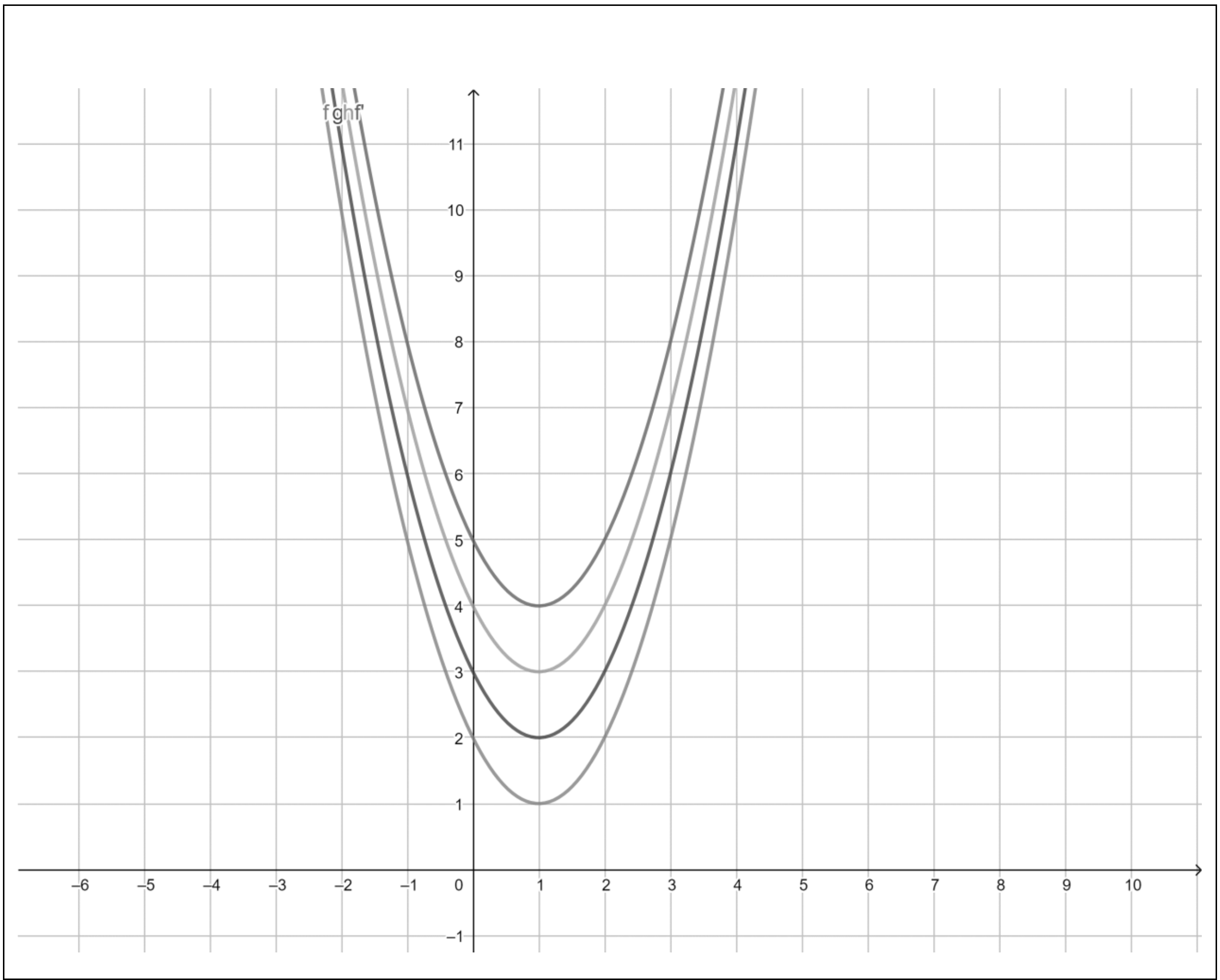


Figure 2.5: The effects of varying c

2.4. UNDERSTANDING AND REPRESENTATION OF QUADRATICS

The objective of learning quadratic functions is to understand the concepts involved, apply the concept, and describe the connection between these concepts (Minarni et al., 2016). Moreover, the role of mathematical understanding in learning quadratic functions is to deepen and develop learners' understanding of mathematical concepts and their connections through different representations. Therefore, the integral of doing mathematics is for mathematical understanding and mathematical representation. The latter will be discussed later; mathematical understanding is characterised as levelled but not linear and a recursive process that occurs when thinking moves between levels of sophistication (Pirie & Schwarzenberger, 1988). These levels of complexity involve the movement between the action, process, object, and schema (Arnon et al., 2014).

Understanding characterised by the action level of sophistication entails using prior knowledge of quadratic equations and linear functions. Therefore, a learner performing an action level of understanding of quadratic function would require a formula to determine the x- intercept of the vertex that is $x = -\frac{b}{2a}$. The learner can substitute into the formula of the vertex, that is, if given $f(x) = x^2 + 6x + 8$. For the learner to determine the vertex, would first note that $a = 1$ and $b = 6$, then substitute $x = -\frac{6}{2(1)} = -3$. However, Parent (2015) found that learners possessed a limited action level since most struggled with the substitution into the formula of the vertex. Instead, they wrote the value of the y-intercept as the y value of the vertex. Also shared by Fonger et al. (2020) that learners often struggle to interpret the role of parameters while learning quadratic functions.

Literature reveals that most learners struggle to tap into the process level of understanding (Kshetree et al., 2021; Mutambara et al., 2019). Some incorrectly viewed the parameter a as the function's gradient, a conceptual obstacle of linear functions (Mutambara et al., 2019). Moreover, Mutambara et al. found that the learners' prior knowledge of the function $y = ax + c$ is the one that causes the conceptual obstacle of the quadratic function concepts. The levels of sophistication assume that the action level of understanding should provide an affordance to reverse the action and make connections with other concepts to develop the different levels of the APOS. In continuation with the example above, this would imply that the learner will acknowledge the x-intercept of the vertex and now will be able to determine the y-intercept of the vertex. For example, if $x = -3$, then $f(-3) = (-3)^2 + 6(-3) + 8 = -1$, therefore, the vertex of $f(x)$ is $(-3; -1)$. As a result of this level of quadratic function conceptual understanding, the learner can now find any vertex for a different function without the use of a formula by completing a square.

The process is then encapsulated into an object-level understanding of quadratic functions. The object level of conceptual understanding permits the learner to reflect on the actions applied to a particular process, become fully aware of the process, and acknowledge that transformations can act on it and construct these transformations. The learner in this understanding can compare and relate two vertices of a quadratic function. That is, if given $f(x) = 2x^2 + 8x$ with vertex is

$(-2;-8)$ and $g(x) = x^2 - 2x$ vertex is $(1;-1)$; see Figure 2.6 below. The learner can also create a linkage between the concepts. This means that the learner is at the object level of understanding. However, Mutambara et al. (2019) found that the relation between the concepts of the quadratic functions is not fully acknowledged or recognised as learners continue to fail to see their connections.

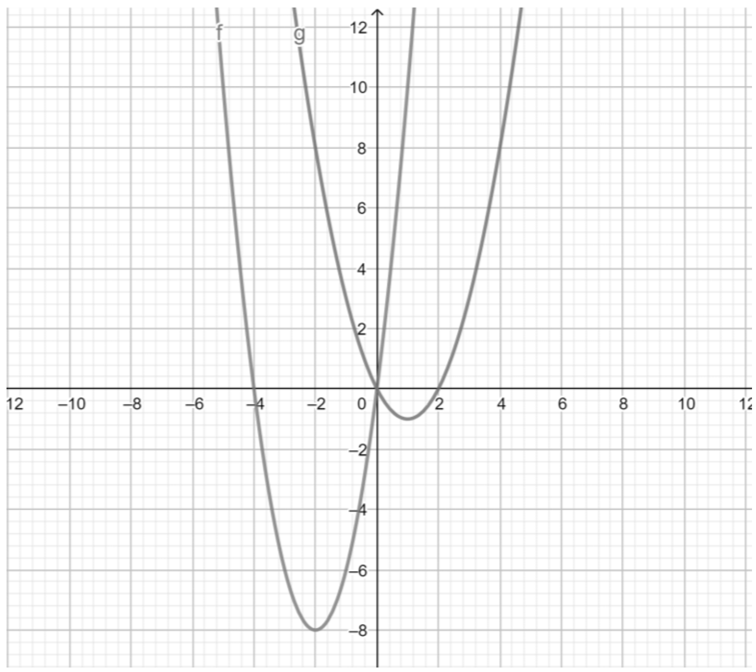


Figure 2.6: Showing the vertices of two quadratic functions in a representation

Lastly, the highest level of sophistication is constructed through the collection of the action, process, and object-level of understanding together with the connections brought by the quadratic function concept. The schema level of understanding for quadratic function advances for conceptual understanding. Conceptual understanding is the process of deducing procedures from general mathematical relationships and the strength to connect mathematical notations and symbolism with the mathematical idea through the recursive process. In this level of conceptual understanding, the learner can now interact with constructs of the quadratic function. However, these levels of conceptual understanding for quadratic function are not independent but are intertwined and interwoven for one to ultimately achieve the level of understanding. Therefore, understanding complex concepts like quadratic functions requires integrating the levels of understanding (Ozaltun-Celik & Bukova-Guzel, 2017). Hong and Choi (2014) concur by stating that after learners have developed a conceptual understanding, they can solve quadratic functions using different strategies, and use their understanding of quadratic functions to represent, analyse and create graphs.

Sierpinska (2013) asserts that understanding should be viewed from three different lenses, i.e., the act of understanding, understanding and the process. According to Sierpinska, the act of understanding refers to a cognitive experience associated with linking what is to be understood with the basis of understanding. In this case, since learners will possess prior knowledge of quadratic equations, they have mental representations and models. The second lens, "understanding," is achieved due to understanding. Thirdly, there are processes of understanding which involve links being done between acts of understanding through reasoning processes. In the context of Dubinsky and Wilson (2013), the act of understanding is the integration of the action, process, object, and schema. Therefore, conceptual understanding is a complete network of internalised mathematical concepts.

However, mathematical understanding alone cannot nurture knowledge of quadratic functions. Hence, learners should also demonstrate mathematical representations to understand quadratic functions fully. This is so because any mathematical concept must be represented in some way if it needs to be developed into a schema (Arnon et al., 2014). Mathematical representation implies expressing a mathematical concept differently, such as graphs, symbols, and numeric forms (Santia & Sutawidjadja, 2019). Therefore, for learners to understand quadratic functions, they should demonstrate mathematical representation. For example, given the following coordinate, i.e., the x-intercepts $(-4; 0)$ and $(-2; 0)$ with vertex (-3; -1). The learner involved in the process of learning the concept will be able to sketch the graph in Figure 2.7 and can be able to derive a formula for the given coordinates in any form of the quadratic that is $y = x^2 + 6x + 8$.

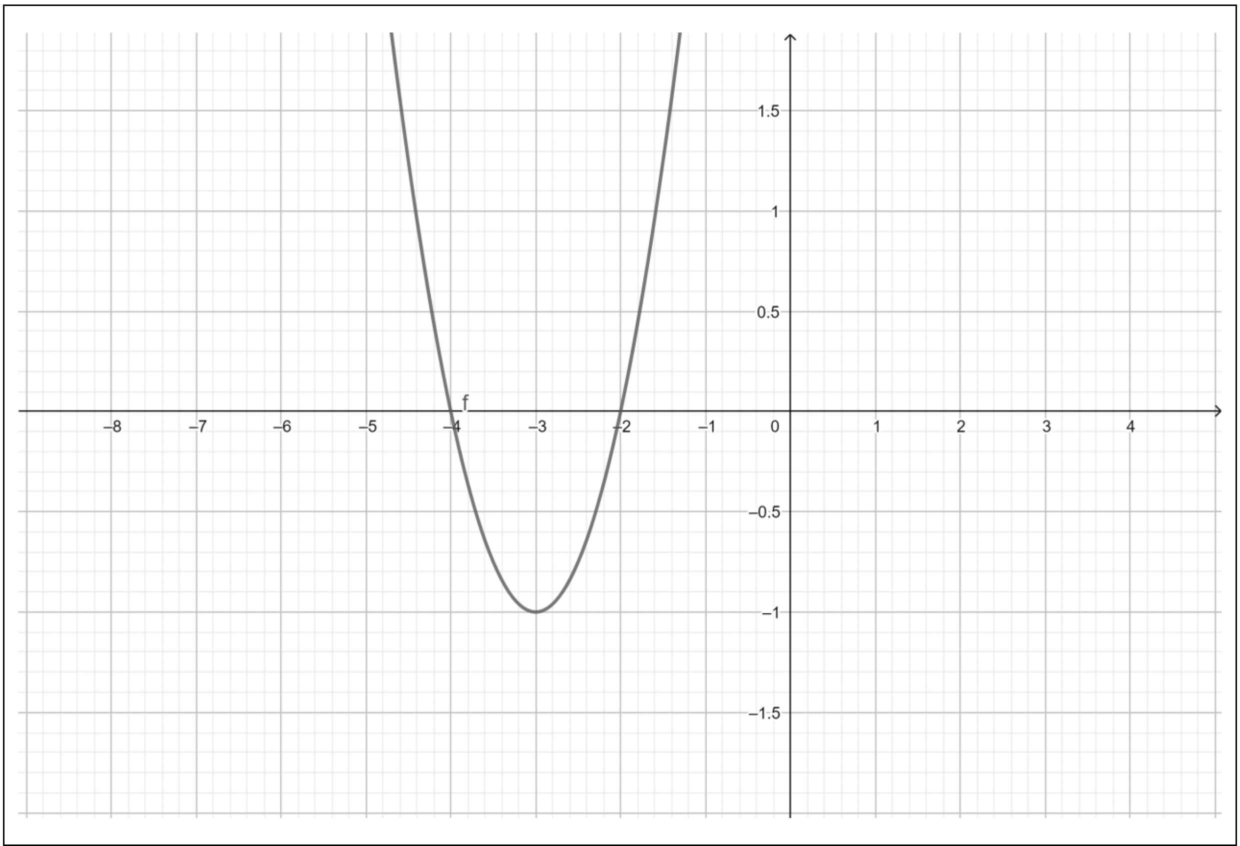


Figure 2.7: The graph of $y = x^2 + 6x + 8$

The different mathematical representations above are an essential trajectory in mathematical understanding development (Parent, 2015). The mathematical representation will enable learners to create and use representation for quadratic functions concept to select, apply and translate among the representations to solve quadratic functions questions and to use representation to model real-life cases of quadratics functions. Numerous research has been conducted on mathematical representation (Bal, 2014; Caglayan & Olive, 2010; Santia & Sutawidjadja, 2019). These studies reported that mathematical representation is the central aspect of success in developing learners' conceptual understanding of quadratic functions. Thus, doing mathematics is dependent on using representation since mathematical ideas are abstract (Dreher & Kuntze, 2015). Therefore, mathematical representation is essential in developing conceptual understanding (Duval, 2006). Moreover, using representation can nurture mathematical understanding only if learners are encouraged to actively create a connection between these representations (Dreher & Kuntze, 2015).

The learning of quadratic functions requires conceptual understanding and connections between different representations. These connections are essential for understanding the other parts of quadratic functions. The curriculum requires learners to be able to create and use tabular, symbolic, graphical, and verbal representations to understand quadratic functions (Parent, 2015). This is seen in Figure 2.8 below, where learners transit from one form of representation to another to develop a conceptual understanding of quadratic functions.

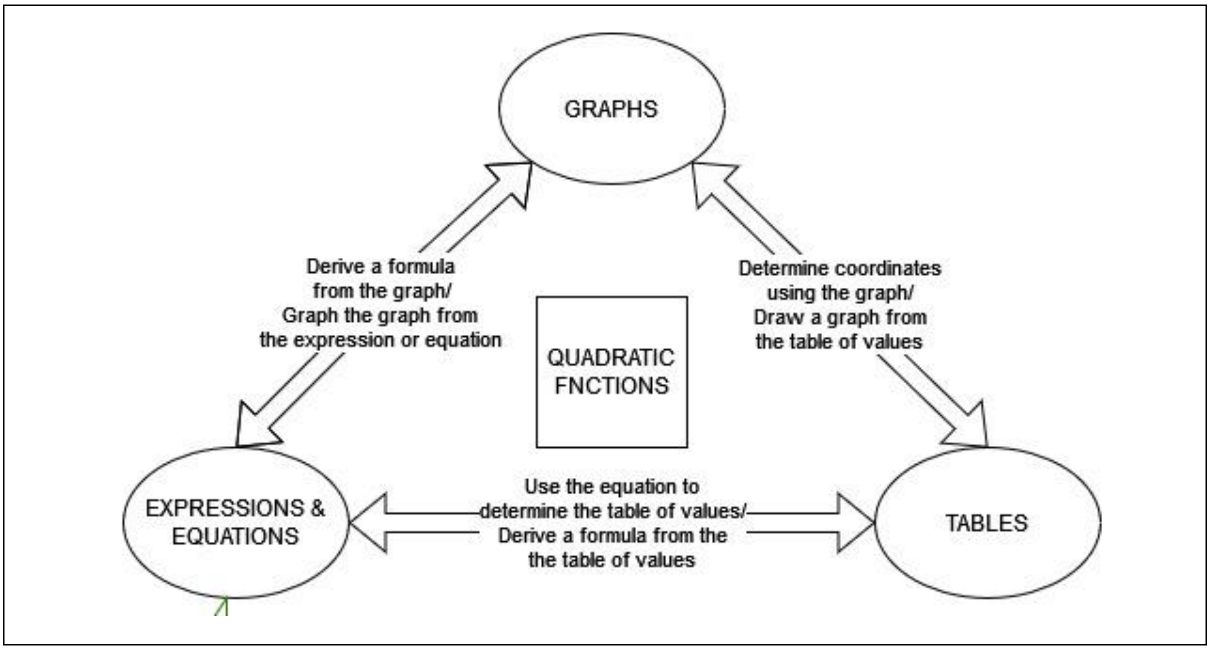


Figure 2.8: Moving between the three representations of a quadratic function

Therefore, working with the different representations of the quadratic function shown in Figure 2.7 above is a way to promote flexible competence (Parent, 2015), which emphasises conceptual understanding. Other researchers have commented on the importance of connections between the various representations of quadratic functions (Ellis & Grinstead, 2008; Knuth, 2000). Flexible understanding means that a learner possesses conceptual understanding rather than procedural understanding and can transit from one representation to another without any hindrances. Research on the importance of mathematical representation found that learners' representation ability is the central aspect of success in understanding mathematical concepts (Bal, 2014).

The success of problem solvers is rooted in the ability to build problem representations in problem-solving situations (Zhang, 1997). Choosing representations allows learners to practice balancing the different aspects of representations (Chapman, 2010). The understanding of quadratic function is built from a strong knowledge of mathematical representations. Kieran (2014) asserts that the concept of the quadratic function is a form of understanding algebra with various representations. However, learners possess little knowledge of mathematical representations involving quadratic functions.

Knuth (2000), cited in Parent (2015), found that learners appear to understand connections between equations and graphs. Also, learners relied on algebraic solution methods versus graphical methods. The observations of the study indicated that learners depended on rote instrumental than relational understanding. The recommendation was that learners be assisted through mathematical representations to develop conceptual understanding. Thus, having conceptual understanding allows learners to apply and adjust the procedure to fit the problem they are busy with (Parent, 2015). This is also supported by a study of Suwarsono and Khabibah (2022) that learners with a limited level of understanding often struggle to represent quadratic functions graphically.

Ozaltun-Celik and Bukova-Guzel (2017) found that the participant struggled to identify and draw a graph for $f(x) = x^2 + 2x - 3$. In interacting with the function, it was noted that the learner could find the point $(1; 0)$ but failed to determine the other. This resulted from the method employed in finding the coordinates that the learner assigned several values of x instead of solving the equation. The method employed by the learner resulted in another point after the substitution of $x = 0$ into the function that is $(0; -3)$. As a result of the learner, the learner drew an incorrect graph. On the notion of identifying the function, the learner said the function is linear. This showed that the learner's action level of understanding was immature.

2.5. ASSESSING LEARNERS' CONCEPTUAL UNDERSTANDING

Assessing learners' conceptual understanding of quadratic function involves taking two points in mind: understanding as a connection made between mental representations, which is different from the result of understanding. To assess conceptual understanding of quadratic functions, Heibert and Carpenter (1992) state that understanding cannot be inferred from a single response to a single activity, meaning that a learner can have presented a correct solution without understanding. In support of Heibert and Carpenter's view, Parent (2015) found that learners can use the formula to determine if the vertex that is $x = -\frac{b}{2a}$, to determine the x-coordinate of the vertex but struggles to find y- coordinate. As such, the correct solutions of the x-coordinate of the vertex cannot infer an understanding of the concept. Instead, several activities are needed to develop a profile of knowledge, and these tasks include efficiency to accurately substitute into $x = -\frac{b}{2a}$, and skill in carrying out the procedure to later on substitute $-\frac{b}{2a}$ into $f(x)$ to flexibly determine the intercept of the vertex. In recognising that understanding is a complex network, thus if we are to assess learners' conceptual understanding, we need to consider their thinking skills as per the activity given. Moreover, we need to consider the deep procedural knowledge that the learner demonstrates to reach the final solution. Rittle-Johnson (2017) asserts that this deep procedural knowledge is the flexibility of the learner to pose instrumental and relational understanding iteratively.

A typical mathematics classroom focuses on a narrow collection of well-defined activities which foster instrumental understanding at the expense of advancing relational understanding (Skemp, 1976). As a result, the assessment conducted in the class is similar to the one that learners once interacted with them as an example. If learners get correct solutions, teachers assume they understand the tested mathematical concept. However, the correct solution does not guarantee possession of understanding since the attainment of conceptual understanding is a long ongoing process that is levelled.

The notion of teaching for correct answers impedes teaching for conceptual understanding. An interesting idea is the non-binary nature of understanding (Nickerson, 1985). Suppose a learner has interacted with the quadratic function concept. In that case, they will have some understanding of this concept due to their prior knowledge of quadratic equations and linear functions, however limited or inappropriate links within their understanding might be. As a result, we cannot ever have a complete understanding, but we can continually develop understanding by developing our schema. In light of these points, I can consider two ways to assess conceptual understanding, i.e., through the lens of learners' conceptual obstacle and mathematical representations.

2.5.1. Assessing learners conceptual understanding of quadratic functions

The conceptual obstacles that learners commit while learning quadratic functions is in synergy with their conceptual obstacles of the concept (Mathaba & Bayaga, 2019). Learners' difficulties result from conceptual obstacles of the false ideas developed in algebraic concepts. Therefore, conceptual obstacles generally surface when learners integrate prior knowledge with a new topic (Rittle-Johnson & Schneider, 2015). Moreover, conceptual obstacles result from the combination of scarcity of logic and learners' ability depending on inappropriate understanding of mathematics concepts. The ability to rely on undeveloped knowledge leads to conceptual obstacles of the concept to be learned. As such, the developed conceptual obstacles give birth to challenges, which re-surfaces when learners carelessly fail to relate mathematical concepts or reflect on the solution (Mathaba & Bayaga, 2019). Thus, conceptual obstacles result from learners' failure to apply algebraic rules when working with quadratic function concepts.

In mathematics classrooms, conceptual obstacles are essential since they add to the learners' learning process as errors (Makonye & Hantibi, 2014), and can be classified into three categories: procedural, conceptual, and arbitrary (Luneta & Makonye, 2010). Procedural errors are challenges that are displayed by learners when they cannot use computation properly. This type of an error, following the APOS theory of understanding, hinders the development of the action level of understanding. Luneta and Makonye (2014) noted that learners with procedural errors struggled to determine the roots of the quadratic function. For example, given $(2x-3)(3-x)=4$, learners failed to notice that this is a quadratic equation. Instead, when they wanted to solve, learners converted the equation into an expression writing it as $2x-3+3-x+4$. The results of the study revealed that learners struggled with the distributive law to write it in a standard form that is $ax^2+bx+c=0$ and then solve for the roots.

Conceptual error happens when learners fail to understand the concept involved in a given task or connect the relationship between concepts (Luneta & Makonye, 2010). Conceptual error hinders the development of the process, object and schema levels of understanding resulting from procedural obstacles. The first conceptual error of quadratic functions is an inability to interpret the information contained in the function. Learners assume that the two functions are the same without acknowledging the importance of parameters, for example, $f(x) = (x + 3)^2 + 1$ and $g(x) = (x - 3)^2 + 1$ since the parameter a is equal to one in both functions (Luneta & Makonye, 2010).

Secondly, failure to note the relationship between quadratic functions and quadratic equations. This conceptual error is seen as a stumbling block when learners are required to draw the graph since they fail to determine the x-intercepts and y-intercept of the quadratic function. The third conceptual error is the failure to note the similarities between the quadratic and linear functions. Learners often assume that the parameters in the linear mean the same thing as the ones in the quadratic function since they are both written using the same variables that are $f(x) = ax^2 + bx + c$ and $g(x) = ax + c$.

The arbitrary error occurs when learners change the question to suit their level of understanding. For example, a learner is given a function and required to determine the values of x for specific restrictions. Given $f(x) = 2(x + 1)^2$ determine for which values of x is $f(x) > 0$. In this case, learners prefer writing to change the inequality to an equation instead of the solution of the inequality (Makonye & Shingirayi, 2014). Therefore, assessing learners' conceptual understanding through the lens of conceptual obstacles involves checking if the learners could show some action, process, object, and schema level of understanding even though they encountered some conceptual obstacle.

2.5.2. Assessing understanding through mathematical representations

Learners' use and transition through mathematical representation forms are essential to learning. However, mathematical representations are internal and external generalisations of mathematical concepts constructed as an internal and external mental network (Castro et al., 2022). Therefore, mathematical representation can assess learners' conceptual understanding of quadratic functions. The learners' ability to transit from one representation to another demonstrates understanding at a certain level based on the APOS theory of understanding. The model for mathematical understanding above shows how we can assess conceptual understanding of quadratic functions based on the genetic decomposition. The model depicts the movement between the different representations as an interactive process.

The interactive process of moving through the representation depicts an improvement in learners' conceptual understanding of the quadratic functions. Consequently, a learner who has improved their conceptual understanding of quadratic functions understands how they interact with the form of the function and the relationships between them. Figure 2.9 depicts what this conceptual understanding might be like if it is to be assessed. This means that learners understand expressions, graphs, tables, and quadratic equations. Additionally, the learner understands the transition in moving from the different quadratic forms and knows what each form's parameter highlights. Suppose one view one of the expressions or equations is as depicted in Figure 2.9. In that case, one can note that learners understand how to compute each form of a quadratic function algebraically to determine the other forms. Additionally, the forms guide the learner in deciding if a given type of form can and cannot do with it to solve quadratic function problems.

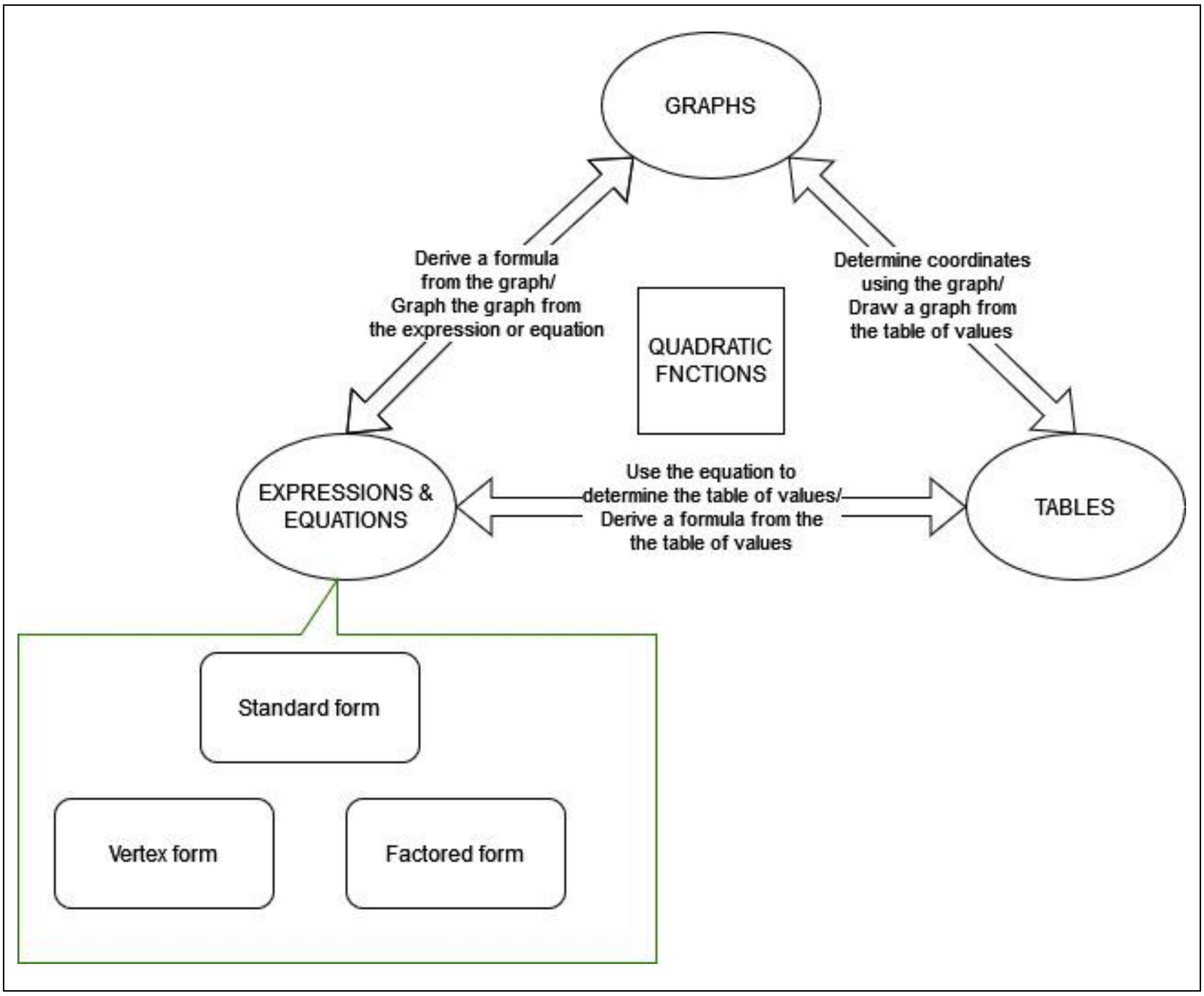


Figure 2.9: Conceptual understanding of quadratic functions

Various studies reveal that many learners have difficulties translating from one form of representation to another (Adu-Gyamf et al., 2019; Castro et al., 2022; Nurrahmawati & Sudirman, 2021). The reviewed research in this section is necessarily based on the concept of quadratic functions per se since literature relating to representations of quadratic functions involving the three forms is scarce. However, from the literature, it is clear that mathematical representation posed challenges for learners. Learners' multiple representation skills are the heart of success in mathematics understanding. The process involved in translation from one representation to another is critical in the learning process involved in quadratic functions.

However, Nurrahmawati and Sudirman (2021) found that representation in a mathematics lesson is not highly realised as the most paramount for understanding and ignores the forms of representations developed by learners. Mathematical representation allows the learner to demonstrate their difficulty concerning the learned topic. Adu-Gyamfi et al. (2019) found that translation inability portrayed by learners was an essential factor in influencing learning process performance. This is achieved by looking at the learner's inability to translate from one representation into another.

Furthermore, many learners misunderstood the translation process involved in algebraic, tabular, and graphic representations (Adu-Gyamfi et al., 2019). For example, learners cannot fully understand the information given in the algebraic form, such as $f(x) \times g(x) > 0$. These conceptual obstacles negatively affect their mathematical representation skills. Nurrahmawati and Sudirman (2021) support the notion by stating that most learners face a conceptual impediment when dealing with algebraic and graphical representations. In contrast, Adu-Gyamfi et al. (2019) indicated that learners found it difficult to deal with translation from algebraic representations. Furthermore, one of the results of research by Castro et al. (2022) asserted that learners could translate from Table to graphical forms and are often challenged by translation from algebraic form.

However, Castro et al. (2022) revealed that learners possessed difficulties in the process of translational form, especially from tabular to algebraic form. Adu-Gyamfi et al. (2019) studied learners' translational tasks between the forms of representations and found three common conceptual obstacles that emerged from the data. These are failure in implementation, failure to interpret, and preservation conceptual obstacle. Out of the stipulated conceptual obstacle identified by Adu-Gyamfi et al. is that the second one is the most applicable quadratic function as learners often find it challenging to interpret the information given in an algebraic form.

2.6. THE LEARNING PROCESS OF QUADRATIC FUNCTIONS

The learning process involved through the quadratic function needs one to be familiar with the content to be learned and the performance trends engaged with the concept. The content to be learned needs to be accessed through the CAPS document, and the performance trends are sourced from the diagnostic reports that are often published after the matric exams. Therefore, this section looks at the learning process through the content and the trends of the concept.

2.6.1. Learning the content of quadratic functions

The learning and teaching about functions, mainly quadratic functions, are determined by the outcomes set in the CAPS document see Figure 2.10 below. Once the outcomes are known, teachers should decide the structure of teaching and learning quadratic function. School curricula worldwide have taken different approaches to include function in their curriculum. More specifically, the South African policy is cited from the CAPS document depicted in Figure 2.10 below (DBE, 2011).

Functions

Grade 10	Grade 11	Grade 12
Work with relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations (tables, graphs, words and formulae). Include linear and some quadratic polynomial functions, exponential functions, some rational functions and trigonometric functions.	Extend Grade 10 work on the relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations (tables, graphs, words and formulae). Include linear and quadratic polynomial functions, exponential functions, some rational functions and trigonometric functions.	Introduce a more formal definition of a function and extend Grade 11 work on the relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations (tables, graphs, words and formulae). Include linear, quadratic and some cubic polynomial functions, exponential and logarithmic functions, and some rational functions.

Figure 2.10: Overview of the function concept

The approach cited above is silent about how teachers should deliver the content to improve learners' conceptual understanding of quadratic functions. As such, teaching and learning to understand a function is still problematic. This is so because of the unclear definition of the concept of a function. The relationship between the conception of function and the mental image of learners is vital in the development of functions and has been widely researched over time (Ubah & Bansilal, 2018; Parent, 2015; Mutambara et al., 2019). The study of function is an ongoing debate in the mathematics research field. It is noted that learners continue to lack understanding of functions notations that are $f(x); y; f: x;$ and $(x; y)$. Parent (2015) noted that learners struggle to note that this is the same question, for the example given the following function $f(x) = 2x^2 + 3; y = 2x^2 + 3; f: x \to 2x^2 + 3;$ and $(x; y) \in \{y = 2x^2 + 3\}$. For learners to develop a rich understanding, they need to be exposed to various representations (Parent, 2015), including symbolic, graphic and algebraic representations.

2.6.2. Performance trends in quadratic functions from 2017 to 2021

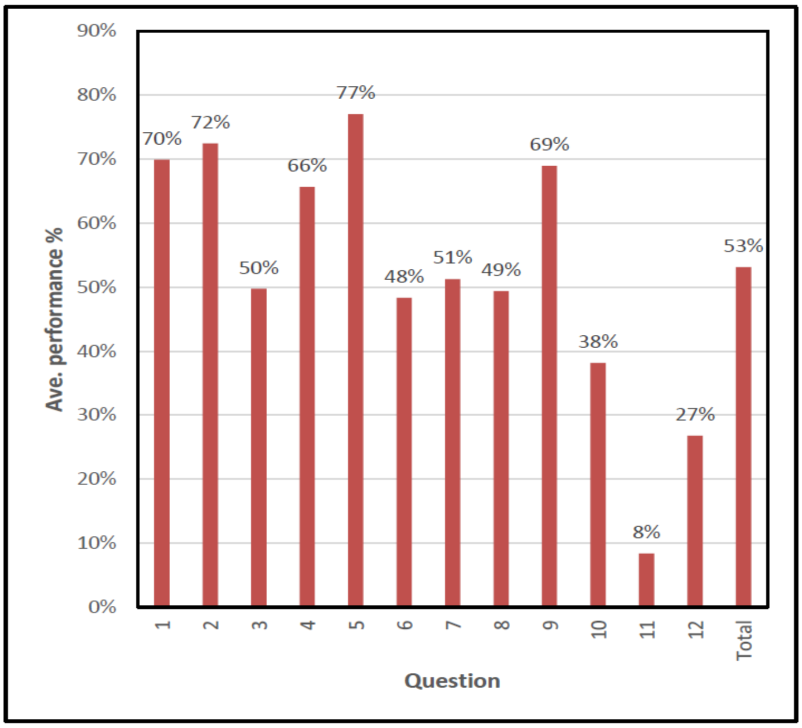
The enrolment of learners in writing the Grade 12 mathematics exam in 2021 increased compared to the other years. Thus, learners' performance in the content varied yearly. The Table in Figure 2.11 indicates variations in the performance over five years. However, performance in 2021 showed a slight improvement in learners in some concepts, but it is clear that this improvement does not guarantee that they understand the content. This is so because learners and teachers in mathematics are over-reliant on past exam papers to drill for marks. The focus on the past exam papers inhibits the teaching and learning of basic concepts in the mathematics classroom.

Year	No. wrote	No. achieved at 30% and above	% achieved at 30% and above	No. achieved at 40% and above	% achieved at 40% and above
2017	245 103	127 197	51,9	86 096	35,1
2018	233 858	135 638	58,0	86 874	37,1
2019	222 034	121 179	54,6	77 751	35,0
2020	233 315	125 526	53,8	82 964	35,6
2021	259 143	149 177	57,6	97 561	37,6

Figure 2.11: Overall achievement rates in mathematics

Therefore, the mathematics exam is divided into two papers: paper one and paper two. My focus is on paper one as the content that the study explored is being examined in paper one. In 2021 mathematics paper one, any learner could correctly answer the knowledge and routine questions and score good marks. However, achieving good marks does not connote understanding the concept. As a result, of lacking understanding of the concept but having good marks, it was noted that learners' algebraic skills were poor, and they also lacked the essential mathematical competencies. Therefore, lacking these skills hinders conceptual understanding other concepts like quadratic functions.

The findings from the report alluded that since teachers and learners in the mathematics classroom are over-reliant on the previous question papers, this demeans the focus on developing learners' profound understanding of concepts. As such, it was found that learners performed poorly in questions that nurture understanding of concepts. This poor performance is depicted in the graph below (Figure 2.12). However, the focus of our study, questions one and seven, should be considered. The two are the main focus because they cover the content of quadratics in the final paper. Moreover, the questions are further looked at by considering the sub-questions (Figure 2.13).



Q	Topics
1	Equations, Inequalities & Algebraic Manipulation
2	Number Patterns & Sequences
3	Number Patterns & Sequences
4	Number Patterns & Sequences
5	Functions & Graphs
6	Functions & Graphs
7	Functions & Graphs
8	Finance
9	Calculus
10	Calculus
11	Calculus
12	Probability & Counting

Figure 2.12: Average performance per question

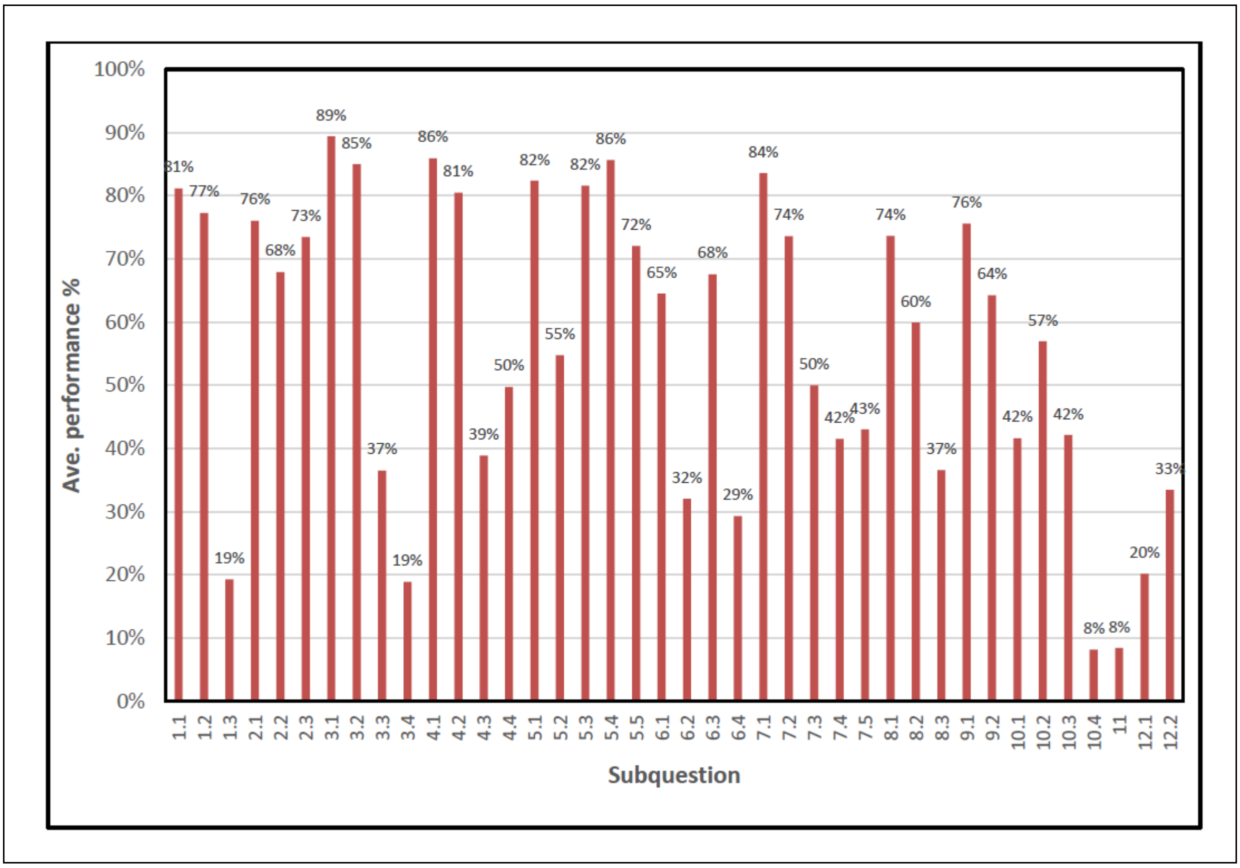


Figure 2.13: Analysis of learners' performance adapted from the DBE (2021)

The learners' performance in question one demonstrates a conceptual obstacle of the concept. The conceptual obstacle includes, among others, i.e., failure to factorise quadratics, rounding off challenges, and treating a quadratic inequality as an equation. The first conceptual obstacle deals with determining the roots of the function. Learners were given $x^2 - 2x - 24 = 0$ and required to determine the roots. It was noted that most learners struggled to factorise the quadratic instead, they wrote $x(x-2) = 24$. Furthermore, their solutions were $x = 24$ or $x = 26$ (DBE, 2021). This conceptual obstacle emanates from how learners learn the topic in class. The learners might have encountered this type of quadratic $x^2 + 2x = 0$ because teachers drill them. They might have forgotten to highlight essential aspects in the development of the concept.

The second conceptual obstacle resulted from failure to use the quadratic formula to determine the roots. Learners were given $2x^2 - 3x - 3 = 0$ and further instructed to leave their solutions to two correct decimal places. However, they failed to substitute into the formula; as such, they wrote $x = \frac{-3 \pm \sqrt{-3^2 - 4(2)(-3)}}{2(2)}$ instead of $x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-3)}}{2(2)}$ (DBE, 2021). Omitting the brackets resulted in wrong solutions, and they struggled to round off. The third conceptual obstacle emanated from the learners' approach to inequalities, viewing them as equations. They were given $x^2 + 5x \leq -4,$ and their solution were $(x + 1)(x + 4) = 0$ followed by $x = -1$ or $x = -4$ (DBE, 2021). The learners did not realise the concept behind the question; therefore, they did not understand the solution.

The conceptual obstacles identified in question one surfaced again in question seven. This is so because the content examined in question one serves as essential knowledge for question seven. Therefore, in question seven, learners failed to write the roots of the function in a coordinate form while given the factored form that is $f(x) = (x + 4)(x - 6)$. Furthermore, they demonstrated difficulty in determining the vertex of the function from the factor. As such, most learners wanted to use the formula $x = \frac{-b}{2z}$ instead of determining roots. Using formulas without conceptual understanding connote conceptual obstacles of concepts. Moreover, learners persistently showed a poor understanding of a range and domain of a function.

2.7. CONCEPTUAL OBSTACLES THAT INHIBIT UNDERSTANDING

Relational and instrumental understanding seem to compete for attention in mathematics classrooms. Teachers are found in the hub, whereby they nurture instrumental instead of relational understanding. For example, in teaching the vertex, they stress using formulae even if the function is in the vertex form. Some studies focus on conceptual versus procedural understanding of quadratic functions (Mutambara et al., 2019; Parent, 2015; Ubah & Bansilal, 2018). Mudaly and Rampersad (2010), cited in Ubah and Bansilal (2018), conducted research with South African Grade 11 learners, where they investigated learners’ conceptual understanding of the graphical representation of quadratic functions. Their results revealed that most learners depended on procedural understanding as their conceptual understanding was weak.

Parent’s (2015) study investigated how learners develop a conceptual understanding of the graph of the quadratic function. The results revealed that learners found it easy to interact with the standard form ($f(x) = ax^2 + bx + c$) then the vertex form ($f(x) = a(x - p)^2 + q$) when engaged in quadratic function questions. For example, learners might struggle to determine the roots of a quadratic function where $a = 1$ and b and c are non-zero, such as $x^2 + 8x + 6 = 0$, might have difficulties interacting with $x^2 + 8x = 0$ where $c = 0$ and therefore not visible in the equation $x^2 + 8x = -6$. Didiş et al. (2011) assert that a quadratic function in which the parameters b and c are zero does not look like a quadratic function to learners and assumes that the parameter does not exist. For example, learners might say that $y = ax^2 + bx$ does not have the y-intercept because the value of c is zero. However, in this function, the value of c exists, and the y-intercept would be $(0; 0)$ (Zavlasky, 1997).

In the continuation of the Parent’s (2015) study, it was noted that learners confused the y-intercept of the standard form with the y-coordinate of the vertex form. Thus, it is indicated that learners who immersed in procedural understanding make several conceptual obstacles (Siyepu, 2015). Zaslavsky (1997), cited in Parent (2015), researched the conceptual obstacles that hinder learners' conceptual understanding of quadratic functions. The conceptual understanding noted are: (1) interpretation of graphical information; (2) relation between quadratic equation and quadratic function; (3) analogy between linear functions and quadratic functions; (4) change in the form of a quadratic function to standard form, vertex form or factored form; and (5) focus on one particular coordinate.

Dede and Soybas (2011) studied the experience of mathematics pre-service teachers with a quadratic function. The data were collected using semi-structured interviews and questionnaires and analysed through phenomenology. The results revealed that learners in those classrooms have incorrect conceptions and ideas about the quadratic function, which yielded their understanding to be termed procedural. Zazkis et al. (2003) studied a horizontal translation of quadratic functions of the form $f(x) = (x + 2)^2$ and its relationship to $g(x) = x^2$. The researchers wanted to investigate learners' difficulties with the translation of quadratic functions. In an interview session, the participants were asked to predict, check and explain the relationship between $f(x)$ and $g(x)$. Data gathered from the interviews were analysed in terms of common trends of mathematical explanations. The findings of the study revealed that learners who failed to sketch the graph of $f(x)$ correctly demonstrated procedural understanding.

A study by Borgen and Manu (2002) investigated learners' understanding of calculus problems. In their research, they posed the question of determining the stationary point of the quadratic function $y = 2x^2 - x + 1$ and then decided if it had a minimum or maximum. The data was collected using a video camera as learners engaged with the problem. Data obtained from the videotape and learners' written work were the basis for the analysis of this study. The study employed Schoenfeld's (1998) four layers of analysis and Pirie and Kieren's (1994) model to analyse the learners' mathematical understanding. The researchers found that one learner lacked an understanding of connecting with related concepts. Additionally, the analysis of Pirie and Kieren's theory showed that learners could not fold back as these learners did not develop conceptual understanding.

The work presented by Sajka (2003) investigated an average learner's understanding of functional equations. In an interview session, the researcher asked a learner a non-standard question. The question was to give an example of a function f such that for any real numbers x, y in the domain, the following equation holds: $f(x + y) = f(x) + f(y)$. Data were analysed using the precept theory by Gray and Tall (1994). This theory consists of three components: a process producing a mathematical object and a symbol representing either a process or an object (Gray & Tall, 1994). Sajka acknowledged that learners had a problem understanding the question at the beginning, and after a long interaction with the researcher, the learner showed some improvements. To achieve this, the researcher presented the following functions to the learner, i.e., $f(x) = x^2 - 2x + 3$ and $g(x) = x^2 + 5$.

A difficulty with some of the learners is other conceptual obstacle of the term variable (Vaiyavutjami & Clements, 2006). For example, in the factored form of the quadratic function, that is $(x-2)(x+4)=0$, some learners assume that the solution of the equation that is $x=2$ and $x=-4$ stands for a different value. They noted that learners held this conceptual obstacle of assuming that the x determined above can be substituted simultenously as $(2-2)(-4+4)=0$ meaning that x can be 2 and -4 at the same time. Didiş et al. (2011) confirm these findings, stating that learners can determine the correct answer without understanding how the zero product affirms that one of the factors will be equal to zero.

Learners' understanding of quadratic functions is rooted in the poor action level of understanding of the quadratic equations. This is so because learners prefer focusing on the positive solution of the equation more often. Didiş et al. (2011) assert that learners do not understand the solutions of the quadratic functions. This was seen from the point that learners gave one answer to $x^2=a$ instead of two. For example, learners are given the function $f(x)=x^2-4$ and required to determine the x-intercepts. They would give the positive value only instead of two that is $x=\pm 2$. Didiş et al. suggest that learners do not understand the meaning of ± in the square root.

Research on how learners understand the functions of quadratics demonstrates that learners prefer drawing functions from equations of functions over generating an equation of a function from its graph. When working with graphs, Zaslavsky (1997) noted that learners made assumptions about quadratic functions based on the graph they saw and did not use their understanding of a quadratic function to assist them in interpreting the function. For example, given the graph below in Figure 2.14, learners may assume that the y-intercept does not exist in the function. Moreover, they may think that the function has a vertical asymptote. Therefore, Zaslavsky also noted that learners could not use implicit information related to the axis of symmetry unless it is drawn.

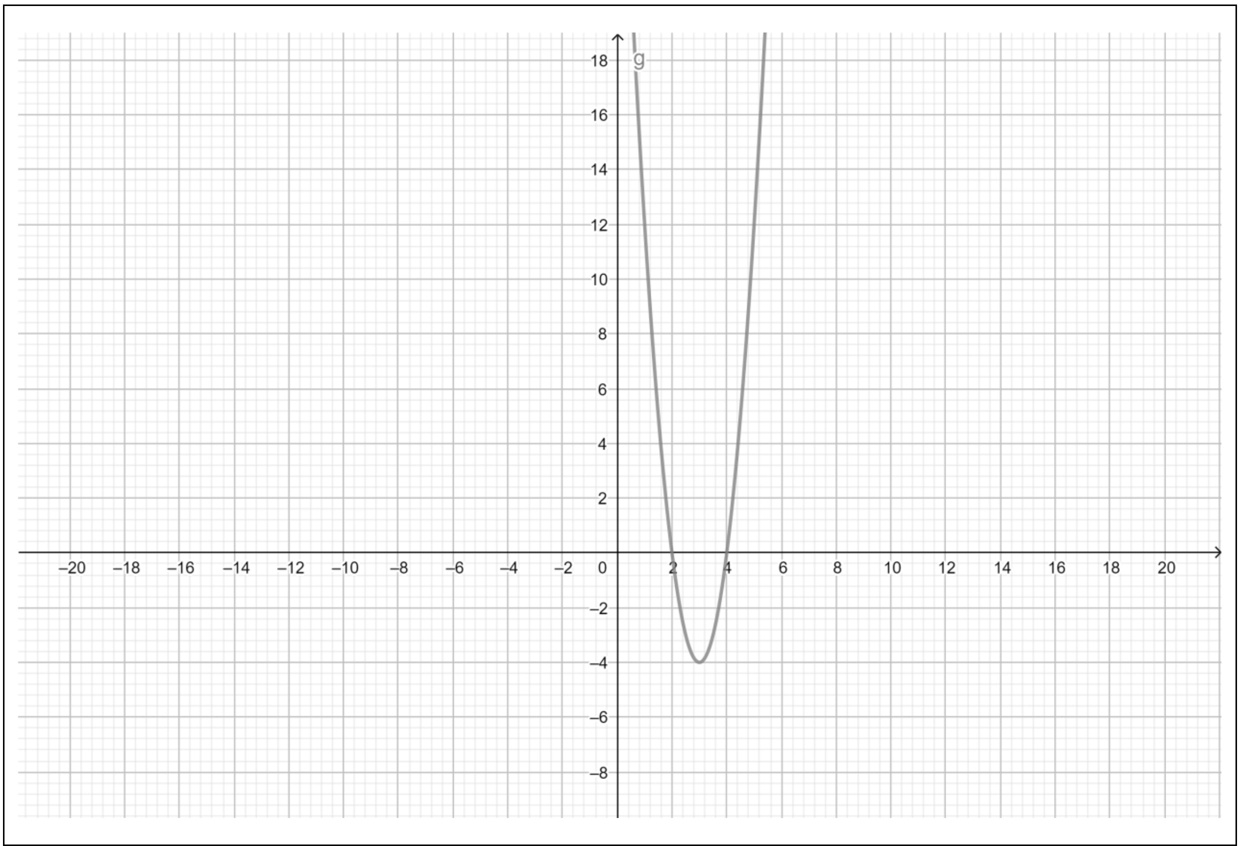


Figure 2.14: Graph of $g(x)$

Learners have difficulties in determining the x-intercepts of the quadratic, which leads them to fail to graph the graph. This idea is also supported by Bossé and Nandakumar (2005) that learners struggle with multiplication facts, which makes it difficult for them to determine the factors for the expression in the form quickly $ax^2 + bx + c$. These difficulties increase when $a \neq 1$, for example, $8x^2 + 10x + 20$ or $20x^2 + 30x + 100$. Given these cases, learners are left with options to use the quadratic formula or to complete the square. However, the use of the other method to determine the x-intercepts does not mean that learners have entirely understood the concept since those who were given this equation struggled $x(x - 2) = 0$. From the equation, Kotsopoulos (2007) found that learners would cancel the x on both sides of the equation by dividing the equation by x, and learners lose track of the root $x = 0$.

Numerous researchers have identified conceptual obstacles that have blocked learners' understanding of quadratic functions. Parent (2015) defines conceptual obstacles as features of learners' knowledge that is repeatable and seen in the solution. Conceptual obstacles mean the learner has purposefully solved an answer while thinking it was right. Parent's (2015) study noted some conceptual obstacles similar to Zaslavsky (1997) work. The conceptual obstacles mentioned are, i.e., various representations of the quadratic function, relational reading and interpretations, the concept of a variable within the equation, and notation within the graph of a function.

Kotsopoulos (2007) found that high school learners encounter many difficulties when factoring quadratic functions. In the same study, Kotsopoulos (2007) pointed out that learners get confused when the quadratic function is shown in a different form, either standard form, vertex form, or factored form. Ellis and Grinstead (2008) built from the work of Kotsopoulos, noting that when learners are working on quadratic functions, the conceptual obstacles that they encounter are: (1) connection between algebraic, tabular and graphical representations; (2) a view of graphs as an object; and (3) struggle with interpretations of parameters.

One of the fundamental studies leading to conceptual obstacles about learners' encounters in quadratic functions is by Zaslavsky (1997). Zaslavsky researched the conceptual obstacles that hinder learners' understanding of quadratic functions. Conceptual obstacles are cognitive and can be explained in terms of mathematical structures. In the study, she identified five conceptual obstacles, i.e., interpretation of graphical information, the relation between quadratic equation and quadratic function, an analogy between linear functions and quadratic functions, change in the form of a quadratic function to standard form, vertex form or factored form, and focus on one particular coordinate.

Baki et al. (2010) conducted research and asked learners to find the vertex of the quadratic function $f(x) = 2x^2 - 12x - 14$ and to draw its graph. It was observed that most learners were not familiar with the function in the form $y = f(x)$. As a result, most of them could not draw the function; what they drew was a linear function. Moreover, Baki et al. found some obstacles, i.e., they did not understand the purpose of the problem and felt confused when they used the formula because they were rooted in using formulas. They tend to memorise the method of finding solutions.

The Department of Basic Education issues diagnostic reports on conceptual obstacles that Grade 12 learners commit in the final mathematics papers. These reports cover quadratic functions, among other mathematics topics. In 2012, DBE reported that learners commonly made mistakes in quadratic functions, where they struggled to: (1) substitute the value of "a" into the factored form when determining the equation for f(x); (2) to interpret the function and give a correct y-intercept; and (3) to find roots. It was suggested that teachers stress the characteristics of parameters a, b and c. In 2014, the DBE reported that learners could not recognise the connection of quadratics to optimisation. In addition, learners did not notice that the x-intercept of the turning point gives the line of symmetry (DBE, 2015).

The 2020 DBE diagnostic report noted that most learners could not read off the coordinates of the turning point from the equation. Instead, they converted from the given form to the standard form $f(x) = ax^2 + bx + c$, then calculated the axis of symmetry and the minimum. Some learners gave the x-intercept of "A as 5" instead of "-5", while some assumed that A is the y-intercept. Therefore, these conceptual obstacles reviewed from different DBE reports result from teaching and learning quadratic functions. Hence, it can be noted that these conceptual obstacles can be treated only if teachers and learners strive to develop conceptual learning through the contextual teaching of quadratic functions.

The literature review reveals that the learners have a limited understanding of quadratic functions. More specifically, the finding showed that learners' difficulties were divided into three tenets, i.e., failure to understand the mathematical notations, the restricted context of activities, and learners' idiosyncratic interpretation of mathematical activities. Moreover, the reviewed studies under this section concerning learners' understanding of quadratic functions showed some learning problems in four categories: (1) quadratic functions depicted by graphs are taken as a picture; (2) quadratic functions are viewed as quadratic equations, and not functions; (3) difficulty of translation from graphical form to algebraic form; and (4) overemphasising on one coordinate of particular points.

As opposed to Zaslavsky (1997), who used a quantitative approach in his study, some studies (Borgen & Manu, 2002; Zazkis et al., 2003; Sajka, 2003) were approached more qualitatively by employing learners' interviews. However, the interview techniques in these studies are problematic as the theoretical framework does not inform them of their studies. Ginsburg (1981) suggested using a one-to-one unstructured interview to discover learners' cognitive processes, conceptions and obstacles. In light of Ginsburg's recommendations, the present study uses one-to-one unstructured interviews to improve learners' understanding of quadratic functions.

2.8. ACE TEACHING CYCLE IN QUADRATIC FUNCTIONS

While there is an identity crisis for functions in the mathematics curriculum, teachers in the classroom are at the centre of making decisions on what to teach (Denbel, 2015). As a result, such decisions are complex and demanding and are more difficult when the purpose and goal of teaching functions in high school mathematics are unclear. Moreover, it is not teachers alone in the struggle on what to do. Instead, textbook authors do not present the notion of function in ways that link this concept to the rest of the mathematics concepts, thus leaving this to the rest of the teacher (Denbel, 2015). The teacher with the South African mathematics approach should ensure that learners eventually develop a concept image matching the comprehensive definition of a function (Denbel, 2015). This is why some teachers opt for procedural natured teaching strategies while interacting with natured conceptual contents.

Yin (2005) reports on learners' understanding of quadratic inequality conceptual obstacle through an in-depth interview. The author stresses that teachers must consider their practice. The more teachers think and develop their teaching strategies, the more they would be superior with their content delivery (Ball & Cohen, 1999). The findings of the study revealed that learners are embedded in procedural knowledge without understanding the concepts. Moreover, learners gave quadratic function domain procedurally without understanding the concept. Thus, learning quadratic functions combines aspects of geometry and algebra, which are crucial for learners' understanding. Unfortunately, learners who engage in quadratic functions experience barriers in learning the concept.

The outcry now is on the teacher to opt for teaching methods that nurture conceptual understanding of functions. The method that can be adopted is the ACE teaching cycle, which demands teachers' conceptual understanding of the concept. Dede and Soybas (2011) studied pre-service teachers concerning quadratic functions. The findings were that the pre-service teachers have incorrect conceptions about quadratic functions. Also, Bansilal et al. (2014) investigated how teachers with poor mathematical understanding taught conceptual understanding. The results revealed that the teachers did not articulate coherent explanations while presenting the knowledge of quadratic function. As a result, literature is silent on how the teaching of quadratic function should be; thus, this is the contributing factor that teachers tend to use procedural understanding anchored pedagogy to teach abstract natured concepts (Arnon et al., 2014; Bansilal et al., 2014; Brijlall & Maharaj, 2015). Therefore, learners' possession of procedural understanding is the derivative of pedagogy. For example, if the teachers stress using a formula in finding the vertex, learners will forever memorise that, and even if the function is in a vertex form, they will use the formula. As a result, teachers should always nurture an understanding of mathematical concepts. Therefore, they should adopt pedagogies that cultivate learners' understanding.

Hence, there is a need to explore learners' conceptual understanding of quadratic functions through the ACE teaching cycle. The teaching cycle is a teaching approach anchored by APOS theory (Arnon et al., 2014), and is based on three constituents, i.e., activities designed to nurture learners' development of mental structures and classroom discussions to allow learners to reflect on the activities they were doing. Lastly, exercises outside the class can be homework to consolidate the knowledge obtained from items 1 and 2 (Brijlall & Maharaj, 2015). Through the exploration of learners', the growth of conceptual understanding of quadratic function can be nurtured using the ACE teaching cycle.

2.9. SUMMARY OF THE CHAPTER

In this chapter, the literature reviewed was under the following sub-headings: the breakdown of the quadratic function concept, mathematical understanding, and mathematical representation of quadratic functions, assessing learners' conceptual understanding, learners’ conceptual understanding of quadratic function, conceptual obstacles about quadratic function, and contextual teaching of the quadratic function. The theoretical framework was also presented, and some of the studies that used the APOS theory were discussed as they provided insight into the research methodology described in the next chapter.

CHAPTER THREE: RESEARCH METHODOLOGY

3.1. INTRODUCTION

In this chapter, I present the research paradigm choice and the research design. The choice of sampling is captured in this chapter. Moreover, I present my role in this study, how I collected data, the procedure followed in the study, and how I analysed my data. The quality criteria and ethical consideration are captured. Lastly, I present the chapter summary.

3.2. THE CHOICE OF RESEARCH PARADIGM

Research is a systematic process of gathering and analysing data for improving practice (Bogna et al., 2020). Research can use the methodological paradigm of qualitative, quantitative, or mixed-method approaches. Quantitative research seeks predictions and explanations. Conversely, qualitative research is an approach for exploring and understanding the meanings of the individual (Bogna et al., 2020). In contrast, the mixed-method approach incorporates quantitative and qualitative research designs simultaneously. Therefore, the reported study was located within the interpretive qualitative approach. In the context of the study, I explored understanding of learners' thoughts as they interact with mathematics tasks.

The work of Johnson and Christensen (2004) guide my choice of research paradigm. Johnson and Christensen explain that a research paradigm is a perspective based on fundamental beliefs, world views, concepts, assumptions, and values. These worldviews become a foundation for understanding and interpreting reality (Bogna et al., 2020). For the present study, I had to comprehend the assumptions of reality through which nature of knowledge is developed in the processes of learning. Such knowledge becomes learners' knowledge, serving as a lens for their mathematical reasoning (Arnon et al., 2014).

Researchers adopt different paradigms, such as positivism, constructivism, pragmatism and transformative (Guba & Lincoln, 1994). I adopted the constructivism paradigm. Constructivism asserts that people (learners) construct their understanding and knowledge by experiencing and reflecting upon those things

(Honebein, 1996). Although knowledge construction in this paradigm is attributed to interaction, Steffe and Thompson (2000) hold that one should not perceive the teacher as the giver of knowledge; instead, the teacher’s role is to mediate mathematical activities.

I hold that the constructivist paradigm asserts the role of interactions as a tool for developing knowledge in a mathematics class. Therefore, it is within the heart of this paradigm that I derive the purpose of the study. Contrary to other world views, such as positivism, which holds that explanations are used to predict and control phenomena, I was concerned with re-learning and reconstructing prior knowledge of quadratic functions. Hence, this continuous process was done to permit new interpretations and constructions of quadratic functions knowledge. Therefore, knowledge within the frames of the constructivist is not passively received but actively developed (Kusuma et al., 2021). Constructivism embraces learners’ prior knowledge of quadratic functions from the previous Grades. Learners bring prior constructions of quadratic functions into the mathematics classroom to their learning space through their interaction with others.

In contrast to the positivist paradigm, which uses scientific methods and statistical procedures to report data, constructivism uses methods that stimulate understanding of the phenomena under exploration (Kusuma et al., 2021). Additionally, analysis is done through content analysis. Hence, I did not pursue to generalise the findings to a group but understand how learners' understanding of quadratic is within the tenets of the APOS theory through the implementation of the ACE teaching cycle.

3.3. RESEARCH DESIGN

Creswell and Poth (2016) define research design as the whole process of research, from conceptualising the problem to writing the report. Moreover, a research design determines the following in a qualitative study, i.e., participants, data collection methods, and data analysis procedure (Merriam & Tisdell, 2015). As such, I adopted an exploratory qualitative case study. A case study is a complete portrayal and investigation of a bounded phenomenon (Merriam, 1998). As suggested by Merriam, the boundaries of the study were Grade 12 learners’ conceptual understanding of quadratic functions concepts, that is, an axis of symmetry, vertex, the location of roots, whether the graph opens up or down, the maximum or minimum point of the graph, the y-intercept, and the transformations through the ACE teaching cycle. The structure of the methodology is explained below and shown in Figure 3.1.

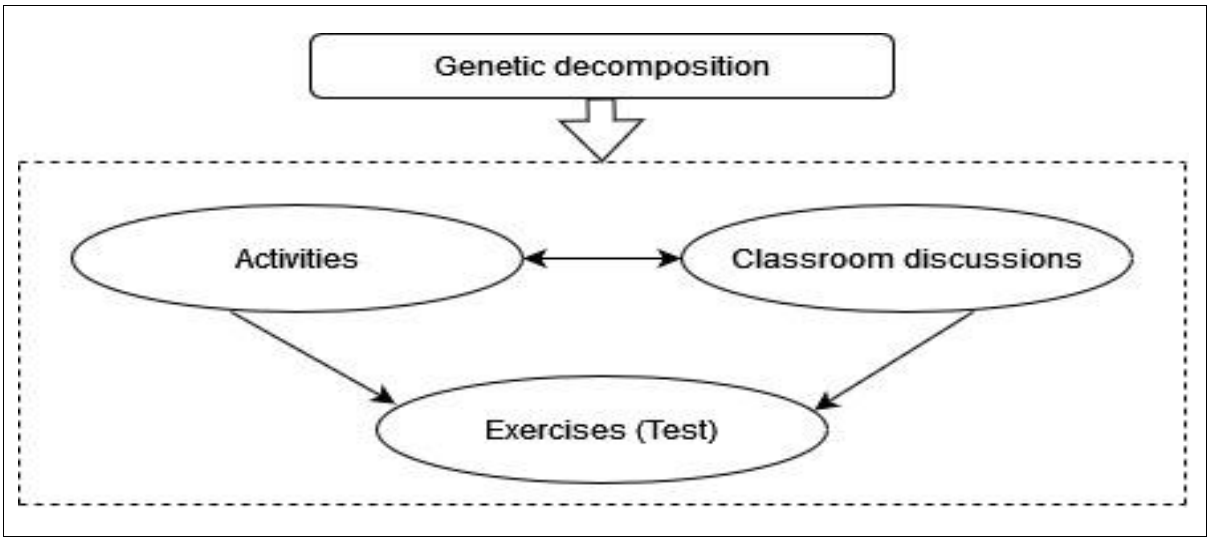


Figure 3.1: The relationship between the ACE cycle and Genetic decomposition

Firstly, I developed a genetic decomposition, that is the APOS theoretical analysis for learning quadratic functions. Secondly, I implemented an ACE teaching cycle in class. During the ACE teaching cycle, I started with the activities phase, followed by the classroom discussions phase. The activities phase was designed to check learners' knowledge of the quadratic function concept, and the class discussions phase was done after interacting with the activities phase. Thirdly, I implemented the exercises phase of the ACE teaching cycle, which was designed to reinforce the activities and classroom discussion phase.

3.4. SAMPLING

Merriam (1998) suggests that sampling should occur before data collection. Purposive sampling is a sampling tied to a specific objective, meaning that one perceives sampling as a sequence of strategies (Campbell et al., 2020). Therefore, I employed purposive sampling as a sampling technique to sample 30 participants. These participants were sampled because of their performance on mathematics from Grade 11 results. All these were exposed to the ACE teaching cycle for two weeks, where their Annual teaching Plan (ATP) was not affected with the process as we met after the normal time for teaching and learning. By so doing, none of the learners was disadvantaged by selection bias. All the participants were identified as data sources.

The manner of choosing the participants is aligned with the sampling method employed. The power of purpose sampling is to select a few grain-sized participants that are studied to yield more insights into the topic (Campbell et al., 2020). The study was bounded by time and location, with the results only about these sampled learners. Hence, the participants were intentionally drawn from my class to minimise outside mathematical influence. As a result, there are no other assumptions about other learners except the participants of the study. The data collection was for two weeks except on weekends, and we met for 50 minutes each day.

I obtained ethical clearance from the research office at the University of Limpopo. Then I submitted a request to conduct the study to the research office of the province, i.e., the Limpopo Department of Education (see Appendix A). Lastly, I sent a request to the school's principal, i.e., the research site (see Appendix C). I received permission to conduct the study from the Limpopo Department of Education (see Appendix G). Learners gave consent to participate in the study (see Appendix B), and the principal and SGB of the school were notified (see Appendix C). To maintain confidentiality, the learners self-selected pseudonyms using letters of the alphabet. For example, "Learner A." Although I did not foresee any reason a learner might wish to withdraw from the study, there was a protocol for this. All the participants had the right to withdraw from the study without penalty. As such, if a participant wanted to withdraw from the study, any previous data collected from the participant would still be used for the research. Luckily, none of the participants withdrew from the study.

3.5. MY ROLE IN THIS STUDY

The study participants interacted with a task that was non-directive since I adopted a constructivist approach. This meant that I was not giving them answers or funnelling them to my procedures. Therefore, I was in the classroom and did not funnel their responses as they interacted with the learning tasks. The content was delivered through a mathematics learning unit (see Appendix D). A learning activity is a unit designed by the teacher to bring about and create an atmosphere for learning in the mathematics classroom. My presence in class during the interaction with the learning unit was to keep learners on task and to scaffold them if they hit a snag. During the learners' interaction with the learning activities, I watched and looked for the “aha!” moments to gain more insight into their understanding of quadratic functions. To ensure validity, I developed a research protocol (see Appendix H) to assist me when interacting with the learners in the classroom.

3.6. THE LEARNING ACTIVITIES

I developed the learning activities using the CAPS document and assessment guidelines following Bloom’s taxonomy, which is a hierarchical learning method. The rationale behind it was that learning at higher levels depends on having attained the knowledge at the lower levels (Web, 2020). I divided the learning unit into three sections, i.e., quadratic functions and parabolas, sketching parabolas and transformation involving quadratic functions. Each section presented in the learning unit was guided by the constructivism theory of learning, which should present the learner as an instrument to guide them in understanding the section. I prepared a pre-test and post-test in the learning unit to gain more insight.

3.7. DATA COLLECTION

With the implementation of the ACE teaching cycle, I collected qualitative data. Creswell and Poth (2016) assert that researchers often consider observational data when considering qualitative data. Merriam (1998) broadens data collection methods by focusing more on collecting data as determined by the researcher’s paradigm, purpose, and the research design adopted. For this purpose, I collected qualitative data through the constructs of the ACE i.e., activities (learning task), classroom discussions, unstructured interviews and exercises (test).

3.7.1. Activities

The activities phase was the first tenet to be applied, involving writing learning tasks on the quadratic functions. During this stage, Task 0 was administered, which assessed learners’ prior knowledge, and thereafter, learners worked on learning tasks. For the learning task and Task 0, I complied with the CAPS document and previous question papers. These activities took place in the class where learners were working individually and in groups. The period was 50 minutes, and learners did not receive any assistance from the teacher. They started by working individually when they wrote Task 0 and in groups when interacting with the learning tasks.

3.7.2. Classroom discussions

The classroom discussions phase followed the activities phase. Here learners had an opportunity to express their ideas and understandings of the learning tasks. They engaged in concepts that posed conceptual obstacles on quadratic functions. Additionally, I purposefully recorded two participants for learner-learner and learner-teacher discussions. The learners' discussion was sampled due to their inability to justify incorrect responses to the activity (Campbell et al., 2020). The classroom discussion was recorded in the classroom during their interactions.

3.7.3. Exercises

The exercise phase of the cycle reinforced the previous activities and classroom discussion phases. Learners continued to build their ideas and understanding of the concept of the quadratic function. A test compromised the exercise phase and assessed their level of understanding of quadratic functions.

The constructs of the ACE teaching cycle served as a tool for collecting data. Therefore, the ACE teaching cycle was guided by the APOS theory. The data collection was as follows, i.e., development of the genetic decomposition, implementation of the ACE cycle, and collection of qualitative data from activities, classroom discussions and exercises. Therefore, I employed the constructs of the ACE teaching cycle to collect qualitative data to respond to the research question, i.e., “How does the ACE teaching cycle improve learners’ conceptual understanding of quadratic functions?”

3.8. RESEARCH PROCEDURE

The learners participated in the study for two weeks, and none was absent nor withdrew from the study. Each session lasted 50 minutes per day. Firstly, learners wrote Task 0, and the scripts were collected. Secondly, they were given learning tasks to work with. The learning unit had three sections to be attended to. Although I was present during the overall research process, my presence was to keep them on task and to encourage them to work on the activities continually. There was a protocol to follow, so I could not compromise the results. I did not assist them unless they hit a snag, and I only had to probe and scaffold. The research procedure is depicted diagrammatically in the model below (see Figure 3.2).

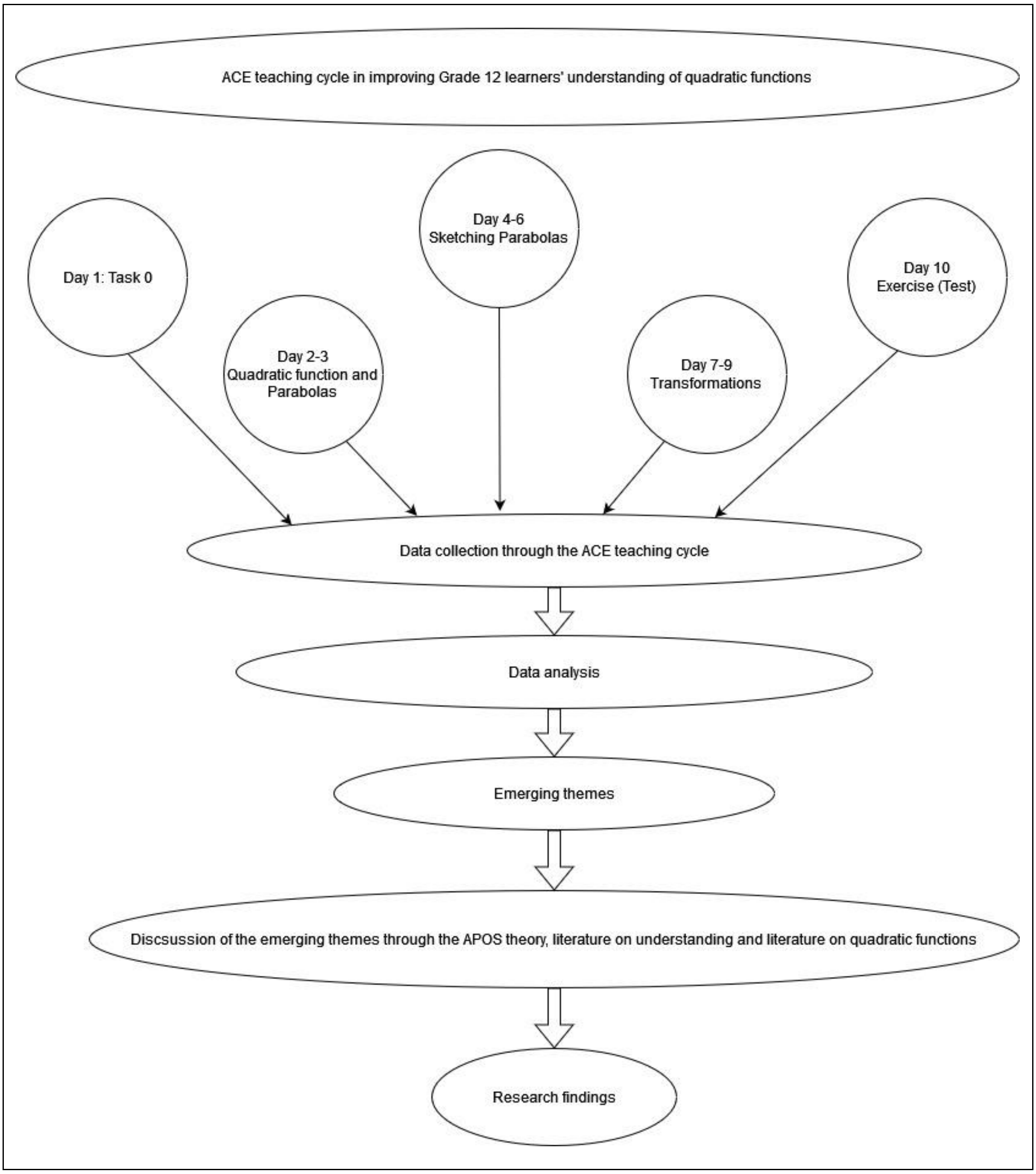


Figure 3.2: Research procedure

3.9. DATA ANALYSIS

Data analysis reduces gathered information interpretations and summarisation to get the meaning of the data (Bikner-Ahsbahs et al., 2015). Therefore, I analysed the collected data using content analysis. Content analysis is an analysis method for interpreting textual data through organisation process codes and themes (Hsieh &

Shannon, 2005), and can be used in a deductive or inductive way; thus, I employed the deductive method in my analysis process.

The deductive analysis is applied when the structure of analysis operates based on prior knowledge, and the purpose is theory testing (Elo & Kyngas, 2008). In this case, I explore learners' conceptual understanding through the levels of the APOS theory and viewed learning knowledge through genetic decomposition. All this was done to improve Grade 12 learners' conceptual understanding of quadratic functions. Deductively, the tenets of the theory were used to determine learners' level of conceptual understanding. Content analysis is divided into three phases, i.e., preparation, organising and reporting.

During the preparation phase, I reviewed the data to allow new insights to emerge. I read all data gathered repeatedly to obtain a sense of the whole. I separated data from the activities, classroom discussions, and exercises during this phase. Next, from the analytic process, I strove to make sense of the data, learn what is going on, and understand the whole. During the process, the tenets of the APOS theory using genetic decomposition were used to locate the data in each tenet.

After making sense of the data, I started to group them. Since the purpose of the reported study was to improve learners' understanding of quadratic functions, therefore the existing constructs of the APOS theory, namely action, process, object, and scheme, were re-tested in this context to illustrate learners' levels of understanding. Lastly, the collected data were reviewed for content and themes that emerged from the data. Furthermore, a report was written on the themes, deciding the level of learners' understanding of quadratic functions based on the APOS theory.

3.10. QUALITY CRITERIA

In ensuring quality criteria, I adhered to issues of rigor and trustworthiness. Guba (1981) asserts that for the verification of qualitative data, the researcher must respond to the four proposed trustworthiness concerns: transferability, conformability, credibility and dependability. Therefore, the confirmation of data in my study referred to the scrutiny of these concerns.

Bryman et al. (2008) assert that credibility ensures that the study results are accurate. In qualitative research, credibility depends more on the wealth of the information gathered. To ensure the credibility of the study, I have three tenets to collect data: activities, classroom discussions and exercises. Therefore, all this was done to foster triangulation of data. Triangulation means collecting and analysing datasets from more than one source to understand more about the studied case.

Transferability refers to how the research findings can be applied to other situations (Bryman et al., 2008). To apply transferability in a qualitative study, I used it with caution. This implies that I transferred the findings to another context with caution to the sampling and the purpose of the study in mind. Furthermore, the study was not meant to generalise its findings. Hence, the criterion of transferability is relevant for a qualitative study if the research intends not to generalise.

Bryman et al. (2008) define dependability as the ability to verify that the findings of the study are consistent and can be repeated if the same data collection method is used. Therefore, dependability was maintained in this study, and the learners' written responses to the activities were collected. Moreover, the scripts served as valid proof of their understanding of quadratic functions. The scripts illustrated what learners can and cannot do. According to Creswell and Poth (2016), conformability deals with the level at which the research findings are supported by the data gathered and are free from bias. To ensure conformability, I was not biased towards the collected dataset. I achieved conformability by not tempering with the data by choosing what I preferred and excluding others. Additionally, the findings are synthesised directly from the data by providing rich quotes from participants.

3.11. ETHICAL CONSIDERATIONS

Creswell and Poth (2016) assert that researchers have a task to adhere to the participants' rights, values, desires, and needs to ensure that the reported study is ethically conducted. Ethics establish boundaries that the study should not harm the participants. Qualitative research is conducted in a practical setting that allows the participation of individuals. Thus, Batchelor and Briggs (1994) reported that research involving people's participation needs awareness of ethics. For my study to be ethical, I addressed the following dimensions: informed consent, anonymity and confidentiality, human dignity, privacy, and responsibility to avoid harm in pursuit of ethical considerations.

3.11.1. Informed consent

The most effective way to address the informed consent issue is through an information sheet given to all stakeholders invited to participate in the research (Bulmer, 1982). The faculty gave me approval to proceed with the study (see Appendix I). Additionally, to adhere to informed consent, I wrote a letter to Limpopo Department of Education (see Appendix A) and to the principal of the school where the study took place (see Appendix C). I did this to seek permission and to alert all relevant stakeholders about the research project. Additionally, I sought permission from parents for their children to participate in this study as they are minors. The parents were also furnished with a letter that included the purpose of the study, the authorisation of the research by the University of Limpopo, the ethical clearance certificate (see Appendix F), and the period of participation in the study. I ensured that the study did not disturb the academic process but improved it. The Limpopo Department of Education permitted me to conduct the study (see Appendix G).

3.11.2. Anonymity and confidentiality

Orb et al. (2001) assert that any research should adhere to the opinion of respect and justice. Anonymity requires that you do not know who the participants of the study are, especially their original names. I ensured that the participants remained anonymous and had free will to participate in the research by filling in a consent form (see Appendix B). Hence, I maintained confidentiality and the protection of all. The school’s name, the teacher, and the learners were kept anonymous to maintain confidentiality. Therefore, for the school’s name, I used “XXXXXXX” high school, for the teacher, I used “T-Math,” and for the learners, I referred to them as, for example, “learner P20.” Moreover, I did not expose the participants to unusual circumstances since it was conducted in a school environment. Hence, their answers were kept confidential at all costs.

3.11.3. Human dignity

Research should adhere to human dignity. Human dignity is intricately linked to individual inviolability. In research ethics, human dignity implies that individuals have interests and integrity which cannot be set aside in research to achieve a deeper understanding of the study (Orb et al., 2001). I protected personal integrity and preserved freedom and self-determination to protect human dignity. Thus, the choice of topic for the study was suitable for the participants, and not to expect any harm when results were reported.

3.11.4. Privacy

Research must be conducted following considerations for data protection of the concerned participants (Creswell & Poth, 2016). Therefore, privacy in the context of the study was narrowed to the two dimensions of privacy: when individuals have impaired or absent capacity to protect their own needs and interests; and when individuals actively contribute to acquiring data for research, for example, by agreeing to be observed or interviewed. Privacy in the study was applied with caution and responsibility, as I maintained that the data is kept in a safe and no one is allowed to access that data.

3.11.5. Responsibility for avoiding harm

Researchers are responsible for ensuring that participants are not exposed to serious physical harm or severe or unreasonable strain due to the research (Creswell & Poth, 2016). In a qualitative study, there is usually insignificant risk of participants being exposed to severe physical harm. However, serious mental strain is a possibility. This may be more difficult to define and predict, and it can be challenging to assess the long-term effects, if any. Therefore, the participants in the study did not experience any form of harm due to the activities not being harmful but beneficial. Their answers were not used to embarrass them; instead, they were used to respond to the research questions posed above.

3.12. SUMMARY OF THE CHAPTER

In this chapter, I outlined the rationale for the research paradigm. An account for choosing a case study design was provided. The sampling of participants of the study was also outlined. Data collection methods, data analysis procedure, ethical consideration, and quality criteria were also addressed. The next chapter focuses on the analysis of the collected data.

CHAPTER FOUR: PRESENTATION AND DISCUSSION OF FINDINGS

4.1. INTRODUCTION

In this chapter, I presented qualitative data analysis observed during the ACE teaching cycle in the classroom. The qualitative data sought to explore learners' conceptual understanding of quadratic functions using genetic decomposition. To thoroughly explore this conceptual understanding of quadratic functions, I employed a cyclic process (see Figure 4.1) guided by genetic decomposition. The genetic decomposition informed each phase of the ACE cycle, as seen in Figure 4.1. During all these phases, the genetic decomposition functioned as a lens through which it hypothesised that learners engaged in a learning task would undergo a specific indicator route to understand that concept.

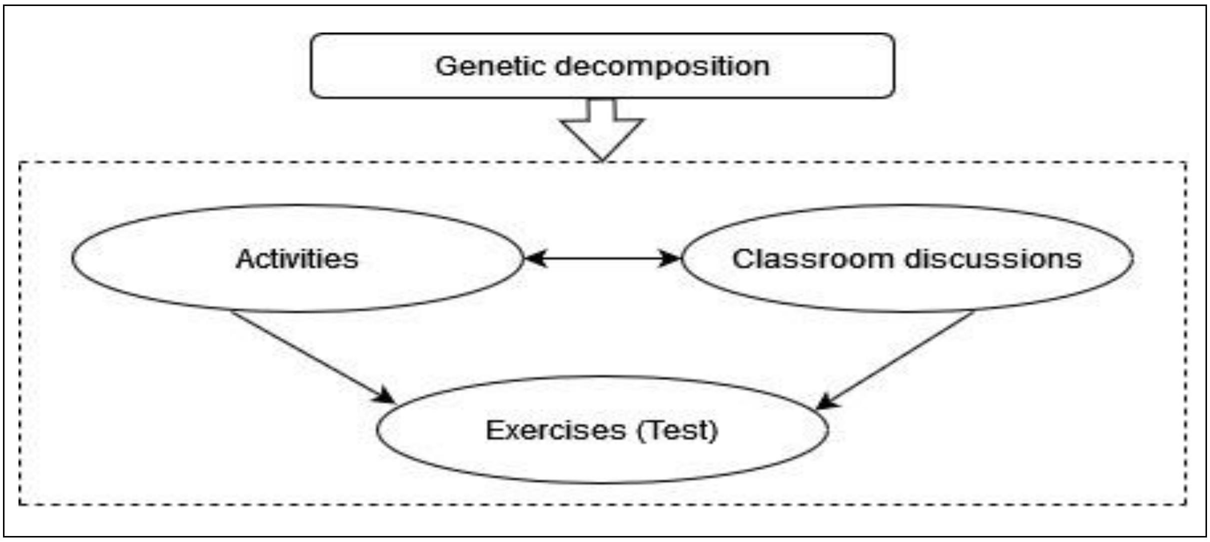


Figure 4.1: The cyclic process of data analysis

Chapter four is arranged in line with Figure 4.1 as dictated by the APOS theory (Arnon et al., 2014). Şefik et al. (2021) hold that a theory must reveal what can be done to improve learning. Thus, to achieve this, the theory must allow four things to occur in learners' construction of knowledge, i.e., prior knowledge, actual development, potential development, and scaffolding to attain understanding (Van Der Stuyf, 2002). Thus, for the APOS to meet these elements required for understanding suggests three processes to be considered, i.e., activities, classroom discussion, and exercises (Arnon et al., 2014). This process continues until understanding is achieved through the attainment of indicators in the genetic decomposition. The precedent genetic decomposition of the quadratic function concept in Figure 4.2 predicts how the concept was learned in the classroom, as seen in Figure 4.1. Trigueros and Possani (2013) noted that a genetic decomposition should prevail in how concepts are learned in classrooms. Therefore, the genetic decomposition was used for two things in this study (1) as a lens to view learners' conceptual understanding of the quadratic function, and (2) as a tool used to dictate the learning of quadratic functions.

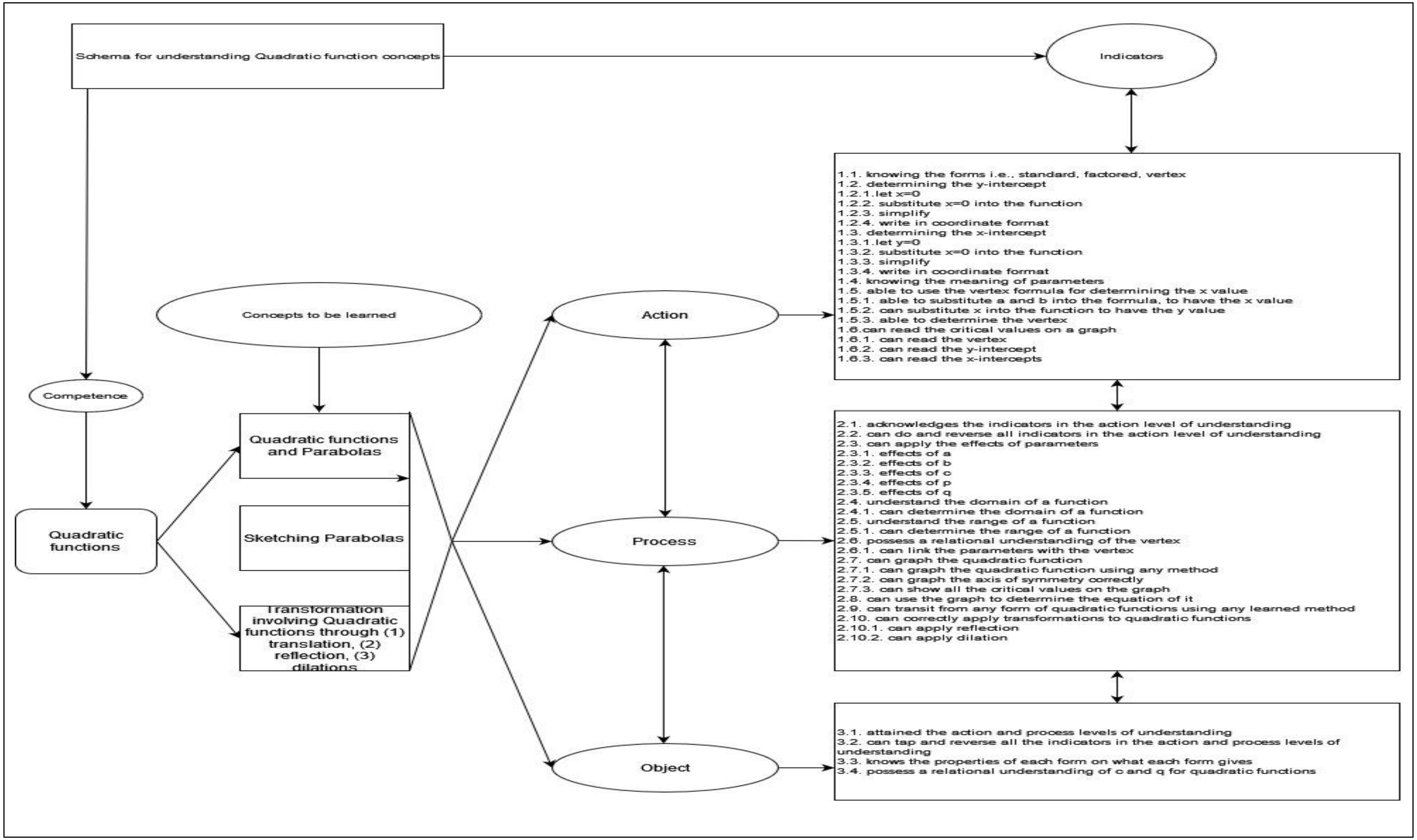


Figure 4.2: Genetic decomposition for quadratic functions

The use of genetic decomposition as a lens is anchored on the work of Arnon et al. (2014). The genetic decomposition was used as a lens to explicitly explain how the attainment of each tenet could be achieved (Arnon et al., 2014). It provides a synopsis of how learners' interaction with the learning task can develop mental structures in their minds. As a result, I have detailed the indicators for attaining the action, process, object and schema. The indicators are numbered; for example, action indicators are 1.1-1.6. The indicators in the process level are numbered from 2.1-2.10, and those in the object are numbered from 3.1-3.4. Therefore, the attainment of 1.1-1.6, 2.1-2.10 and 3.1-3.4 constitute a schema level of understanding.

To begin chapter four, I divided it into three sections to thoroughly explore learners' understanding of quadratic functions. Firstly, I simultaneously analysed and discussed the qualitative data by unpacking three tenets of the quadratic function through the activities, classroom discussions and exercises. Secondly, the analysis and discussions are synthesised, giving the principal findings of the study. Lastly, I presented the chapter summary.

4.2. DATA ANALYSIS AND DISCUSSION

In this section, I simultaneously analysed and discussed data from activities, classroom discussions and exercises. I started by exploring learners' prior knowledge of quadratic functions in the activities phase. I used a task (Task 0) to assess learners' prior knowledge of quadratic functions, which was compiled in line with the precedent genetic decomposition. As put by Şefik et al. (2021), the genetic decomposition depicts how the concept should be learned. Therefore, the precedent genetic decomposition depicted how quadratic functions should be learned for conceptual understanding. Moreover, the data collected from Task 0 paved the way for additional data collection in the activities phase. As such, the other data in the activities phase was collected using the learning unit, which had a learning task that covered the following aspects of quadratic functions, i.e., the forms of quadratics, the shape of the graph, the intercepts, sketching the function, translation, reflections, and dilations.

The data collected from the activities phase informed the classroom discussions phase. In this phase, I present the transcribed classroom discussions and the interview session. Lastly, I present a reflection on the impact of the ACE teaching cycle on learners' conceptual understanding of quadratic functions, looking through the exercises. In this phase, I wanted to reinforce the activities and classroom discussions. During the exercise phase, learners wrote an exercise (Test) which assessed the most challenging concepts in quadratic functions identified from the activities and classroom discussions. The analysed and discussed data from the activities, classroom discussions, and exercises were synthesised to profile learners' conceptual understanding based on the genetic decomposition, related literature on quadratic functions, and the literature on understanding. Therefore, the concept of conceptual understanding and conceptual knowledge will be interchangeable, as they mean the same (Star, 2005).

4.2.1. Activities phase

In this section, I presented the learners' conceptual understanding across the activities phase. To thoroughly explore their conceptual understanding of quadratic functions, learners were given Task 0 to write. This assessed their prior understanding of quadratic functions. The task assessed three concepts of a quadratic function: intercept, transformation, and graph orientation. The data from Task 0 informed the exploration of learners' conceptual understanding during the learning tasks. Thus, for me to be able to explore their conceptual understanding of these concepts thoroughly, I had to mark all tasks that the learners wrote. Thus, marking the tasks gives a synopsis of conceptual understanding as an answer can be correctly or incorrectly marked. Yet still, that answer requires analysis to determine its level of understanding according to the precedent genetic decomposition (Figure 4.2).

4.2.1.1. Learners' prior understanding of quadratic functions (Task 0)

I compiled an activity that assessed learners' prior knowledge of quadratic functions using previous question papers, CAPS document, the assessment guidelines pertaining to Grade 12 for mathematics and the precedent genetic decomposition. Prior knowledge is vital in a mathematics classroom, as it is a collective knowledge of the learner when entering a learning environment (Akinsola & Odeyemi, 2014). In this study, this collective knowledge is referred to as the conceptual knowledge of equations and functions done in earlier grades. Thus, prior knowledge influences learners' interaction with latest content (Akinsola & Odeyemi, 2014). Therefore, to achieve this expected result of prior knowledge being able to influence the present content, the knowledge should be assessed at the start of instruction. This is so because prior knowledge sets out the actual development of learners and gives the teacher an idea of what to do to improve their potential development (Yildiz & Celik, 2020).

Consequently, I assessed learners' prior knowledge of quadratic functions in this study. Task 0 (see Table 4.1) had three questions to explore learners' understanding, which served as an introductory activity to this discourse. The task opined to nurture the actual development of learners. The questions of Task 0 demanded that learners tap into their prior knowledge of quadratic functions. For example, the first one requires them to demonstrate their conceptual understanding of intercepts. Bansilal et al. (2014) noted that conceptual understanding intercepts is considered a smaller algorithm but nurtures the overall schema of quadratic functions when encapsulated into action, process, and object. Therefore, it is from conceptual understanding intercepts that learners can interact with graph orientation and transformations (Bansilal et al., 2014). In a related study by Emmanuel (2012), it was found that learners posed conceptual obstacles as they failed to note the distinction between $y = ax^2 + bx + 3$ and $y = ax^2 + bx + 7$. The researcher concluded that this seems to result from learners' failure to make connections between the symbolic to graphical. Consequently, this transition requires a conceptual understanding of intercepts.

Table 4.1: Learners prior understanding of quadratic functions

Task 0
1. Determine the y-intercept for the following equation: $y = -3(x-4)^2 + 100$
2. Clearly explain in words all the transformations that must be applied to $y = x^2$ to obtain the graph of the function below $y = -\frac{1}{4}(x+6)^2 + 12$
3. Sketch each quadratic function and fill in the blanks below:
3.1. $y = (x-2)^2 + 3$

Vertex; Axis of symmetry; x-intercepts; y-intercept

3.2. $y = -(x+5)^2 - 2$

Vertex; Axis of symmetry; Max/Min value; Range

3.3. $y = 0{,}5(x-4)^2 + 5$

Vertex; Axis of symmetry; Step pattern; Domain

Task 0 was written by all the learners and marked on a 24-point scale, allowing me to determine average marks on each question. Conversely, Pirie's (1988) view of understanding where learners are not categorized, I employed a different view as I categorises conceptual understanding based on four tenets. Therefore, I employed Dubinsky's (2002) assertion that understanding can be categorised into action, process, object, and schema. Consequently, I had to assign marks to obtain per question for each learner for me to thoroughly delve into their solution through the lens of genetic decomposition to categorise their conceptual understanding of quadratic functions. The average marks are shown in Table 4.2 for all the learners. The Table depicts the class average marks obtained per question. The marks give a snapshot of learners' conceptual understanding of each assessed concept.

Table 4.2: Learners' average marks on Task 0

Concept	Total marks	Class average
Intercept	3	1,7
Transformations	3	0,1
Graph orientations	3	0,4

The Table above reveals concepts that learners grappled with the most to grasp their conceptual understanding entirely. From the Table above, learners' prior knowledge of quadratic functions posed challenges in understanding the y-intercept, transformation, and graph orientation. Learners have a fragmented understanding of the y-intercept and graph orientation as most scored average marks of two and eight, respectively. Hattikudur et al. (2012) observed that conceptual obstacles differed in context when dealing with functions. In their study, they explore learners' conceptual understanding of the y-intercept. Hattikudur et al. found that learners posed fragmented knowledge of the y-intercepts. In contrast to conceptual understanding intercept and graph orientation, the concept of transformations posed severe difficulties to learners, as most scored an average of less than one.

The concept of transformation impedes learners' conceptual understanding of quadratic functions. As Zazkis et al. (2003) noted, transformations exhibit severe conceptual obstacle among learners. The researchers asserted that the conceptual obstacles result from transformation being abstract and requiring representations to understand. The ideas found while navigating literature are in cooperation with the average class marks. Consequently, the class average marks from Task 0 give a synopsis of learners' prior understanding of quadratic functions. Therefore, the class average marks are used to grade knowledge of these concepts as conceptual understanding according to the APOS theory can be evaluated. The evaluation of conceptual understanding is anchored by literature (Arnon et al., 2014; Childers & Vidakovic, 2014; Groves, 2012; Hiebert & Lefevre, 1986; Kilpatrick et al., 2002; Rittle-Johnson, 2017; Skemp, 1976).

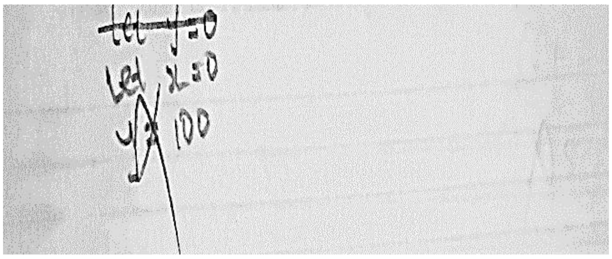
Therefore, delving deeper into the synopsis, I began exploring learners' conceptual understanding of the first question of Task 0, which assessed their conceptual understanding of the intercepts. The learners' conceptual understanding of intercepts is vital in quadratic functions as knowledge mitigates misconceived knowledge found by Parent (2015), who asserted that learners often confused the y-intercept of the graph and the turning point. In a related study, Ubah, and Bansilal (2018) studied the understanding of quadratic functions. In their study, the researchers conform to Parent’s findings by stipulating that the learners' failure to note the distinction between the parameters impedes the full attainment of the concept. Learners in the first question were given a function in the vertex form, i.e., $y = -3(x - 4)^2 + 100$, and were required to determine the y-intercept. This question of determining intercepts needed them to let the value of x be zero, i.e., $x = 0$. Then after having this notion, they would substitute the value of x into the function and have the value of y. However, learners would be required to conclude their solution and acknowledge that they are determining intercept and not solving for y. This will imply that they would give the coordinate of the y-intercept. This process nurtures the conceptual understanding of quadratic functions as learners note the difference between equations and functions (Aziz et al., 2018). Therefore, the question requires a learner to conduct procedures flexibly (Kilpatrick et al., 2002), as a he or she was not opined to use a specific method to determine the y-intercept. The expected solution to the question is shown in Table 4.3.

Table 4.3: Expected solution of conceptual understanding the y-intercept

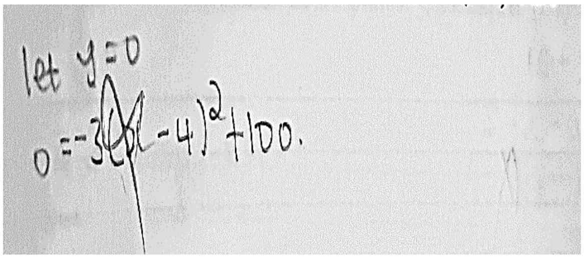
$y = -3(x-4)^2 + 100$	Indicator *1.2 (1.1, 1.6)*
$let\ x = 0$	Indicator *1.2.1*
Sub $x = 0$ into $y = -3(x-4)^2 + 100$	Indicator *1.2.2*
$y = -3(0-4)^2 + 100$	Indicator *1.2.3*
$y = -3(-4)^2 + 100$	
$y = 52$	
$\therefore$ y-intercept $(0; 52)$	Indicator *1.2.4*

The question demanded that learners understand that they should use their prior understanding of equations but acknowledge that they are determining intercepts and not solving for y. The learners' understanding of functions and equations is vital for developing conceptual understanding (Li, 2010). The question stressed the first tenet of genetic decomposition, i.e., quadratic functions and parabolas. The learners' attainment of this question would mean that they should display the following indicators, i.e., 1.1 knowing the forms, 1.2 determining the y-intercept, and 1.6 can read critical values from the function and graphs (see Figure 4.2). Therefore, the attainment of this tenet would nurture their conceptual understanding of intercepts. Hattikudur et al. (2012) noted that intercepts served as a ladder to fully understanding functions since the concept cuts across all functions in mathematics. However, learners grappled with the conceptual understanding of intercepts. Struggling learners included learners C5, F10, Q21 and U25. I have presented their understanding of the y-intercept in Figure 4.3.

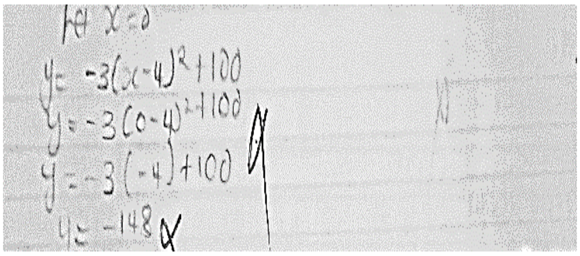
Learner C5



Learner F10



Learner Q21



Learner U25

Figure 4.3: Learners' incorrect conception of the y-intercept

Learners interacted with the question of intercept, and from there, four solutions learners were sampled. These are the ones who posed serious conceptual obstacles in this question of the intercept. Although most of them scored zero on the question, this does not mean that they do not have any sort of understanding of intercepts. As put by Nickerson (1985), the non-binary nature of understanding, i.e., if learners have interacted with quadratic functions means that they will have limited links within their conceptual understanding. Therefore, this non-binary nature demands the learners to understand that they might score zero, but their links for conceptual understanding vary. For example, learner U25 wrote that the y-intercept is 100 without taking into consideration the form of the function (Figure 4.3). The learner failed to efficiently portray indicator 1.1, i.e., understanding the forms of quadratic functions. He or she did not consider that the function was not in standard form. This notion made the learner write that $y = 100$ without interrogating what 100, subsequently this thought inhibited the conceptual understanding of parameters and the forms of quadratic functions. This type of misconceived idea is like learner C5's solution.

Delving deeper into these learners' solutions based on Sierpinska's (1994) definition of understanding, I can say that these learners' acts of understanding are fragmented since they failed to provide enough links to the quadratic forms, which inhibited the attainment of indicator 1.1. However, comparing the two learners, C5's level of conceptual understanding differs from U25's. In learner C5 solution, the learner failed to note that the question nurtured indicator 1.2 of understanding. The question required learners to tap into their prior knowledge of determining y-intercept. Instead, learner C5 applied indicator 1.3, i.e., determining the x-intercepts, which posed conceptual obstacles of comprehending the question. For example, learner C5 started by writing that "let $y = 0$" and thereafter the participant cancelled it. Learner C5 started all over again by writing "let x =0" but did not substitute it into the function. This was explained by Díaz et al. (2020), that learners have an instrumental understanding but often failed to use it when required. Similarly, learner C5 seems to have an instrumental understanding of determining the intercept but is underdeveloped.

Consequently, the conceptual obstacle posited by C5 seems to be inhibited by an underdeveloped conceptual understanding of quadratic forms and parameters. The conceptual obstacles posed was also noted by Parent (2015), as learners consistently used one form due to a lack of understanding of the other forms. However, learners C5 and U25 viewed the function as a standard which explains the reason for them equating the y-intercept to be $q = 100$. This thought is guided by the fact that learners thought "$q = c$." Thus, failure to understand the forms of quadratic functions inhibits the attainment of indicators 1.1, 1.2 and 1.6. According to precedent genetic decomposition, the two learners posed various limited action conceptions of the forms of quadratic functions. They neglected the importance of parameters, which inhibits the development of the whole action level of conceptual understanding, which impedes the process and object level of conceptual understanding. The fragmented knowledge of forms and parameters hinders the development of a coherent understanding network. As noted by Nielsen (2015), learners found it easy to work with the standard form '$f(x) = ax^2 + bx + c$' than the vertex form '$f(x) = a(x - p)^2 + q$, when engaged in quadratic functions questions. Nielsen's view is replicated in this study, as these learners thought that the quadratic functions were in a standard form. Thus, the learners might have gotten the correct answer if the functions were in standard form. This notion is guided by the view that these learners thought $c = q$, meaning that they possess some understanding of the parameter c, but it is muddled with parameter q.

In addition, from the first question of Task 0, I explored learner Q21's conceptual understanding of the y-intercept (see Figure 4.3). Exploring learner Q21's conceptual understanding of y-intercept, the learner's first step portrayed indicator 1.2.1, which was a good start, showing that the learner understood the question, unlike learner F10, who wrote $y = 0$ exhibiting 1.3 indicators and then stopped writing. Learner F10 demonstrated that they do not have a 1.2 indicator of understanding as she was confused and tapped into incorrect indicators of understanding. However, the two learners' understanding of their first steps indicated that learners Q21 and F10 have various levels of conceptual understanding regarding the intercept. Learner F10 failed to differentiate the method used to determine the y-intercept and x-intercepts. As such, this led to learner F10's understanding of y-intercept to be underdeveloped as the learner posed fragmented knowledge of intercept. Putri (2021) denoted this as instrumental understanding: learners usually use procedures without fully understanding them. Putri's type of understanding was clear from learner F10's solution, as she tapped into indicator 1.3 instead of 1.2.

Unlike the fragmented instrumental understanding of intercept posed by learner F10, learner Q21 showed some bits of understanding. Therefore, the excellent start of learner Q21 of the question implied that the learner held a complete instrumental understanding of the determining intercepts as he achieved indicators 1.2.1 and 1.2.2 and failed to attain the 1.2.3 indicator. While interacting with learner Q21's solution, I noted that the learner ignored exponent 2 as she progressed with her workings. However, Hiebert and Carpenter (1994) would have acknowledged the conceptual obstacles and said that this learner is posing some understanding but not to the extent. Thus, if the learner posed understanding to the extent, it would have meant that the participant managed to correctly substitute zero into the function and obtain the y-intercept as $(0; 88)$. However, delving deeper into learner Q21, the participant's operational understanding of the function after zero substitution is underdeveloped. As seen from the work of Cangelosi et al. (2013), learners often have an undeveloped understanding of additive and multiplicative inverse. This notion was replicated in the work of learner Q21, as the participants failed to distribute the operators correctly.

Consequently, the ignorance of such algebraic operational skills inhibits complete understanding of the determining intercepts, i.e., attaining a complete indicator of 1.2. As pointed out by Kotsopoulos (2007), that learners would perform instrumental procedures without having a relational part while determining intercepts. The view of Kotsopoulos is immature when analysed by the ideas of Hiebert and Carpenter (1994). In their ideas of understanding, the two researchers are not disputing conceptual obstacles but acknowledging that they form a profile for the learner. Therefore, the conceptual obstacle posed by these learners would place learner Q21 at a higher order level of understanding intercepts even if the solution is incorrect, according to Hiebert and Carpenter.

Moreover, learner Q21 posed challenges in multiplying the negative signs. This results in hampering the development of the process conception due to failure to simplify correctly after correctly substituting into the function the value of $x.$ Learner Q21 did not check and verify their solution whether the value that she got was correct or not. Her solution clearly showed that she struggled with working with additive and multiplicative inverses. The conceptual obstacle in correctly applying the operators was also observed in Booth et al. (2014). The researchers found that participants could not correctly simplify expressions in quadratic, especially when working with negative terms. A related study by Cangelosi et al. (2013) found that learners posed computational incompetency with operations. For example, the participants viewed -2^2 as $(-2)^2$. They thought that -2^2 and $(-2)^2$ are the same. Thus, these findings were replicated in this study as learner Q21 thought that $(-4)^2$ is equal to -4. What is fascinating about the learner's solution is the definitive answer which is -148 from her method. This solution emanates from pitfalls in arithmetic, which forms the basis for understanding quadratic functions. The pitfalls identified from this question are unclear understanding of parameters, overreliance on one form of the function, lack of arithmetic skills, and difficulty collaborating with arithmetic operators. In a related study by Ruli et al. (2018), it was noted that the pitfalls observed in the current study were a result of epistemological conceptual obstacles of functions that are held as prior knowledge, and it needs treatment by using understanding them as they add to the learning. Therefore, these pitfalls positioned the four learners' levels of understanding in various limited action levels of understanding of quadratic functions. Consequently, learners F10, C5 and U25 are positioned at a lower limited level of understanding while learner Q21 is at a higher level. As a result, these learners fragmented knowledge gaps. Hence their understanding cannot be fully developed into the process, object, and schema conception.

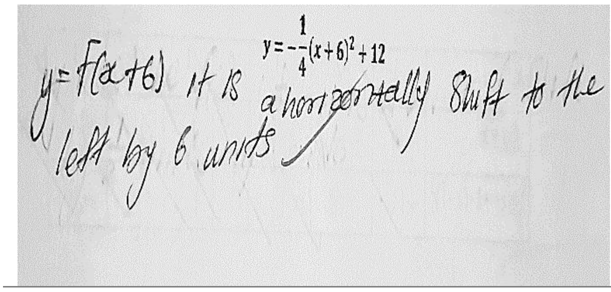
The second question of Task 0 was based on the concept of transformations. Learners were required to explain in words all the transformations that must be applied to $y = x^2$ to obtain $y = -\frac{1}{4}(x+6)^2 + 12$. The expected solution to this question is shown in Table 4.4.

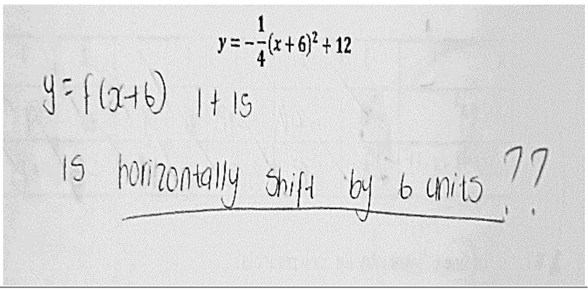
Table 4.4: Expected solution of conceptual understanding the effects of transformation

From $y = x^2$ to obtain $y = -\frac{1}{4}(x+6)^2 + 12$	Indicators *2.3* (*1.1, 1.4, 2.1, 2.2, 2.9, 2.10*)
The graph is vertically compressed by $a = \frac{1}{4}$ units	Indicator *2.3.1*
The graph is reflected over the x-axis	Indicator *2.10.1*
The graph is shifted to the left by 6 units to the left	Indicator *2.3.5*
The graph is shifted 12 units upwards	Indicator *2.3.4*

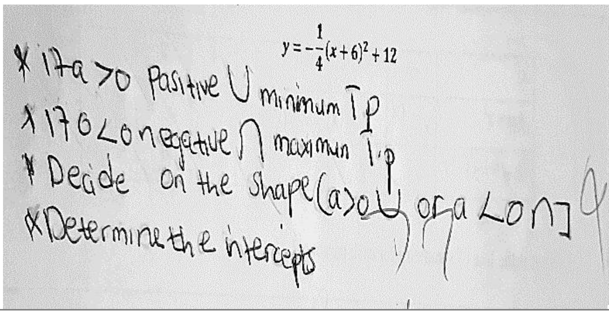
The question integrated the tenet of action and process levels of understanding to achieve object conception. The question demanded that learners reverse the action level indicators of understanding and link them to the process to attain object levels of comprehension. Baker et al. (2000), cited by Zazkis et al. (2003), noted that vertical transformations nurture action conception. In contrast, horizontal transformations require a network of action and process for their attainment. The findings by Zazkis et al. conform to Eisenberg and Dreyfus (1994) that an object understanding is a prerequisite for attaining the transformation concept, thus making transformations of quadratic functions challenging to grasp as it is an abstract concept. This question was challenging to most learners since most did not score good marks while others did not write anything. Amongst the participants who tried to answer the question, I sampled learners N18, P20, X28 and Z30's workings on this question to explore their understandings of transformations (see Figure 4.4) thoroughly.

Learner N18	Learner P20





Learner X28	Learner Z30



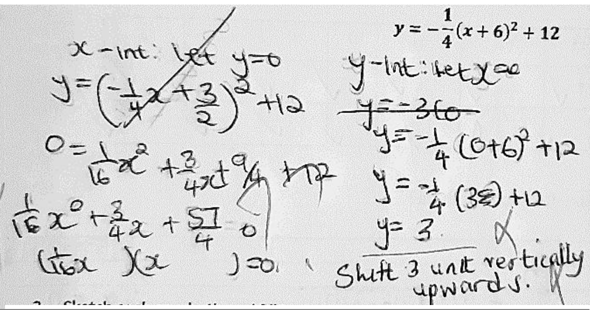


Figure 4.4: Learners' conceptual understanding of transformations

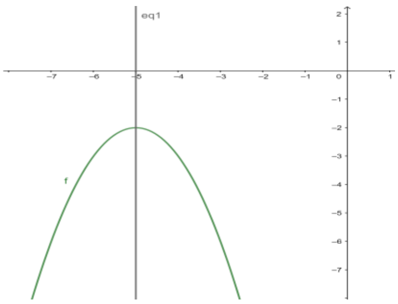
The vignettes presented four learners' solutions on the aspect of transformation involving $y = x^2$. However, two learners have a muddled understanding of the transformation. These are learners N18 and P20. However, both learners did not fully explain the transformation that is required for $y = x^2$ to be $y = \frac{1}{4}(x+6)^2 + 12$. In the vignettes, learner N18 was able to correctly notice that there is a horizontal shift of $y = x^2$ by 6 units to the left, showing the attainment of indicator 2.3.5, while learner P20 did not specify the position of the shift. The absence of relational understanding of the learner inhibited higher levels of cognition and posed conceptual obstacles to learners, as said by Skemp (1976). This type of conceptual obstacle concurs with Adu-Gyamfi et al. (2019), whose study found that the participants could not fully translate quadratic functions, revealing conceptual obstacles of the transformation concept. The notion is replicated in this study as learner P20 struggled to state the position of the shift, demonstrating fragmented knowledge with horizontal shifts, which inhibits him from fully attaining indicator 2.3.5. As seen by Castro et al. (2022), learners posed difficulties with translations.

In the same vignettes, I explored learner Z30's understanding of transformations. The learner started by determining the x and y intercepts. This demonstrated an inability to comprehend the question, as the learner failed to explain the transformations and opted to determine the intercepts. This notion of determining what you understand would be explained by Skemp (1976) as exhibiting instrumental understanding. The inability to understand questions in mathematics inhibits the development of action, process, object, and schema understanding (Arnon et al., 2014).

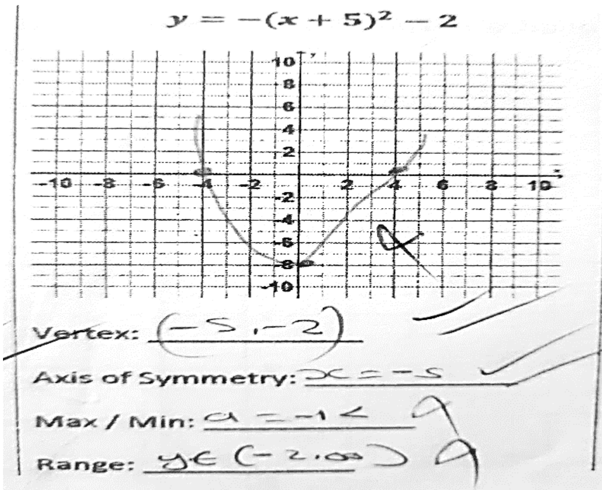
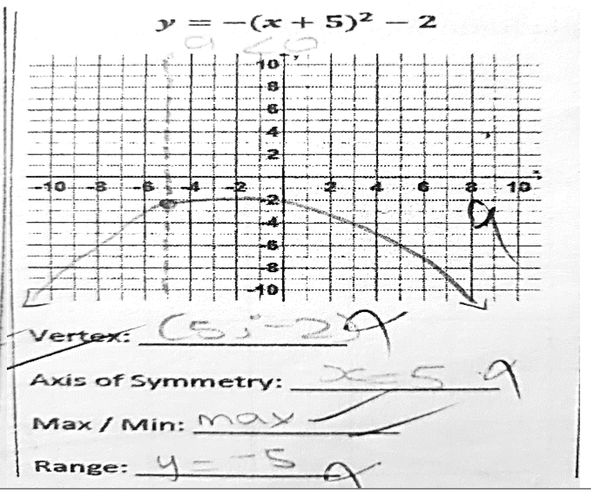
Moreover, given what the learner opted for, surprisingly, the learner even struggled to determine the x-intercepts because of failure to factorise the function, and could not determine the y-intercept due to pitfalls in arithmetic skills. The inability to understand the question led the learner to look at the shifts from her solution, i.e., $y = 3$, instead of reading from the function itself. Therefore, judging the transformation from that stance hindered the development and attainment of indicators 2.3, 1.1, 1.4, 2.1, 2.2, 2.9 and 2.10. Consequently, learner Z30 posed application, comprehension, and factual knowledge conceptual obstacles of transformation.

The last question on Task 0 tested learners' understanding of sketching the quadratic functions, writing the vertex, an axis of symmetry, intercepts, max/min, and range. This question had three sub-questions. However, in this section, I presented the second question due to learners' performance on the question. In the question, learners were given the function, i.e., $y = -(x + 5)^2 - 2$, and were required to sketch, write the vertex, write the equation of the line of symmetry, give the min/max values, and give the range of the function. The expected solution to this question is captured in Table 4.5 below.

Table 4.5: Expected solution on understanding graph orientations

$y = -(x+5)^2 - 2$. Determine the vertex, and axis of symmetry, and state whether it has a max/min, range. 	Indicator *1.1, 1.2, 1.3, 1.5, 1.6, 2.1, 2.2, 2.5, 2.6, 2.7* Indicator *2.7.1, 2.7.2, 2.7.3*
Vertex $(-5; -2)$	Indicator *1.5, 1.6, 2.6*
Axis of symmetry $x = -5$	Indicator *1.5, 1.6, 2.6*
Maximum T.P	Indicator *2.6*
Range $y \leq -2$	Indicator *2.5.1*

As such, most learners in this question posed conceptual obstacles in comprehending the quadratic functions orientation. The conceptual obstacles were caused by the fact that $a \neq 1$ inhibited their understanding of sketching the graph, determining the axis of symmetry, and determining the range and domain. As noted by Nielsen (2015), most learners posed challenges with parameter a especially if the value is not one. This notion held by Nielsen is confirmed in this study as learners failed to explain the effects a. Amongst the participants who struggled with the question, I sampled learners BB4, C5, N18, P20 and X28 (Figure 4.5).

Learner BB4	Learner C5
	
Learner P20	**Learner X28**

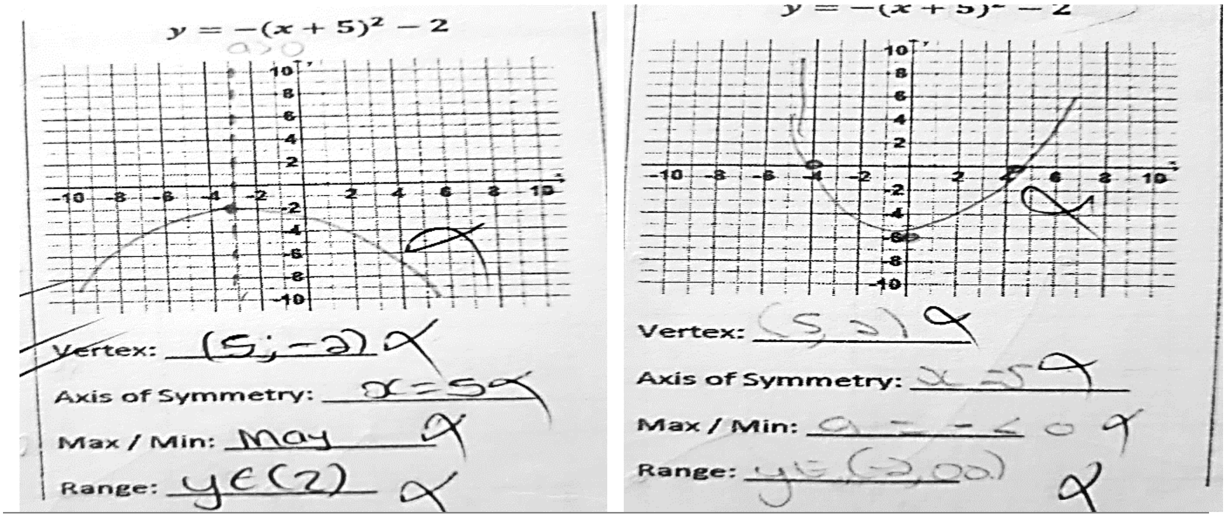


Figure 4.5: Learner conceptual obstacle on sketching the quadratic function

The learners interacted with the question by showing various understandings regarding comprehending graph orientation of quadratic functions. For example, exploring learner X28 understandings of graph orientation while interacting with the function $y = -(x + 5)^2 - 2$ the participant posed conceptual obstacles. The learner sketched the graph facing upwards instead of downwards, disregarding the parameter 'a' (Figure 4.5). The sketch depicts negligence of the importance of parameters, especially a inhibits the attainment of indicator *2.3*. Ellis and Grinstead (2008) studied the standard form of a quadratic function and acknowledged that the a parameter is interpreted as influencing the shape of the graph. However, learner X28 ignored the effects of parameter a when graphing. Moreover, Ellis and Grinstead add that participants would assume that changing the value of a in the function does not alter the vertex's location. This notion is seen in learner X28's solution as he drew the graph facing upwards instead of downwards. This type of conceptual obstacle was also observed in learner BB4's solution. As such, these learners ignored the meaning of the parameter 'a' while sketching the graph; therefore, they had a limited action conception involving the parameter 'a.' Hence, the learners' understanding based on the precedent genetic decomposition failed to tap into a whole action level of understanding, which inhibited the development of process, object, and schema level of understanding of parameters. Consequently, this posed difficulties in grasping 1.1, 1.2, 1.3, 1.5, 1.6, 2.1, 2.2, 2.5, 2.6 and 2.7

indicators of understanding, hindering the development of coherently built mental structures.

On the same question, some learners posed conceptual obstacles with the axis of symmetry, while they showed some limited understanding of the shape of the graph. For example, learners' C5 and P20 understanding of the form seemed cluttered and fragmented. The vertex form gives the vertex of the graph, and the x-coordinate of the vertex gives the equation of the axis of symmetry. However, the two learners have pitfalls in possessing complete indicators for the vertex of a function. If learners C5 and P20 had these indicators, they would have written down the vertex as $(-5;-2)$ and the line of symmetry as $x=-5$ from the function itself. However, learner P20 gave an incorrect vertex as $(5;-2)$ instead $(-5;-2)$, surprisingly the learner struggled even to plot the vertex. The failure to correctly plot the critical values of a function hinders the development of 1.5, 1.6 and 2.6 indicators of understanding. As noted by Bossé et al. (2011), learners posed a conceptual obstacle on reading coordinates on a graph. For instance, the coordinates $(5;-3)$ were read as $(-3;5)$. However, learner C5 could represent the line of symmetry on the graph but failed to write the equation of it (Figure 4.5). From learner C5's understanding, he does not fully comprehend the equation of the symmetry line since he could correctly represent it on the negative side of the Cartesian plane and failed to represent it algebraically. This conceptual obstacle supports the findings by Mpofu and Pournara (2018). The researchers noted that learners posed conceptual obstacles to transit from one form to another. The failure to perform such understandings inhibits the development of a complete action level of understanding, which inhibits the learners' understanding in attaining these indicators, i.e., 1.1, 1.2, 1.3, 1.5, 1.6, 2.1, 2.2, 2.5, 2.6, 2.7.

Additionally, the question required learners to give the range of the function, i.e., $y=-(x+5)^2-2$. To explore the understanding of range, I sampled learner BB4's workings. Learner BB4 failed to write the range of the function, i.e., $y=-(x+5)^2-2$ in a correct format using the brackets. He could note the range but failed to present it in writing. As such, the learner failed to attain a complete action level of understanding as it was inhibited by failure to grasp indicator 2.5.1. Using the brackets in mathematics is tricky for learners since they misuse them without understanding them. The shape of brackets, that is, either round bracket, (, or, square bracket,] which means two different things. In mathematics, the use of *"("* means excluded while *"]"* means included. Learner BB4 had a challenge with the issue of brackets (Figure 4.5). He failed to explain or give the range with the brackets correctly. Instead, he assumed that the range of the function is not inclusive of both values of -2 and ∞. Aziz et al. (2019) noted such conceptual obstacles. The researchers pointed out that the participants had difficulty noting variables given the range. They found that several participants used x to denote range instead of y. Therefore, such conceptual obstacle impedes the full grasp of the concept of range. Thus, this inhibits its representation using correct notation.

4.2.1.2. The forms of a quadratic function

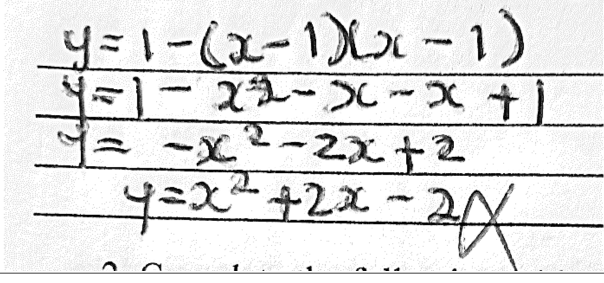
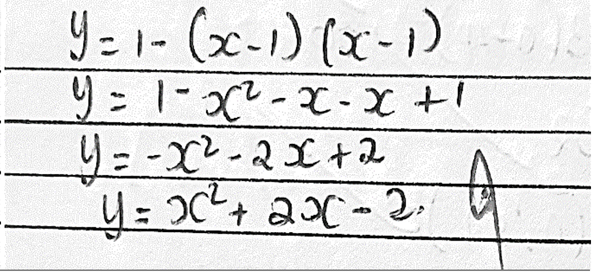
The quadratic function is one of the forms $f(x) = ax^2 + bx + c$ where the variables a, b and c are integers and the value of $a \neq 0$ (Nielsen, 2015). More formally, a quadratic function is a function with a degree of two, and its graph is called a parabola, which is recognised for its U-shaped (Pender et al., 2011). The quadratic function can be expressed in three different forms, i.e., standard form $f(x) = ax^2 + bx + c$, factored form $f(x) = a(x - x_1)(x - x_2)$, and vertex form $f(x) = a(x - p)^2 + q$ (Ousby et al., 2008). Mutambara et al. (2019) noted that these forms demonstrate some graphical information related to the location of particular points on the graph. They asserted that the standard form reveals the location of the y-intercept $(0; c)$, the vertex form indicates the turning point of the graph $(p; q)$, and the factored form gives the x-intercept of the function $(x_1; 0)$ and $(x_2; 0)$.

However, navigating through the literature reveals that most learners are over-reliant on the standard form, which inhibits understanding of all quadratic functions forms (Díaz et al., 2020; Didiş et al., 2011; López et al., 2016; Metcalf, 2007; Mutambara et al., 2019; Ubah & Bansilal, 2018a). Consequently, such overreliance limits their understanding of quadratic functions to comprehension of y-intercept only. The limitation posed by these learners leads to several conceptual obstacles, such as denoting the value of a to mean the gradient of the function and thinking that quadratic functions cannot be written in a vertex or factored form (Parent, 2015). For example, in this study, the first task during the *activities phase* required learners to interact with the vertex form. Learners were given $y = 1 - (x - 1)^2$ and asked to check if this is a parabola and re-writing it in the standard form. The expected solution to this question is shown in Table 4.6 below.

Table 4.6: Expected solution to understanding the forms of quadratic functions

$y = 1 - (x - 1)^2$	Indicators *1.1, 2.1, 2.2, 2.9*
Quadratic or not?	
The function is quadratic in the vertex form	Indicator *1.1*
Re-writing it in a standard from:	
$y = 1 - (x - 1)^2$	Indicators *1.1*, *2.9*
$y = 1 - (x - 1)(x - 1)$	
$y = 1 - (x^2 - 2x + 1)$	
$y = 1 - x^2 + 2x - 1$	
$y = -x^2 + 2x$	

Most learners could conceive that this is a parabola due to the presence of degree two on the brackets. The learners' ability to conceive that this was a quadratic function exhibits the presence of indicator 1.1 understanding. However, re-writing it in the standard form was challenging for most learners. López et al. (2016) noted that learners held instrumental understanding but often failed to exhibit it if needed. This assumption held by López et al. was replicated in this study as learners knew the properties of quadratic functions but failed to apply the knowledge, i.e., for re-writing the function into the required form. Amongst the participants that struggled to re-write the function in the standard form are learners A1, AA2, B3 and R22 (Figure 4.6).

Learner A1	Learner AA2
	

Learner B3	Learner R22

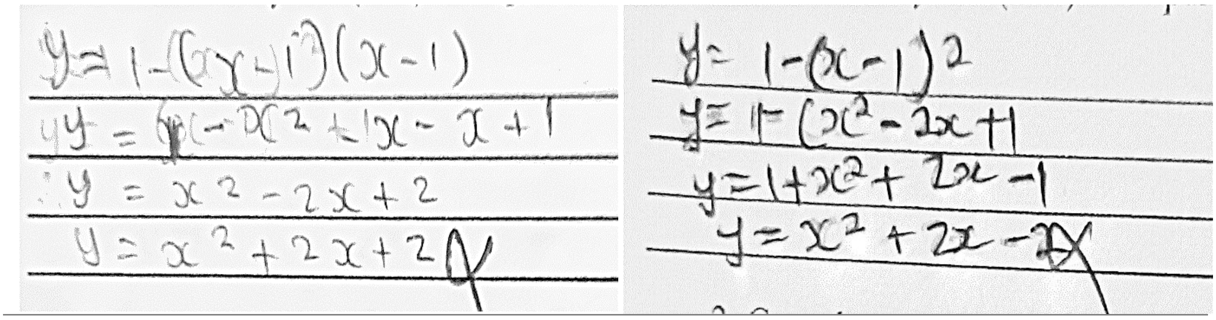


Figure 4.6: Learners conceptual obstacles of connection between vertex and standard form

The question required learners to transit the function into a standard form from the vertex form. Learners posed conceptual obstacles with the concept of transition from one form into another. Thus, exploring learners' understanding of transitioning from one form into another, I started by looking into learner AA2's solution. For example, learner AA2 acknowledged that degree two on the brackets implies expanding the brackets twice when working with the question. Such a method of expanding the function asserts that the learner posits a whole developed skill of algorithms. As noted by Fitzmaurice and Hayes (2020), learners' poising type of skills implies that their understanding is in the developed stage of understanding. However, going deeper with the learner AA2's solution, it can be seen that the learner posed a conceptual obstacle as she could not correctly apply the product rule as she purposeful neglected the negative sign between the constant 1 and the brackets. The issue of product rule was also observed in learners A1, B3 and R22. Leong et al. (2010) also saw the conceptual obstacles relating to factorisation. In their study, the researchers asserted that learners view quadratic functions as abstract.

The abstract nature inhibits the entire understanding of transitioning from one form to another due to a lack of sufficient algebraic skills to tackle the transition. Consequently, these learners posed a challenge with simplifying the function involving negative values. The failure to link quadratic functions and algebra inhibits learners' understanding of functions. Consequently, the absence of such understanding inhibits the attainment of indicator 2.9. As noted by Díaz et al. (2020), the researchers found that the participants in their study could not integrate the understanding of quadratic functions and algebra.

Moreover, in a related study, Didiş et al. (2011) found that the failure to integrate the two concepts, i.e., quadratic functions and algebra, was caused by the absence of relational understanding underlying the connections between the forms. The absence of relational understanding in understanding the forms of quadratic functions inhibits the full attainment of indicator 2.9. Furthermore, the learners seemed to pose pitfalls in arithmetic and algebraic operational skills to simplify functions. Therefore, Metcalf (2007) found that the absence of these operational skills inhibits the development of complete understanding. As such, in this study, the lack would impede the action level of understanding, which hinders the grasp of the process, object and schema understanding of the transition from one form to another.

Furthermore, learners seemed not confident with a negative parameter *'a'* as they tried to make it positive at the end. This notion inhibited the grasp of indicator 2.3. Learners are used to solving quadratic equations where $a = 1$ and are not accustomed to $a \neq 1$. They seem to have a limited instrumental understanding of the forms of quadratics. This instrumental understanding seemed to be caused by the overreliance on one form of the quadratic functions and treating questions where the parameter is always $a = 1$. As also observed in the work of Kotsopoulos (2007). Kotsopoulos found that learners were accustomed to working with one form of a quadratic function. This idea held by Kotsopoulos seemed to be reoccurring in this study as the participants presumed that the function was given in a standard form.

Looking at the learners' solutions again, I found that learners posed conceptual obstacles with the meaning of an equal sign. For example, learner AA2 was muddled with the meaning of an equal sign, as the learner wrote $y = -x^2 - 2x + 2$ and later wrote $y = x^2 + 2x - 2$, in which the two cited functions from her solution are not equal (see Figure 4.7). This held notion of misused equal sign inhibits relational understanding of quadratic functions. Blanton et al. (2018) noted that learners posed an instrumental understanding of an equal sign as the participants persistently continued to pose computation difficulties with the meaning of the equal sign. I also observed the misuse of the meaning of the equal sign in the other sampled learner. For example, learner R22 wrote $= 1 + x^2 + 2x - 1$ as $= x^2 + 2x - 2$. The learner's negligence of not seeing that '$1 - 1 = 0$' concerns the learner's arithmetic action conception. Thus, the level of understanding with the transition from one form to another is fragmented with arithmetic and algebraic operational skills that hinder the development of the process and object conception.

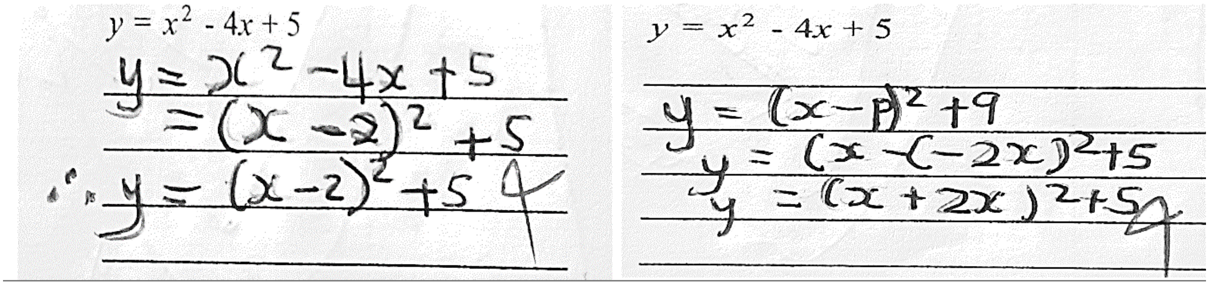
I explored another question to thoroughly have a clear picture of learners' understanding of quadratic function forms. The question required learners to transit from the standard form into the vertex by completing the square. The learners were given the function, i.e., $y = x^2 - 4x + 5$ and required to write it in the vertex form by completing the square. The expected solution is captured in Table 4.7 using the method of completing the square.

Table 4.7: Expected solution on completing the square

$y = x^2 - 4x + 5$ $y - 5 = x^2 - 4x$ $y - 5 + 4 = x^2 - 4x + 4$ $y - 1 = x^2 - 4x + 4$ $y - 1 = (x - 2)^2$ $y = (x - 2)^2 + 1$	Indicator *1.1, 2.1, 2.2, 2.9*

Most of the learners seem to be challenged by this concept of transitioning from one form to another. This was caused by fragmented understanding in completing the square. To understand the transition from one form to another using completing the square method, I explored learners D7, DD8, G11 and U25's understanding of transit from standard to vertex form in Figure 4.7.

Learner D7	Learner DD8



Learner G11 **Learner U25**

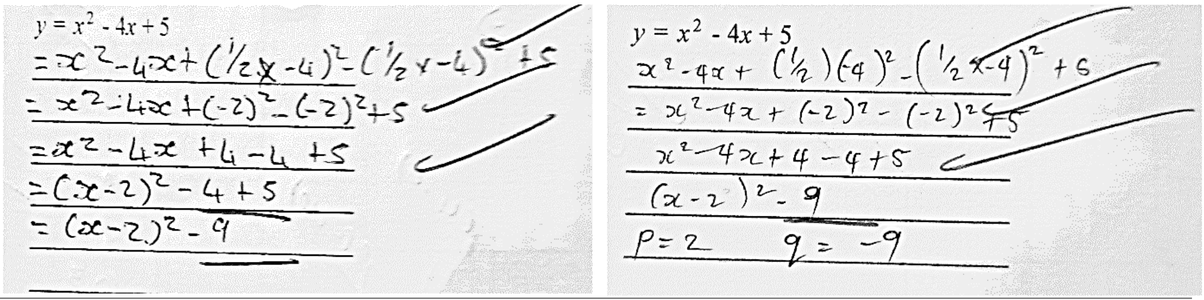


Figure 4.7: Failure to complete the square, leading to conceptual obstacles

The question required learners to tap into their prior knowledge of algebra to transition from the standard form into the vertex form by completing the square method. However, learners seem to be fragmented with knowledge of completing the square. For example, learner D7 encountered challenges transitioning from the standard form to the vertex form due to a conceptual obstacle in completing the square. These conceptual obstacles in completing the square were also observed from learner DD8. The conceptual obstacles emerged from the denoting c in the standard form to be q in the vertex form. Therefore, such conceptual obstacles inhibited the learners from grasping the 1.1 indicators of understanding completely. Hence, these conceptual obstacles were also replicated in the solutions of learners D7 and DD8. Both learners wrote $x^2 - 4x$ as $(x-2)^2$ demonstrating undeveloped understanding of algebraic concepts since they could not note that $x^2 - 4x \neq (x-2)^2$. The meaning of an equal sign seemed persistent in this study, as some participants continued to misuse it. This meant that the learners possessed an instrumental understanding of the meaning of the equal sign. Muchoko et al. (2019) noted that learners often lacked the relational meaning of an equal. The knowledge of algebra is necessary to nurture an understanding of quadratic functions. However, Muchoko et al. noted that learners often confuse the link between the two concepts. As such, the confusing link between the two notions inhibits the full attainment of the action level of understanding of quadratic functions. Hence, the analysis revealed that learners posed a muddled understanding of completing the square and pitfalls from prior knowledge of algebra. Zaslavsky (1997) posits that the different form of quadratics reveals specific information about the location of the y-intercept $(0; c)$, the turning point of the graph $(p; q)$, and the roots of the function $(x_1; 0)$ and $(x_2; 0)$. Consequently, learners D7 and DD8 ignored the notion held by Zaslavsky. Instead, *they equated* c and q. Such fragmented understanding inhibits the attainment of complete understanding of these indicators, i.e., 1.1, 2.1, 2.2, 2.9, which are necessary for the development of understanding in transition from one form to another. Thus, both learners struggled to develop their understanding of the transition from one form to another. While exploring learners' G11 and U25 understanding, I noticed that the learners might possess some understanding of completing the square. However, they seem to have an unstable understanding of arithmetic and algebraic operational skills, which inhibit the full attainment of the method, i.e., completing the square. Such fragmented understanding posed challenges to their understanding of transitions, hindering nurturing of action, process, object, and schema understanding.

4.2.1.3. The axis of symmetry

The axis of symmetry is drawn vertically on the parabola through the x-coordinate of the vertex or turning point of the graph, which divides the graph into two halves. Moreover, it can be viewed as a function or as a number derived from a formula; given the standard form, it can be derived as $x = -\frac{b}{2a}$. The literature noted that learners assert that an axis of symmetry as numbers instead of functions demeans the relational part of the concept (Díaz et al., 2020; Didiş et al., 2011). Consequently, demeaning the concept to be a number instead of a function raises conceptual obstacles with the concept. Therefore, presented with this, learners would struggle with understanding the axis of symmetry. In this study, participants interacted with the concept of the axis of symmetry. Some could determine the equation and seemed to show some understanding traits; while others, such as learners E8, F10, G11, O19 and Z30, posed conceptual obstacles with the concept. Learners were given two functions, i.e., $f(x) = x^2$ and $g(x) = 1 - (x - 1)^2$ to determine the axis of symmetry, respectively. The expected solution to the question is captured in Table 4.8 below.

Table 4.8: Expected solution to understanding the concept of line of symmetry

Determine the domain and range of $f(x) = x^2$ and $g(x) = 1 - (x - 1)^2$.	Indicator *2.7*
$f(x) = x^2$ $x = 0$	Indicator *2.7.2*
$g(x) = 1 - (x - 1)^2$ $x = 1$	Indicator *2.7.2*

To track learners' understanding of the axis of symmetry in this study, I explored the response of learners O19 and Z30. The solutions of the two learners are captured in Figure 4.8.

Learner O19	**Learner Z30**

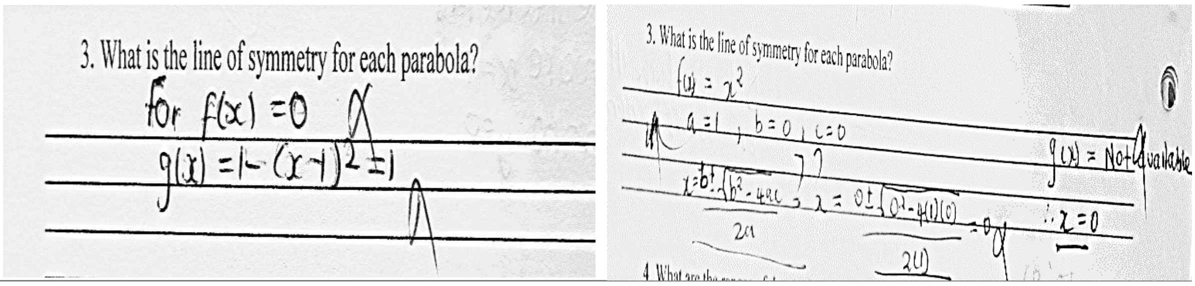


Figure 4.8: Learners' lack of meaning of the axis of symmetry

Learners were required to demonstrate their understanding of symmetry from $f(x)$ and $g(x)$. Although $f(x)$ seems not to demand more when one needs to determine the symmetry equation, learners posed conceptual obstacles. For example, from the vignettes, learner O19 wrote the equation of the axis of symmetry as $f(x) = 0$, while $f(x) \neq x$. Delving deeper into learner Z30's understanding, she started to write the values of a, b and c. Such conceptual obstacles inhibited the full grasp of 2.7 indicators of understanding. This hindered the grasp of 2.7 indicators of understanding as the learner failed to note that the value of c is not essential while determining the axis of symmetry. As a result, learners O19 and Z30 seem to be fragmented in their understanding of the meaning of $f(x)$. Thus, this fragmentation is a result of functions being dual in nature (Kenney, 2005). Kenney noted that functional notations are difficult for learners due to their nature as an operation and an object simultaneously. Moreover, learners thought that $f(x) = 3x + 1$ is a rule which inhibited its full grasp of being a process and object. Consequently, this fragmented knowledge of the notation used in functions resulted in learner O19 equating $f(x)$ with x. However, in an actual sense, the two aspects are not equal as $f(x) = y$ and $y \neq x$ in functions. Similarly, Muchoko et al. (2019) noted that learners posed immature understanding regarding an equal sign. This was seen in learner O19's solution as the learner equated y and x.

Consequently, the notion of equating y and x inhibited the complete development of the 2.7.2 indicator of understanding as the learner failed to note that the line of symmetry is given by x and not y. The learner writing that the equation of symmetry is $f(x) = 0$ reveals that the learner's understanding of intercepts is fragmented. As noted in a study by Ndlovu et al. (2017), one of the participants given $f(x) = a(x \pm p)^2 + q$ interpreted the vertex form and gave $(x \pm p)$ as the vertex instead of $x = \pm p$. Such a conceptual obstacle demonstrated an underdeveloped understanding of axis symmetry. Thus, such a knowledge gap impedes full attainment of the concept; as a result, this is the reason why learner O19's understanding was underdeveloped.

The fragmented knowledge of the axis of symmetry cuts across learner Z30's levels of understanding. Hence, learner 019 equated the axis of symmetry with *'y,'* while it is given by the *'x'* value of the vertex; therefore, learner O19, by writing that $f(x) = 0$ meant that $y = 0$*,* which is not the equation of the line of symmetry but rather the y-intercept. The learner wrote the axis of symmetry for the second function again as $g(x) = 1$. As such, this means that the learner does not have a complete understanding of the meaning of $f(x)/g(x)$ and intercepts. Learners often misunderstand using notations in quadratic functions, inhibiting the understanding of functions. Therefore, learner O19's understanding of the axis of symmetry seems fragmented. The fragmentations result from pitfalls in understanding intercepts and the meaning of notation in functions.

Furthermore, going deeper with learner Z30's solutions of determining the axis of symmetry, the learner started by writing the values of a, b and c as stated above.

Looking at her solution, the learner wrote the quadratic formula instead of $x = -\frac{b}{2a}$. The learner's first step cannot be disputed as the vertex formula requires using those values. Therefore, according to Fitzmaurice and Hayes (2020), skill algorithms dimensions of understanding would mean that the learner exhibited the first dimension of understanding. Therefore, from the dimensions of Fitzmaurice and Hayes, it would mean that learners posit some understanding for determining the axis of symmetry. However, Díaz et al. (2020) would classify the understanding as instrumental instead of relational understanding. Moreover, this type of understanding posed by learner Z30 is not yet fully developed and inhibits the understanding of the axis of symmetry. Therefore, learner Z30's understanding of the axis of symmetry is fragmented due to the absence of relational understanding of when to use the quadratic formula. The learner used the quadratic formula, i.e., $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ instead of the vertex equation of determining the x value that is $x = \frac{-b}{2a}$, which demonstrated undeveloped understanding of equations and functions. Subsequently, this means that the learner is fluent with the conception of vertex and the axis of symmetry (Groves, 2012). Such a skills algorithm, as put by Díaz et al., would mean that the learner failed to tap into correct knowledge schemas denoting the participant to have posed undeveloped understanding of the axis of symmetry and underdeveloped knowledge of equations.

Consequently, analysing the learners' understanding from Díaz et al.'s (2020) perspective would mean that the learner is in possession of instrumental understanding. Thus, as it is evident that the participant is fluent with procedures, if learner Z30 had been given the formula to determine the line of symmetry, she could have managed to determine the line equation without understanding the concept. This indicates that the learner could use procedures without comprehending what they are determining. Looking at the other function, i.e., $g(x) = 1 - (x - 1)^2$, learner Z30 assumed that this was not a quadratic function. Their justification may be based on the fact that the function is not written in the form they are familiar with, as seen by Parent (2015) that learners are accustomed to one form of quadratic functions. Therefore, learner Z30's understanding of forms and axis of symmetry is muddled, inhibiting the grasp of 1.1 and 2.7 indicators of understanding.

Moreover, to thoroughly examine the axis of symmetry concept, I explored another question that required an understanding of the concept. The question involved the concept of the vertex. The concept requires learners to tap into the knowledge of the axis of symmetry before they attend to the concept. The learners were given the function in a standard form, i.e., $f(x) = 2x^2 + 2x - 12$. They were required to transform the function into $h(x)$ first before determining the vertex of the new function. The expected solution on their understanding vertex is shown in Table 4.9.

Table 4.9: Expected solution for learners understanding the vertex

The graph of $h(x) = f(x + p) + q$ has a maximum value of 15 at $x = 2$. Determine the values of p and q The turning point of $f(x)$: $(1; 18)$ The turning point of $h(x)$: $(2; 15)$ $\therefore p = -1$ and $q = -3$	Indicators *1.1, 1.4, 1.5, 1.6, 2.1, 2.2, 2.3, 2.6, 2.7, 2.10, 3.1, 3.2, 3.3, 3.4*

Therefore, in pursuit of the learners' understanding of the vertex, I looked into learner G11's understanding of the concept (Figure 4.9).

Learner G11

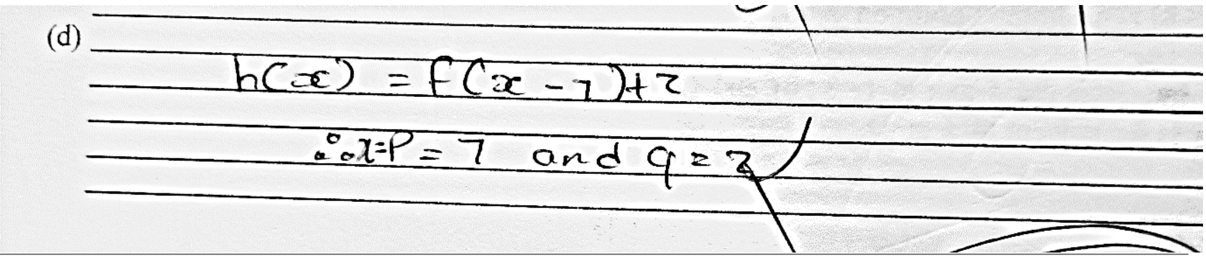


Figure 4.9: Conceptual obstacle of the axis of symmetry

The question was advancing higher indicators of understanding. However, some learners could not tap into the required indicators of understanding. For example, learner G11 misunderstood the concept of the axis of symmetry. Tracking G11's solution for the axis of symmetry is puzzling, as the learner failed to understand the question which was given as $h(x) = f(x - 7) + 2$. Instead, learner G11 viewed the question as a vertex form which is the reason why he wrote $x = p = 7$. This conceptual obstacle inhibited the full attainment of the action, process, and object levels of understanding. The inhibition was caused by the failure to grasp the following indicators for understanding, i.e., 1.1, 1.4, 1.5, 1.6, 2.1, 2.2, 2.3, 2.6, 2.7, 2.10, 3.1, 3.2, 3.3 and 3.4. Consequently, the learner's failure to attain these indicators implied that he was not efficient enough to grasp the concept entirely.

The difficulty in understanding the complete picture of the forms of quadratic functions seems to be persistent in this study. As stated by Kotsopoulos (2007), learners are accustomed to one form of quadratic function, inhibiting them from fully understanding quadratic function concepts. Thus, this inhibited complete understanding of the vertex and posed challenges to the learner. Therefore, the learner had a muddled conception of the vertex form since he failed to explicitly see that the question was not necessarily in the vertex form. The failure to correctly write the value of "p" but directly copying it from the questions supports the idea of learners' fragmented knowledge of the importance of parameters, as noted by Zaslavsky (1997). This notion inhibited the complete grasp of the 2.9 indicators for understanding. The learners' solution clearly indicate that they memorised that the value of p gives the axis of symmetry found in the $(x-p)$ of the vertex form. Their understanding of the concept was instrumental and inhibited the full attainment of the action concept of the action and process levels of understanding of the vertex concept. This meant that the learners posed an instrumental understanding of parameters and the axis of symmetry.

This results in the learners' understanding of the quadratic functions in a fragmented stage. Consequently, learner G11's understanding of the axis of symmetry seemed to be fragmented as from '$h(x) = f(x-7) + 2$' it is clear that '$f(x-7)$' is not raised to the exponent of two, which differs from their prior knowledge of quadratic function. This indicated that the learner did not wholly understand quadratics. As such, the learner would not recognise quadratic equations in algebra. The learner struggled to understand that $f(x-7)$ meant $2(x-7)^2 + 2(x-7) - 12$. Therefore, learners misunderstood the meaning of $f(x)$ or $f(x-7)$. As seen by Hasanah et al. (2021), the understanding of one participant was limited, as the learner substituted $x = 1$ into $f(x) = x^2 + 2x + 6$ as $f(x) = (1)^2 + 2(1) + 6$. The learner failed to note that $x = 1$ was supposed to be substituted on both sides. Therefore, such ignorance posits the learners' understanding of $f(x)$ to be undeveloped. These findings by Hasanah et al. were replicated in this study as they misunderstood the meaning of $f(x-7)+2$ and thought that $h(x)$ was given in vertex form.

4.2.1.4. The domain and range of a quadratic function

The domain deals with the x-axis, and this concept looks for values for which the function is defined along the x-axis. In contrast, the range of a function deals with the y-axis. Again, this looks at the function where it is defined. However, the knowledge of domain and range in the school mathematics fosters instrumental understanding as the CAPS document limits us to foster this understanding. In this study, learners were given the function, i.e., $f(x) = -2x^2 + 4x + 16$ and were required to determine the domain and range of the function. The expected solution to this question is captured in Table 4.10.

Table 4.10: Expected solution of understanding the domain and range

Determine the domain and range: $f(x) = -2x^2 + 4x + 16$	*Indicators 1.1, 2.1, 2.2, 2.4, 2.5*
Domain: $-\infty < x < \infty$	*Indicator 2.4.1*
Range: $y \leq 18$ or $y \in (-\infty; 18]$	*Indicator 2.5.1*

However, some learners managed to portray correct indicators regarding their understanding of the domain concept while understanding the range posed difficulties. This meant that most of them could exhibit indicator 2.4.1 for understanding the domain. At the same time, the concept of the range challenged others. The learners who posed conceptual obstacles include CC6, F10, G11 and K15 (Figure 4.10).

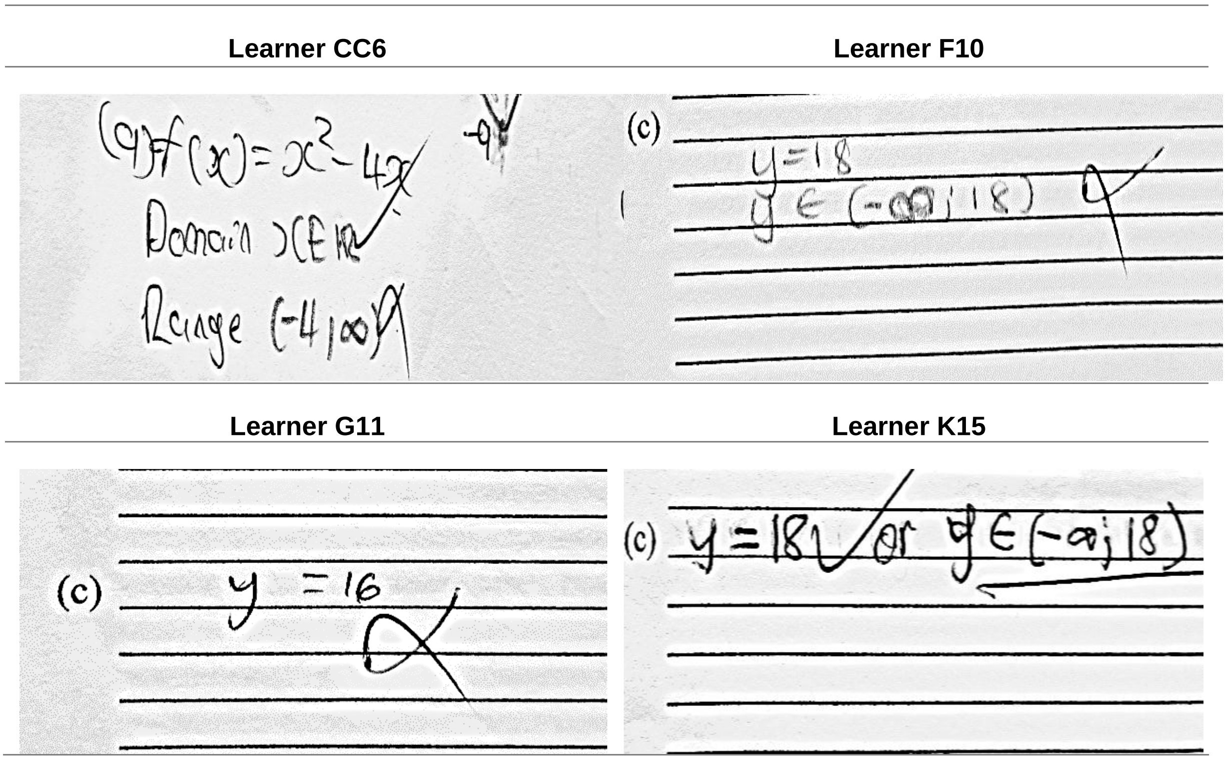


Figure 4.10: Learners' conceptual obstacle of the range concept

The learners interacted with the concept of domain and range because of its nature. They seem to have pitfalls with an especially understanding range. However, for most of them to correctly write the definition of domains for $f(x)$ does not connote a complete understanding of the concept. As put by Díaz et al. (2020), learners are holders of instrumental understanding, which is not fully developed to nurture relational understanding. In this case, the instrumental understanding of the domain did not nurture the knowledge of the range. The learners who determined the domain correctly and failed to exhibit range included learner CC6.

However, according to Fitzmaurice and Hayes' (2020) dimensions of understanding, the learner failed to link the property proof and application dimensions of understanding. Therefore, the absence of such links based on the dimensions of understanding laid by Fitzmaurice and Hayes inhibits the grasp of amalgamated concepts. From the vignettes, it can be noted that learner CC6 struggled to determine the value of the line of symmetry; instead, the learner mistakenly wrote -4, which was the wrong value of the range. The learner failed to notice that -4 was the value of b. As noted by Nielsen (2015), learners struggle to work with quadratic functions, especially if one of the values of a, b and c is zero.

Moreover, the learner incorrectly misused the brackets without understanding their meaning. The misuse was also seen in the solutions of learners F10 and K15. These learners struggled to use the brackets to give the range of the function correctly. This notion meant that learners CC6, F10 and K15 posed a limited understanding of using mathematical brackets to represent intervals. While tracking learner G11's understanding of the range concept, the learner wrote that $y = 16$. The answer from this learner meant that he viewed $q = c$. The learner used the value of the y-intercept instead of the y value of the vertex to give the range. The notion of equating parameters inhibits understanding the effects of parameters thoroughly. As noted by Fonger et al. (2020), learners often struggle to interpret parameters of quadratics, which hinders understanding of the concept. The learner wrote $y = 16$ as the solution to the question. They seem to lack the meaning of the use of an equal sign. This is related to Muchoko et al.'s (2019) findings, as seen from their work, that the learners posed relational conceptual obstacle concerning an equal sign.

In this study, learner G11 seemed to have a relational conceptual obstacle of an equal sign as he thought that the y value of the range equals the y-intercept of the function. The learner's solution resulted from the wrong answer in determining the graph's vertex. I assume that the learner knows that q gives the range, but in her case, it is not q but rather c. Thus, the learner *equates q and c*. This implies that the learner possesses some limited understanding of q that it gives the range. The learners' failure to understand the range inhibited the 2.5.1 indicator for understanding. Therefore, these pitfalls inhibited the full grasp of the indicators for understanding the range concept. Thus, these pitfalls positioned learner G11's understanding of range in a limited lower action level. This limited lower action level seemed to be caused by fragmented knowledge pertaining to the understanding of the y intercept and the vertex.

4.2.1.5. The y-intercept

The analysis of the y-intercept started with the exploration of the learners' work on how they responded to the questions. Understanding the y-intercept is considered easy, especially when it is 'c' in the standard form. Pender et al. (2011) state that each form reveals specific properties and that the standard form gives the y-intercept of the function. However, the conceptual obstacle of the y-intercept surfaced when learners started to interact with the vertex form, demonstrating a lack of understanding of the y-intercept. Hattikudur et al. (2012) asserted that a lack of knowledge of the y-intercept inhibits learners' understanding of drawing graphs. The participants were given a function in a vertex form, i.e., $y = 1 - (x - 1)^2$ and were required to graph the function. The learners started by determining the y-intercept. The expected solution to determining the y-intercept is shown in Table 4.11. The question demanded that learners should exhibit prior knowledge of substitution and simplifying. This knowledge, which is required in this question, advanced to attain 1.2 indicators of understanding.

Table 4.11: Expected solution to understanding the y-intercept

$y = 1 - (x - 1)^2$	Indicators *1.1, 1.2, 2.7*
y-intercept	Indicator 1.2
Let $x = 0$	Indicator 1.2.1
Substitute $x = 0$ into $y = 1 - (x - 1)^2$	Indicator 1.2.2
$y = 1 - (0 - 1)^2$	Indicator 1.2.3
$y = 0$	
$\therefore$ y-intercept: $(0; 0)$	Indicator 1.2.5

The analysis and discussion on the understanding of the y-intercept captured Z30's work in Figure 4.11. The learner posed a conceptual obstacle of the concept of the y-intercept.

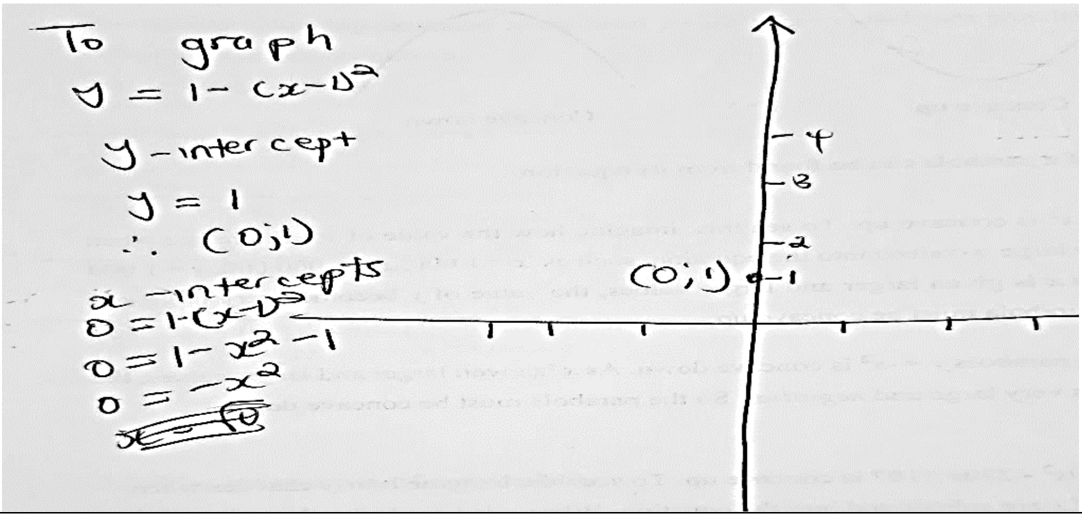
Learner Z30


Figure 4.11: Z30's conceptual obstacle of the y-intercept

The learner was given the vertex form, i.e., $y = 1 - (x - 1)^2$ and was required to graph the function. Graphing the functions requires the learner to determine the intercepts, i.e., x and y-intercept. However, in this section, I explored learner Z30's understanding of the y-intercept of the function. As noted by Malahlela (2017), learners posed an understanding of what should be done to graph a function. However, Malahlela found that learners posed an understanding of intercepts, but it was fragmented as the participants confused the y-intercept of the function and y-intercept of the vertex. For example, learner Z30 thought that the y value of the vertex is the y-intercept as in the standard form c is the y-intercept. It can be seen from learner Z30's solution that the participant possessed an understanding of intercepts. However, this understanding is fragmented as he substituted zero into y while he wrote "x-intercepts". This clearly indicates that the learners' understanding of intercept is not yet entirely developed to advance for the indicators of understanding pertaining to the level.

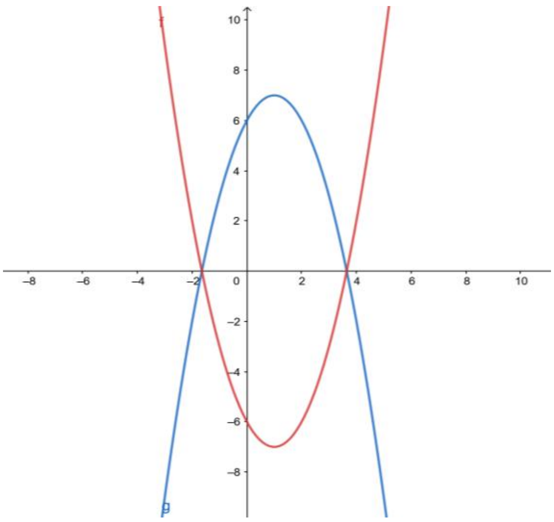
Consequently, the confusion of losing track of what he should have determined and opting for x-intercepts inhibited the full attainment of 1.2 indicators for understanding and further hindered 1.1, 2.7 and 3.4 indicators. Hence, from the learner's solution, he was fragmented with the knowledge of parameters. Conversely, the absence of such understanding impedes a full grasp of the intercepts concept in quadratic functions. As a result, the learner could not fully develop into the *1.2* indicators for understanding. The learner confused the properties of what the standard form gives versus the properties of the vertex form as noted by Hasanah et al. (2021) when studying learners' understanding of intercepts. Hasanah et al. found that learners posed traits of understanding the intercepts but could not plot them correctly. For example, they accurately determined the intercept of $f(-1)$ from $f(x)2x+5$ as$(-1;3)$ but plotted $(-1;-3)$. Such ignorance implied that they failed to have a full grasp of understanding intercepts.

4.2.1.6. The transformation involving quadratic functions

In exploring learners' understanding of transformations of the quadratic function, I tracked learners' solutions which nurtured the concept. However, transformations involving quadratic functions posed challenges to the learners due to conceptual obstacles about the concept. For example, Anabousy et al. (2014) hold the same view regarding the difficulties posed by transformations. Their study noted that many learners do not understand the procedures for transforming functions due to instrumental rather than relational understanding. However, the absence of relational understanding suggests why learners could not see the link of $f(x)$ versus $f(x+k)$. Moreover, Anabousy et al. noted that the conceptual obstacles of transformation are rooted in misconceived knowledge of functions. Eisenberg and Dreyfus (1994) assert that there is more involved in processing the transformation $f(x+k)$ than $f(x)+k$. Thus, a type of transformation process requires relational instead of instrumental understanding. The learners must delve deeper into what happened from $f(x)$ to have $f(x+k)$ before applying the computational process.

In a related study, Zazkis et al. (2003) held that the conceptual obstacle in transformation is rooted in instrumental understanding. The study began by viewing the function $y=x^2$ and being transformed into $y=(x-3)^2$. The study held that most learners would posit the conception that $y = (x-3)^2$ is shifted three units to the left due to the presence of the negative sign. However, in an actual sense, it is shifted to the right by three units. Therefore, in this study, the task that required learners to exhibit their understanding of transformation firstly focused on translation. Learners were given the function $g(x) = 6 + 2x - x^2$ and were required to reflect it over the x-axis. This question advanced for attaining 1.1, 1.4, 1.5, 1.6, 2.1, 2.2, 2.10, 3.1 and 3.4 indicators of understanding. The expected solution for learners to understand the reflection is captured in Table 4.12.

Table 4.12: Expected solution on reflection concept

Determine $g'(x)$ by reflecting $g(x) = 6 + 2x - x^2$ over the x-axis $g'(x) = -[g(x)]$ $g'(x) = -(6 + 2x - x^2)$ $g'(x) = -6 - 2x + x^2$ 	Indicator *1.1, 1.2, 1.3, 2.1, 2.2, 2.4, 2.5, 2.6, 2.7, 2.10*

Most of the learners failed to answer this question; instead, they drew the parent function incorrectly and reflected the incorrect graph. The learners were working in groups of four to respond to this question. Among the learners who showed limited understanding of transformations are AA2, B3, P20 and Z30, and their solution is given in Figure 4.12.

Group A (AA2, B3, P20 and Z30)

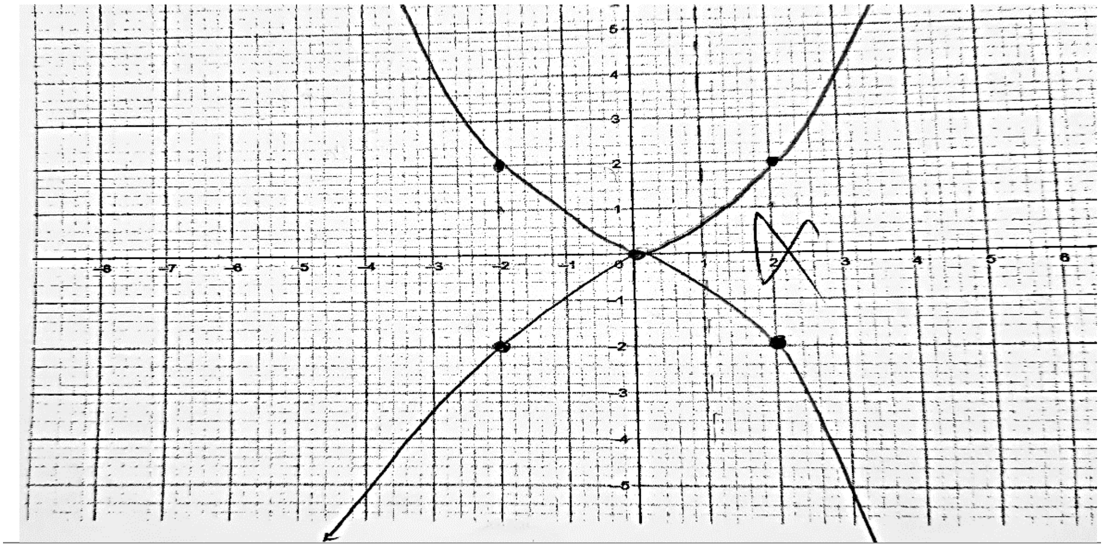


Figure 4.12: Group A’s incorrect translated graph about the x-axis

The learners interacted with the concept of transformation. However, for them to be able to transform the function, they were required to determine $g'(x)$ first and then graph both the functions, i.e., $g(x)$ and $g'(x)$. Baker et al. (2000) state that an object conception is necessary for understanding transformation. The object conception of the quadratic function explains the abstract issues pertaining to transformations. As put by Zazkis et al. (2003), there is more required from the understanding of $y = x^2$ to the conceptualising $y = (x - 3)^2$. Transformation in this study posed challenges to learners. This type of question is seen by Zazkis et al. as being simple to grasp compared to horizontal shifts.

In this study, the participants seem to be muddled with the conception of transformation. For example, learner group A failed to graph the function, i.e., correctly $g(x) = 6 + 2x - x^2$. The learners' failure to correctly graph $g(x)$ inhibited the grasp of 1.1, 1.2, 1.3, 2.1, 2.2, 2.4, 2.5, 2.6, 2.7 and 2.10 indicators of understanding. Consequently, the absence of these indicators makes it impossible for them to determine $g'(x)$. As seen by Mpofu and Pournara (2018), learners found it difficult to relate symbols to graphical. This notion explains the fact that these learners have drawn $g(x)$ incorrectly. The learners' failure to correctly draw the graph emanates from an underdeveloped understanding of parameters.

Complementarily, as seen by Ellis and Grinstead (2008), learners ignored the parameters' effects. In their study, they found that these learners believed that changing a parameter does not alter the vertex location. Consequently, in this study, the learners did not consider the parameter *'a'* of its sign and just drew the function upwards. The negligence of parameter "a" in quadratic is why others usually treat the value of "a" as the same in linear and quadratics. The negligence of such understanding inhibited the attainment of 1.1, 1.2, 1.3, 2.1, 2.2, 2.4, 2.5, 2.6, 2.7 and 2.10 indicators of understanding. As noted by Zaslavsky (1997), learners neglected the difference between "a" in linear versus "a" in quadratics.

In a related study, Ellis, and Grinstead (2008) noted linear interference in quadratics as learners' confusing parameters. Group A failed to correctly represent the y-intercept of the function on the y-axis and did not determine and accurately plot the x-intercepts. The failure to correctly plot all the critical coordinates on the Cartesian plane makes it difficult to understand the transformation concept as a whole (Baker et al., 2000). The absence of such critical points on the graph while plotting it impedes the understanding of reflection. This led to a limited understanding of plotting the graph, further leading to the narrow conception of reflection. This limited knowledge led learners to posit an undeveloped understanding of reflection.

In another task, the learners worked in pairs and were required to work on transformation regarding the vertex. The question required them to determine the vertex after transforming the function from $f(x)$ to $h(x)$. They were given the function $f(x) = -2x^2 + 4x + 16$ and were required to determine the values of p and q after transforming the function $h(x) = f(x + p) + q$ at a maximum value of 15 at $x = 2$. Most learners struggled to understand the question. Amongst the learners, L16, Q21, T24 and Z30 struggled with understanding the question. The question advanced for attaining 1.1, 1.2, 1.3, 2.1, 2.2, 2.4, 2.5, 2.6, 2.7 and 2.10 indicators of understanding. Therefore, the expected solution for learners to understand the vertex is shown in Table 4.13.

Table 4.13: Expected solution on learners understanding the vertex concept

The graph of $h(x) = f(x+p) + q$ has a maximum value of 15 at $x = 2$. Determine the values of p and q The turning point of $f(x)$: $(1; 18)$ The turning point of $h(x)$: $(2; 15)$ $\therefore p = -1$ and $q = -3$	Indicators *1.1, 1.4, 1.5, 1.6, 2.1, 2.2, 2.3, 2.6, 2.7, 2.10, 3.1, 3.2, 3.3, 3.4*

To track learners' understanding of this question, I tracked learners L16 and Z30's workings in exploring their understanding of the concept (Figure 4.13). The question stressed the higher level of understanding, i.e., object. However, it needed to acknowledge previous indicators of understanding, i.e., 1.1, 1.2, 1.3, 2.1, 2.2, 2.4, 2.5, 2.6, 2.7 and 2.10.

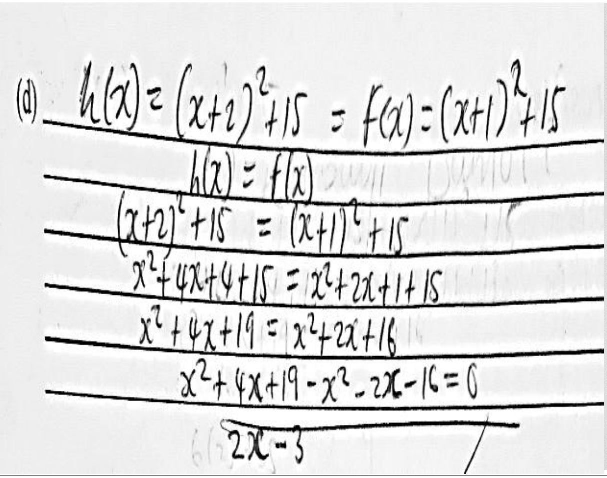
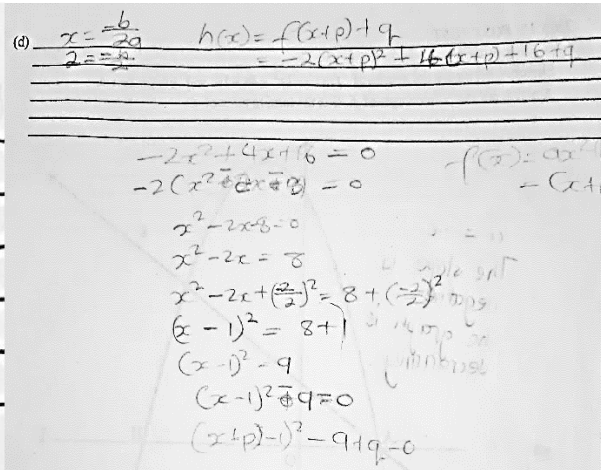
Pair A (L16 and Q21)	**Pair B (T24 and Z30)**
	

Figure 4.13: Learners' conceptual obstacle of transformation

The four sampled learners attended to the task and managed to write something. However, their solutions are incorrect and pose some traits of understanding. For example, learners T24 and Z30 tried to derive the vertex form from the standard, and later changed it to determine the vertex form using completing the square method. However, delving deeper at their workings, the learners seemed to be trying to complete the square as they tried to factor out 2 and later divided both sides by 2

and then transposed 8 to the right. The notion of completing the square meant that the learners did not know what they were doing due to the question format. Adu-Gyamfi et al. (2019) show that learners cannot understand the information given in algebraic notation. For example, learners could struggle to extract the meaning of $f(x) \times g(x) > 0$. Consequently, from their work, the learners seemed unsure of what they were doing and could not reach any conclusion with their method.

In contrast, learners L16 and Q21 approached the question differently as they started by equating $h(x)$ *and* $f(x)$. Even though the two functions are not equal, the learners overlooked that. The pair perceived the equal sign operationally instead of as a relational symbol. However, Bryd et al. (2015) assert that for success in mathematics, one must have an operational and relational understanding of an equal sign. Nevertheless, the learners' solution indicate that the pair started to substitute $(x + p)$ into the $f(x)$. The substitution of this kind meant that these learners were not accustomed to what they were doing as they were confused with the forms. As stated by Kotsopoulos (2007), learners often confuse the forms of quadratic functions, which impedes understanding of the concept.

Furthermore, these learners wrote $h(x) = -2(x + p)^2 + 4(x + p) + 16 + q$, replicating Adu-Gyamfi et al.'s (2019) idea that learners posed difficulties with notations. However, looking into the learners' solution, the pair stopped writing at $h(x) = -2(x + p)^2 + 4(x + p) + 16 + q$, and started to transit $f(x)$ into a vertex form. However, the learner still failed to complete what he started to do. Díaz et al. (2020) noted that learners are holders of instrumental understanding and cannot use it in need. Therefore, all these conceptual obstacles inhibited the grasp of 1.1, 1.2, 1.3, 2.1, 2.2, 2.4, 2.5, 2.6, 2.7 and 2.10 indicators of understanding. Therefore, these learners' understanding of transformation to determining the vertex was at a limited action level due to the solution that he got, i.e., $2x - 3$, which is linear and not quadratic. However, the pair failed to note that what they got as their solution '$2x - 3$' is longer quadratic but linear. The analysis of these four learners led to the realisation that they have not yet attained the complete indicators in the action and process levels, which inhibited the grasp of 3.1, 3.2, 3.3 and 3.4 indicators of understanding.

4.2.1.7. *Graph orientation for quadratic functions*

The graph orientation looks at several characteristics when the quadratic function is drawn on a Cartesian plane. The graph of the orientation of a parabola delves into the critical points of the function being correctly presented, the axis of symmetry, the maximum or minimum value of the graph, and the effects of the shifts involved with the function (Akhsani & Nurhayati, 2020). The graph orientation of quadratic functions nurtures both instrumental and relational understanding. However, while exploring what effects graph drawing, Sumartini et al. (2019) found that learners are more immersed in instrumental understanding than relational understanding. They further found that most learners, when asked to draw $f(x) = (x-1)^2$ few thoughts of it as a shift, and the majority wanted to work with a standard form. The learners who transit the function from vertex into standard were to use the formula $x = -\frac{b}{2a}$ to determine the vertex of the graph, which nurtured instrumental understanding. Delving into the results of Sumartini et al. and analysing them through López et al.'s (2016) understanding of quadratic function would mean that the learner posits instrumental understanding but failed to use it since they would have determined the vertex from the function without calculation.

In this study, the graph orientation began by exploring the parabola's shape if the function drawn opens upwards or downwards. Understanding parameters is necessary to nurture a relational understanding of quadratic functions (Ozaltun Celik & Bukova Guzel, 2017). The shape of the graph is given by understanding the value of a in the function $f(x) = ax^2 + bx + c$, $f(x) = a(x-p)^2 + q$, or $f(x) = a(x - x_1)(x - x_2)$. If the value of a is positive, that is $a > 0$, then the graph opens upwards, and if the value of a is negative, that is $a < 0$, the parabola opens downwards (Nielsen, 2015). The analysis of the graph orientation starts by delving at how the learners responded to the task where they were required to determine the standard form from the vertex form and then decide if the function is concave up or down.

Most learners in this task were able to state if the graph is concave up or down. While some struggled with the activity, i.e., learners C5, J14, L16 and X28, most presented similar answers since they were working in a group (Figure 4.14).

Group B (C5, J14, L16 and X28)

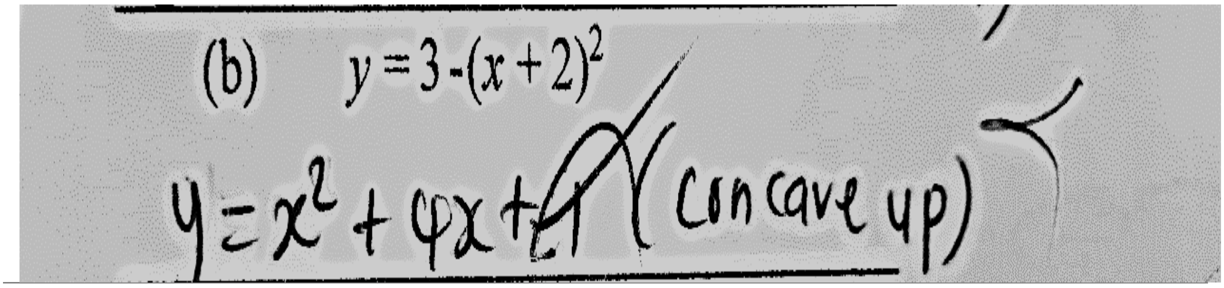


Figure 4.14: Group B's misinterpretation of the concavity of the function

Learners were given a function in a vertex form, and they were required to decide if the function opened upwards or downwards. The question required them to tap into their prior knowledge of quadratic function forms. Pender et al. (2011) state that each form reveals certain properties. However, learners consequently failed to see this notion asserted by Pender et al. These learners struggled to decide if the function $y = 3 - (x + 2)^2$ is concave up or down. The difficulty emanated from the fact that the function was not presented in a usual format, i.e., $y = a(x + p)^2 + q$; instead, it was given as $y = q + a(x + p)^2$. However, the value of a in the question was not positive. While interacting with the question, the group tried to convert this vertex form into a standard form to decide if it opened up or down. This method explicitly asserts that these learners' understanding of parameters is fragmented. As said, learners are usually not accustomed to $a \neq 1$ (Kotsopoulos, 2007). The transition from vertex into standard form proved that the learners were accustomed to working with one form of quadratics.

Similarly, Nielsen (2015) noted that learners will always try to work with the standard form rather than the others. This notion of Nielsen was replicated in this study as the learners wanted to transit into standard form before concluding if the function is concave up or down. The issue of stressing one form depicts an overreliance, and grounds learners' knowledge in possession of instrumental understanding. As noted by Skemp (1976), this is memorised understanding. However, we cannot dispute this kind of understanding as it serves as a ladder for attaining relational understanding. Exploring the learners' understanding, I found that the group simple wrote $y = x^2 + 4x + 1$, neglecting the sign of parameter 'a.' Similarly, they ignored the negative sign at the value of 'b.' As such, the learners' knowledge of quadratics seemed to be at a limited level of understanding of product rules and algebraic operational skills to simplify functions. The notion of ignoring the presence of these parameters supports the idea by Owen (1992), who found that learners usually assume that the absence of a specific parameter does not denote the function in a quadratic. Therefore, from the analysis, it can be noted that the learners ignored the fact that the value of "a" gives the concavity of the quadratic.

Moreover, other groups assumed that the value of 'a' gives the gradient of the graph. The understanding these learners held regarding the importance of "a" was also noted by Ellis and Grinstead (2008), Eraslan (2008) and Parent (2015). The group that held that conceptual obstacle in their discussion is captured in Figure 4.15.

Group C (A1, AA2, B3 and Z30)

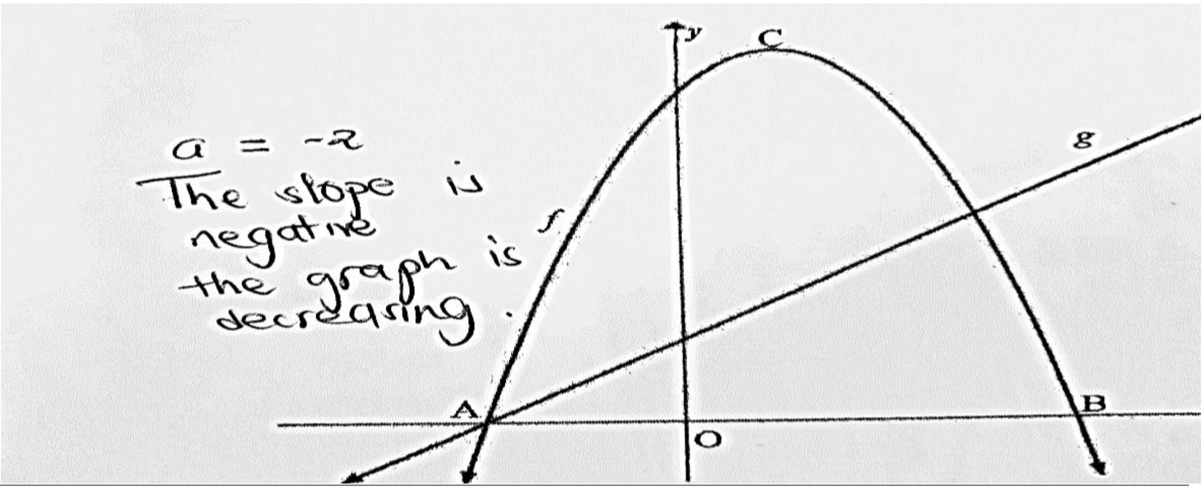


Figure 4.15: Group C's limited understanding of parameter a

The group viewed the value of *'a'* as the function's gradient, as they discussed before answering the given question. I had to capture this difficulty because it posed conceptual obstacles in interacting with the proceeding question in the learning task. These learners confuse "a" in the linear function with the "a" in the quadratic function. This example conforms to what Ellis and Grinstead (2008) asserted about learners' tendency to make the linear and quadratic functions have an analogy. In their findings, the researchers contended that learners thought of parameter a as the gradient of the quadratic function.

In contrast, the effects of parameter a in quadratics influence the shape and the stretching of the graph. Therefore, the group neglected the fact that we do not have a gradient in the quadratics function. This meant that the learners posed a limited understanding of the properties of quadratic and linear functions learned before quadratics.

Consequently, I navigated through the sections, i.e., 4.1.1.1 to 4.1.1.7, to scaffold learners' understanding of quadratic functions. I found that learners tend to think about quadratic functions as isolated concepts. This notion of separating quadratic functions from other concepts was noted in the work of Parent (2015), who found that learners failed to conceive the relation of another concept to a quadratic function. Subsequently, I discovered that the conceptual obstacles in this discourse resulted from the failure to link quadratic functions to other concepts. As noted earlier in this discourse, learners relied on instrumental instead of relational understanding to tackle quadratic function questions.

However, Fitzmaurice and Hayes (2020) do not dispute learners' instrumental understanding. They acknowledge this understanding and assert that it is essential for nurturing relational understanding. Hence, learners' over-dependency on instrumental knowledge was seen in this study, as the participants persistently relied on formulas and overreliance on one form of a quadratic function. These findings conform to Parent's (2015) results. Additionally, learners posed fragmented knowledge regarding the y-intercept of the function versus the y-coordinate of the vertex. As a result, learners' understanding of the y-intercept of the function and the y-coordinate of the vertex seemed to pose severe conceptual obstacles.

Moreover, the participants seem to lack relational understanding of an equal sign and their knowledge of arithmetic was fragmented. Furthermore, I found that learners thought that the axis of symmetry is a number and not a function, hence posing a relational conceptual obstacle of the line of symmetry. Furthermore, I found that interference of knowledge of linear functions impeded learners' understanding of quadratic functions. As noted by Ozaltun Celik and Bukova Guzel (2017), their participants experienced the same conceptual obstacle pertaining to parameter a. Similarly, Ellis and Grinstead (2008) noted that learners identified the parameter a as the slope of the function.

The analysis and discussion showed that some conceptual obstacles that impede learners' understanding of quadratic functions were treating two different functions as equivalent. Moreover, learners failed to note the effects of parameter a in the quadratic functions. As seen by Subani et al. (2022), learners thought that changing parameter a changes the y-intercepts. Thus, this was a conceptual obstacle held by the participants in the works of Subani et al. Therefore, this conceptual obstacle was replicated in this study, as learners assumed that altering parameter a affects the other parameters. Consequently, presented with these conceptual obstacles so far led to the realisation that learners' knowledge is at a disequilibrium state between graphical and algebraic thinking. Therefore, the algebraic and graphical disequilibrium state inhibited the full attainment of the action, process, and object indicators of understanding.

4.2.2. Classroom discussions

The classroom discussion phase, a tenet of the ACE teaching cycle, was implemented during the use of the learning unit. The unit was compiled using a curriculum assessment policy statement, assessment policies for Grade 12, and previous question papers for mathematics, thus interacting with the learning unit in the classroom during the activities phase. As a result, classroom discussion was informed by the underdeveloped conceptual understanding observed during the activities phase. The classroom discussions involved learner-learner, small group, teacher-learner, and whole class discussions. These classroom discussions sought to allow learners to reflect on the learning process. Borji et al. (2018) noted that the classroom discussion would enable learners to reflect on the task they interacted with in the activities phase. In this phase, I observed and recorded the instances where learners struggled to comprehend quadratic functions. The participants interacted with the tasks that were non-directive, as the activities were designed from the perspective of a constructivist view. The learning activities afforded learners with gains to comprehend quadratic functions fully. During the classroom discussions, I was awaiting the “aha!” moments to guide the phase, as asserted by Borji et al. Therefore, all this was done to gain more insights into learners' understanding of quadratic functions.

While learners interacted with the various concepts of quadratic functions, they demonstrated their understanding in the activities phase. As a result, I used the classroom discussion phase to delve deeper into learners' conceptual understandings of quadratic functions through their conversations about the y-coordinate of the vertex and y-intercept concepts. Conceptual understanding the vertex and the y-intercept of the function persistently impedes learners' full grasp of the quadratic function concept. As noted by Ellis and Grinstead (2008), learners posed a conceptual obstacle with vertex. The participants believed that the turning point was impacted by parameter a. In a related study, Childers, and Vidakovic (2014) found that learners' understanding of vertex was fragmented. These fragmentations were due to their comprehension of the vertex as a maximum or minimum in relation to the axis of symmetry, as a turning point, as an intercept, and as an intersection.

Consequently, to mitigate these constraints in comprehending vertex, I used the APOS theory to delve deeper into learners' conceptual understanding of the concept. Therefore, to delve deeper, I followed the work of Childers and Vidakovic to explore the silent issues pertaining to learners' knowledge of vertex and y-intercept of the function. Childers and Vidakovic proposed a helpful notion for understanding vertex. They suggested that exploring learners' personal meanings of the vertex is beneficial to help them fully comprehend it (Childers & Vidakovic, 2014). Therefore, I adopted this notion proposed as Childers and Vidakovic in this study to delve deeper into learners' understanding of vertex and y-intercept of the function.

The classroom discussions phase is divided into learner-teacher, learner-learner, and teacher-learner discussions. The classroom discussion involves the learners' understanding of the following concepts: understanding the vertex, not understanding the vertex, and the difficulties with the y-intercept. To understand the discussions more deeply, I have captured the learners' tasks that sparked the discussions with the expected solution to the question.

4.2.2.1. Understanding the vertex

The vertex analysis looked at how the learners working in pairs attended to the problem. The learners were given three functions, i.e., $f(x) = 2 + x - x^2$, $g(x) = -3(x-4)^2 + 100$, and $h(x) = (x-1)(x-3)$. From these functions, they were required to determine the vertex and the y-intercept. However, in this section, I explored learners' understanding of determining the vertex on $h(x) = (x - g1)(x - 3)$. I chose to analyse and discuss their understanding of the vertex on the factored form as most participants thought it was not a quadratic function. As noted by Mutambara et al. (2019), most participants posed a conceptual obstacle with the factored form as they confused it with the vertex form. This view in the study by Mutambara et al. is similar to the findings by Kotsopoulos (2007), that learners often think that quadratic functions are given in one form, that is, the standard form instead of vertex or factored form. However, learners posed an instrumental understanding of the standard form even though they preferred it. For example, the participants in the study by Kotsopoulos thought that $x^2 + 3x + 7 = x + 4$ was not in the standard form. Consequently, such conceptual obstacles imply that the participants posed an underdeveloped knowledge of the forms of quadratic functions.

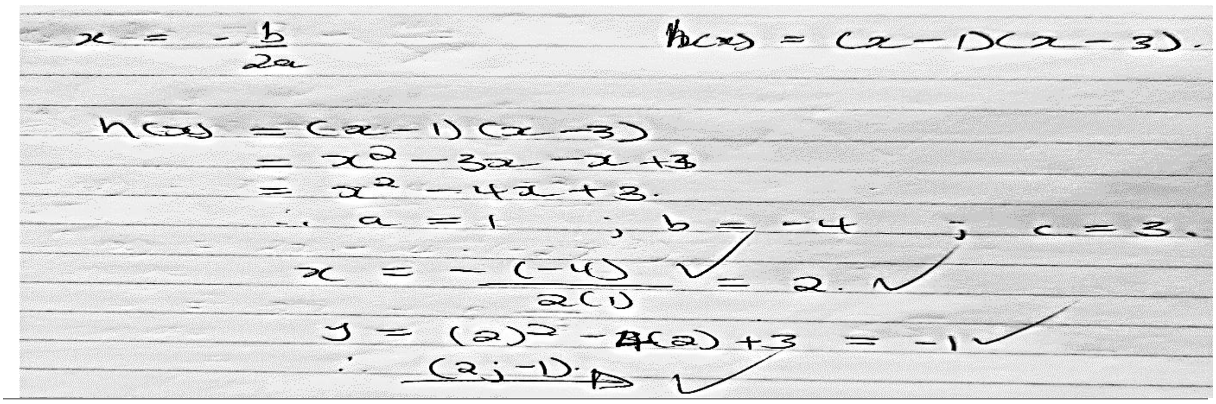
Therefore, it is from the literature that I delve deeper into learners' understanding of the vertex from the factored form. Parent (2015) noted that various quadratic forms reveal critical points related to a function. For example, the factored form indicates x-intercepts of the function. I needed to explore learners' understanding of vertex from the factored form. The question demanded that learners exhibit higher indicators of understanding: action, process and object. In this study, learners were given a function in a factored form and were required to determine the vertex of the function. The expected solution for understanding the vertex is captured in Table 4.14. The question nurtured the attainment of 1.1, 1.5, 2.1, 2.2, 2.6, 2.7, 3.1, 3.2, 3.3 and 3.4 indicators of understanding.

Table 4.14: Expected solution on understanding the vertex

$h(x) = (x-1)(x-3)$	Indicators *1.1, 1.5, 2.1, 2.2, 2.6, 2.7, 3.1, 3.2, 3.3, 3.4*
$h(x) = x^2 - 4x + 3$	Indicators *1.1*
$a = 1; b = -4$	Indicator *1.5*
$x = -\frac{b}{2a}$	Indicator *1.5.1*
$x = -\frac{(-4)}{2(1)} = 2$	
Substitute $x = 2$ into $h(x)$	Indicator *1.5.2*
$h(2) = (2)^2 - 4(2) + 3 = -1$	
$\therefore vertex\ (2; -1)$	Indicator *1.5.3*

Learners interacted with the question in pairs, and some managed to give the correct vertex of the function. Consequently, finding the vertex was easy for learners T24 and W27. The learners determined the correct vertex of the function, and their answer is shown in Figure 4.16. The learners were purposeful sampled due to their answers to the question as they managed to demonstrate understanding of the vertex. However, the purposive sampling was not necessarily on the solution's correctness, but the notions proposed by Childers and Vidakovic (2014) to delve deeper into the answer to explore the silent issues pertaining to learners' conceptual understanding of vertex as they discussed while working on the solution.

Pair C (T24 and W27)



$x = -\frac{b}{2a}$ $h(x) = (x-1)(x-3).$

$h(x) = (x-1)(x-3)$

$= x^2 - 3x - x + 3$

$= x^2 - 4x + 3.$

$\therefore a = 1 ; b = -4 ; c = 3.$

$x = -\frac{(-4)}{2(1)} = 2.$

$y = (2)^2 - 4(2) + 3 = -1$

$\therefore (2; -1).$

Figure 4.16: Conceptual understanding the vertex of the factored form

The learners interacted with the question and determined the correct answer for the vertex of the function. The learners started by writing the correct formula to determine the x-coordinate of the vertex as $x = \frac{-b}{2a}$, thus this was a correct start for determining the vertex as they cited the correct formula. The learners' citation of the correct formula exhibited procedural flexibility. Rittle-Johnson (2017) defines procedural flexibility as knowing more than the procedures and using them adaptively. Thus, this fashion is bidirectional as the procedures lead to subsequent procedural knowledge. Procedural flexibility goes beyond instrumental as it is intertwined with relational understanding (Rittle-Johnson, 2017). This is achieved as understanding the vertex strengthens the comprehension of the axis of symmetry. Therefore, learners attaining a full grasp of vertex implies that they have also understood the axis of symmetry. As a result, this is the bidirectional nature of the concept. Delving deeper into the learners' work, the participants transitioned from the factored form into the standard form. The learners started by simplifying $h(x) = (x - 1)(x - 3)$ into $h(x) = x^2 - 3x - x + 3$ and later simplified to $h(x) = x^2 - 4x + 3$. Yet. This transition was necessary but conformed to Kotsopoulos (2007) findings that learners usually prefer using the standard form over the vertex or factored form. Nevertheless, the transition was necessary to conform to the formula cited in the opening of their work which is $x = \frac{-b}{2a}$. Thus, learners transitioning from factored to standard form demonstrated competency as they would not know the formula alone but also gained the conceptual meaning of the vertex.

Subsequently, the learners could correctly transition from factored into standard form after determining the values of $a = 1$ and $b = -4$. Closer look into the solution strategy, it can be noted that the learners even determined the value of $c = 3$, which is not necessary for the formula of $x = \frac{-b}{2a}$ but they showed strategic competence with the formula. Groves (2012) states that strategic competence is the ability to solve mathematical problems. Nevertheless, the learners were operating in an adaptive reasoning state (Groves, 2012), as in their substitution, they only substituted the values of a and $b,$ respectively. Thus, from their solution, the pair could note that they are determining intercepts, not necessarily solving for x. This skill sets the link between equations and functions (Knuth, 2000). Knuth found that most learners had a limited understanding of the connection between equations and graphs. However, these findings were avoided in this study as learners T24 and W27 correctly determined the intercept instead of the value of x only. Thus, if the pair determined x only, it would have meant that they were determining the axis of symmetry, replicating what Childers and Vidakovic (2014) found. As they found that some thought that the vertex is the axis of symmetry (Childers & Vidakovic, 2014). Moreover, I have transcribed the two sampled learners' discussions regarding the vertex concept. Their discussion is captured in Table 4.15 below.

Case 1: T24, W27, and T-Math's conversation about determining the vertex

Table 4.15: Conversation between T24 and W27 about vertex

1. **T24**: We have to determine the equation of the line of symmetry W27.
2. **W27**: Alright, so we are going to use this formula $x = -\frac{b}{2a}$?
3. **T-Math**: Why do you have to use this formula $= -\frac{b}{2a}$ instead of $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$?
4. **T24**: Sir!!! That's not the correct formula for the x value of the axis of symmetry. No, we need the x-intercept of the vertex and not the roots of the function.
5. **T-Math**: Okay. Then proceed.
6. **W27**: Sir… Please don't trick us. We need to have the equation in the standard form, and then after, we can have the values of a, and b.
7. **T-Math**: What about c?
8. **W27**: Sir, we cannot substitute c, since our formula requires the values of a and b only.

***T24** started writing on the paper…*

9. **T24**: *(While writing on the paper) …* Our x-intercept of the vertex is $x = -\frac{b}{2a} = -\frac{(-4)}{2(1)} = 2$. Then with $x = 2$, we can now work out to determine the y value of the vertex by substituting back into the function.
10. **W27**: Our $y = -1$. Therefore, the vertex of $h(x) = (x-1)(x-3)$ is $(2;-1)$. *(Pointing to the answer with a pen).*

The answer determined was correct.

From the learners' discussion in Table 4.15, one can tell that the learners have attained 1.1, 1.5, 2.1, 2.2, 2.6, 2.7, 3.1, 3.2, 3.3 and 3.4 indicators for understanding. Therefore, attainment of these indicators advanced for learners to exhibit adaptive reasoning of the vertex concept. As noted by Groves (2012), learners operating in this state of understanding can reflect, explain and justify their arguments. Consequently, their exhibiting adaptive reasoning based on the precedent genetic decomposition meant that they had attained action-process and process-object conception of a vertex. Hence, their arguments posit strong adaptive reasoning with the vertex, even if challenged through my scaffolding. Calor et al. (2022) assert that scaffolding in groups is an important skill to foster a higher understanding of the concept to be learned. Subsequently, I used the scaffolding approach to enable a higher understanding of the vertex concept, and the pair did not seem to be confused or feel that their thoughts were incorrect. As a result, through scaffolding, it is noted in line 3, when I asked them why they are using $x = \frac{-b}{2a}$ instead of $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, the learners are able to justify their choice of the formula in line 4. Thus, such justification gives evidence of their mathematical proficiency in the vertex. Therefore, the learners can convince you as the third person that they have procedural flexibility of the vertex concept. In this task of the vertex, they have managed to work and reach the final correct answer. However, challenging the pair in which formula can be used to determine the vertex, they cited a correct one and gave reasons for using it. Such exhibited the presence of procedural, conceptual, strategic competence and adaptive reasoning of vertex conception. Therefore, such understanding implies that the learners do not necessarily possess instrumental understanding (Skemp, 1976) but a coherent whole mental structure that is mathematical proficiency of vertex concept (Grove, 2012). Therefore, mathematical proficiency means that the learners have a schema for the axis of symmetry and vertex. Therefore, such proficiency is called procedural flexibility (Rittle-Johnson, 2017). The learners' discussion indicate that the pair knew the use of these formulas, i.e., $= -\frac{b}{2a}$ and $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ as seen in line 3. Therefore, this exhibited procedural flexibility of the learners when determining the vertex of the function (Rittle-Johnson, 2017). The skills of knowing which formula to use and why that learners justify and posed developed knowledge of equation and function (Knuth, 2000).

Consequently, the participant exhibited 1.1, 1.5, 2.1, 2.2, 2.6, 2.7, 3.1, 3.2, 3.3 and 3.4 indicators of understanding. Thus, the pair had developed an action, process and object conception of the vertex. In lines 9 and 10, the two learners knew what to do after they had determined the x-value of the vertex, and again this denotes the development of strategic competence of the vertex, as the learners were able to determine the $h(2)$ as $h(2) = (2)^2 - 4(2) + 3 = -1$, thus, they did stop there at the values of $x = 2$ and $y == -1$, but they determined the intercepts as $(2; -1)$. Consequently, such procedural flexibility denotes their attainment of mathematical proficiency, as they have demonstrated procedural, conceptual, strategic competence, and adaptive reasoning of vertex conception. Therefore, skills exhibited in the answer positioned the two sampled learners in a schema level of understanding of the vertex. The learners demonstrated a schema of an axis of symmetry and later connected the axis to the vertex. From the learners' discussion, it can be seen that they did not replicate the conceptual obstacle identified by Childers and Vidakovic (2014) regarding the values of c. In their study, Childers and Vidakovic found that learners believed that the vertex's location was affected by the value of c. As a result, the conceptual obstacle identified by Childers and Vidakovic was not replicated in this study as the learners did not substitute the value of c in the formula $x = \frac{-b}{2a}$ while determining the x coordinate of the vertex.

4.2.2.2. Conceptual obstacle of the vertex of the standard form

However, while observing other learners, I was puzzled by how P20 and K15 understood the concept of the vertex. To demonstrate not understanding of the vertex, I transcribed the conversation between K15 and P20's discussion about the vertex. The learners were interacting with $f(x) = 2 + x - x^2$ to determine the vertex. I have presented the expected solution in Table 4.16. The function was given in a standard form of a quadratic function. Parent (2015) noted that learners prefer the standard over the vertex and factored form. Thus, from Parent's view, learners typically find it easy to work with the standard. Conversely, the pair in the study posed conceptual obstacles while they were given the standard form. Hence, the expected answer for determining the vertex from the standard form is captured in Table 4.16. The question advanced for attaining 1.1, 1.5, 2.1, 2.2, 2.6, 2.7, 3.1, 3.2, 3.3 and 3.4 indicators of understanding.

Table 4.16: Expected solution on learners' conceptual understanding of the turning point

$f(x) = 2 + x - x^2$ $a = -1;\ b = 1$	Indicators *1.1, 1.5, 2.1, 2.2, 2.6, 2.7, 3.1, 3.2, 3.3, 3.4*
$$x = -\frac{b}{2a}$$	Indicator *1.5*
$$x = -\frac{1}{2(-1)} = \frac{1}{2}$$	Indicator *1.5.1*
Substitute $x = \frac{1}{2}$ into $f(x)$	
$f\left(\frac{1}{2}\right) = 2 + \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = \frac{9}{4}$	Indicator *1.5.2*
$$\therefore vertex\ \left(\frac{1}{2}; \frac{9}{4}\right)$$	Indicator *1.5.3*

Moreover, I also presented their solution in Figure 4.18. From the learners' work, two conceptual obstacles surfaced, which are demonstrated as a lack of understanding of the y-intercept and vertex.

Pair D (K15 and P20)

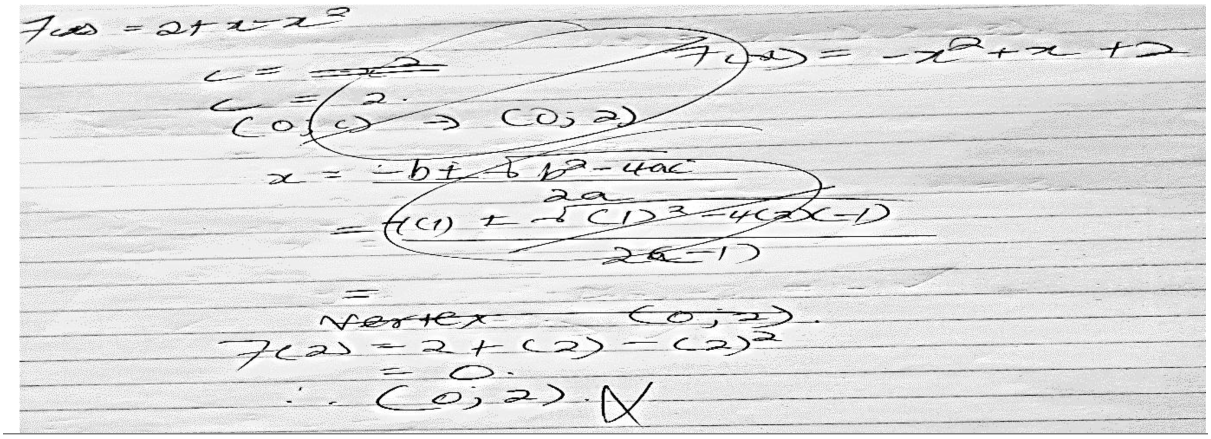


Figure 4.17: Learners conceptual obstacle the vertex on a standard form

The learners interacted with the question in a standard form of the quadratic function, as put by Parent (2015) that the standard form reveals the y-intercept of the function. Delving deeper into the learners' solution, the pair started by writing the value of $c = 2$, and later wrote $(0; c) \rightarrow (0; 2)$. Such a blunder while determining the vertex instead of opting for the y-intercept is similar to the findings by Childers and

Vidakovic (2014), who found that learners viewed the vertex as the y-intercept and could not determine it.

Similarly, in this study, the learners viewed the vertex as the y-intercept and were unable to compute it as they kept on cancelling and writing incorrect methods. For example, the pair wrote the value of $c = 2$ and cancelled their method. Consequently, conceptual obstacle inhibited the learners' procedural flexibility in determining the vertex. The inhibition of procedural flexibility impedes attaining adaptive reasoning as they fail to justify their answer (Groves, 2012). Subsequently, the absence of adaptive reasoning is the reason that these learners were writing and cancelling. Additionally, the pair cited the formula of the quadratic function as $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ and substituted the values of $a = -1, b = 1$ and $c = 2$ into the formula and later cancelled what they wrote. The learners' work posit evidence that the pair lacked mathematical proficiency in the vertex. The absence of proficiency in understanding the vertex explains that the learners cited incorrect formulas, which posed an insufficient mathematical connection between equations and functions (Wijayanti & Abadi, 2019). Furthermore, the learners failed to correctly tap into procedural flexibility of using the formula $x = \frac{-b}{2a}$. This failure to use the correct formula is seen from the step where the learners cited the quadratic formula that is $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ instead of $x = \frac{-b}{2a}$. Thus, such misconceived understanding of the usage of formulas impedes the strategic competence of these learners (Kilpatrick et al., 2002). Conversely, due to a lack of adaptive reasoning and strategic competence, the learners brought back the value of $c = 2$, to be substituted into $f(x) = 2 + x - x^2$ as $f(2) = 2 + (2) - (2)^2 = 0$, and later, determine the vertex as $(0; 2)$, even though this was the solution that was cancelled in the first steps of the question. Even so, their answer was incorrect; the pair acknowledged that they were determining the intercepts by writing the vertex as $(0; 2)$. As a consequence, exhibiting such skill in writing the final solution in an intercept format support the idea posed by Knuth (2000) of understanding the connection between functions and equations. Therefore, to extract the silent issues on the learners' answer, I transcribed their discussion about the vertex in Table 4.17.

The learners' discussions about the concept of vertex began with the participants showing fragmented knowledge of the standard form of a quadratic function. Learners K15 and P20 got confused with the standard form when they wanted to determine the vertex of the function. The pair was confused with the vertex and the y-intercept of the function. Their discussion started from the debate on whether the form was in the standard form or not. The learners' discussion is captured in Table 4.17 below as transcribed. In the latter part, I had to discuss with learner P20 to interrogate his understanding of the vertex form.

Case 2 below presents the conversation between P20 and K15 about the vertex.

Table 4.17: Conceptual obstacle of the vertex concept

1. **K15:** Well, is not the last one our vertex… *(Pointing to the constant of the function)* That would indicate the vertex. We have a vertex. That would be the value of c.
2. **P20:** I'm not following what you are saying. What do you mean?
3. **K15:** How do you determine the value of c…. Oh, eix, I mean the vertex.
4. **P20:** With the area of symmetry.
5. **K15:** You mean the axis of symmetry, $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.
6. **P20:** Yes, that is correct….
7. **K15:** With the axis of symmetry, right, but c be the other half of the value neh….
8. **P20:** You mean this…. *(Pointing to the y-intercept of the function $f(x) = 2 + x - x^2$.)*
9. **K15:** Ayiii, not that one….
10. **P20:** Oh, the y-intercept neh….
11. **K15:** Yes, that one….
12. **P20:** That is what it has to go through… *(While pointing at the y-intercept that it goes through the y-axis).*
13. **K15:** I think that will help us find the axis of symmetry by substituting it into the function *(While perusing her book pages)*. Oh, now I see what you were talking about *(Stopping the paging and putting his hand down a little bit convinced)*.
14. **P20:** So, the one for $f(x) = 2 + x - x^2$ the y-intercept of the vertex is $q = 2$ *(Indicating that the value of q is c).*
15. **K15:** Yeah…. I agree with that.
16. **P20:** Our vertex is $(0; 2)$ *(Pointing at his solution, see Figure 42).*
17. **K15:** Okay….

They ended up with an incorrect answer for the vertex.

The opening statement by learner K15, "Well, is not the last one our vertex… *(Pointing to the constant of the function)* that would be indicating the vertex. We have a vertex. That would be the value of c…" demonstrates a lack logical thought about the vertex. Moreover, the pointing of c as the vertex indicates a misconceived conception of the y-intercept. Thus, failure to know the meaning of parameter c

inhibits the attainment of the vertex concept. Consequently, the inhibition is caused by the existing schema of the y-intercept that constrained the attainment of a vertex. Thomas (2008) noted that some of our conceived schemas either afford or constrain the association of new concepts. Thus, this notion was seen in the discussion that the learners' preconceived schema of y-intercept constrains the understanding of vertex. The conversation between the learners above shows an undeveloped limited action conception of adaptive reasoning (Groves, 2012), which constrains the understanding of the y-intercepts and the vertex. The discussion that the learners had inhibited their algebraic reasoning of quadratic functions. Damayanti et al. (2019) noted that algebraic reasoning is used to generalise arithmetic algorithms to notice actual patterns.

Conversely, the pair failed to reason algebraically as they did not notice the connection of intercepts and vertex. These learners failed to correctly tap into their reasoning capacity as their strategic competence of determining the vertex was undeveloped. The undeveloped understanding of the vertex resulted from an inability to use the correct procedure which impedes the grasp of adaptive reasoning. The learners' lack of procedural flexibility led to affordance in executing a wrong procedure steered conceptual obstacle of the vertex. Furthermore, in line 3, learner K15 changed the focus of the question to determine the value of 'c' instead of the vertex, for the attainment of the object conception of the vertex. Thus, such meandering results impede the processes responsible for nurturing the understanding of vertex. Schoenfeld (1988) noted that learners' prior knowledge posed fragmentation to attaining new concepts.

Consequently, in this study, the pair developed an inaccurate schema for the y-intercept, which impeded the vertex concept. Furthermore, in line 15, the learner wanted to use the quadratic formula to determine the vertex. Hence, using the quadratic functions constrain the understanding of the vertex. Subsequently, the learners' skill of opting for the quadratic formula meant that they did not comprehend when the formula should be used. Thus, they must first understand the question to clear the learners' misconceived thoughts. Intaros et al. (2014) state that the key to understanding concepts is mitigating conceptual obstacles in the question asked.

As a result of these misconceived conceptions of vertex, it explains why the learners in line 4 referred to the axis of symmetry as the area. As Nielsen (2015) noted, the word area was also used by learners to refer to the axis of symmetry incorrectly; instead, they used area of symmetry. In contrast, learner K15 corrected the word but cited an incorrect formula. Therefore, the learners posed a fragmented instrumental understanding of the quadratic formula. Díaz et al. (2020) noted that learners often fail to use instrumental understanding when required, even if they have it. This indicated that their conception of vertex was undeveloped as they would the two terms interchangeably, i.e., area and axis. In line 14, the learners equated q and c, neglecting the parameters' differences. Based on the APOS theory, their level of understanding is at the undeveloped action conception. Therefore, these learners' conceptual obstacles inhibited the full attainment of 1.1, 1.5, 2.1, 2.2, 2.6, 2.7, 3.1, 3.2, 3.3 and 3.4 indicators for understanding. Consequently, due to these conceptual obstacles, I had to discuss with learner P20 regarding the understanding of the vertex.

4.2.2.3. The learner-teacher discussion about the vertex and y-intercept

Learner P20 was purposively sampled for the one-on-one discussion session due to his conversation with learner K15. The criteria to sample the learner was due to the fact that the participant posed severe conceptual obstacles of the concept of vertex and y-intercept. Therefore, to delve deeper into the learner's understanding of the vertex and y-intercept I had to scaffold the participant. Contrary to the traditional approach to learning mathematics, which provides grades which make no sense to the learners (Zimmerman et al., 2011) I employed a different approach where the grades are categorised by the APOS tenets to indicate learners level of understanding. Subsequently, such an approach posed learners with meaningless grades that participants could not understand. In this study, I employed the APOS approach, which uses classroom discussion to allow learners to reflect on their conceptual obstacles (Arnon et al., 2014).

In this section, I adopted the self-reflection phase, a tenet of the cognitive model of self-regulation. Zimmerman et al. assert that self-reflection involves the participants' responses and includes self-evaluative judgements and adaptive selfreactions to learning mathematics. Thus, in this study, I used the self-reflective process to nurture learner P20's understanding of vertex and y-intercept. I sought to improve his/her self-reflection responses through conceptual obstacle correction. Thus, I wanted learner P20 to self-reflect more effectively on his conceptual obstacles of vertex and y-intercept through a one-on-one discussion. Subsequently, the discussion with learner P20 was guided by these questions below:

- What do you understand about the concept of vertex?
- How is the vertex different from the y-intercept
- How do you find the y-intercept of any quadratic function"?
- How do you find the y-intercept?
- Given $g(x) = -3(x-4)^2 + 100$, can you determine y-intercept?

The one-on-one discussion began by making the participant aware that he was in this process, not necessarily because he wrote correct or incorrect solutions while determining the vertex. It was vital for me to allude to this since I did not want to demotivate the learner. The learner was given the function, i.e., $g(x) = -3(x-4)^2 + 100$, and was required to determine the y-intercept and the vertex of the function. The learner was given the function in a vertex since he posed conceptual obstacles with the standard form while working in pairs. The expected solution for the two questions is captured in Table 4.18.

Table 4.18: Expected solution the y-intercept and the vertex on a vertex form

Determining the y-intercept: $g(x) = -3(x-4)^2 + 100$ y-intercept Let $x = 0$ $y = -3(0-4)^2 + 100$ $y = -3(-4)^2 + 100 = 52$ $\therefore$ y-intercept $(0; 52)$	Indicator *1.1, 1.2, 1.6* Indicator *1.2.1* Indicator *1.2.2* Indicator *1.2.3* Indicator *1.2.4*
Determining the vertex: $g(x) = -3(x-4)^2 + 100$ $\therefore vertex\ (4; 100)$	Indicator *1.1, 1.6, 2.1, 2.2, 2.3, 2.6* Indicator *1.6*

Therefore, they interacted with the questions and managed to present their answers. Thus, their solutions are captured below as the interview involved a conversation based on the learner's solution, i.e., Figure 4.18.

Determining the y-intercept	**Determine the vertex**

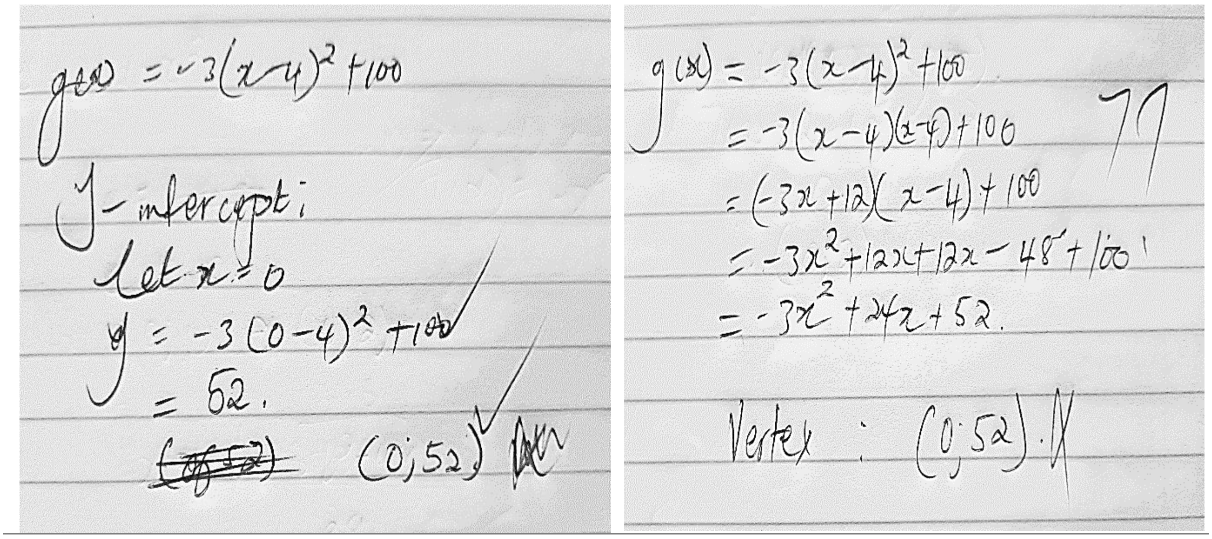


Figure 4.18: The solutions of learner P20 leading to the one-on-one

The learner interacted with the question and was able to give the y-intercept but failed to determine the vertex of the function, as noted by Hattikudur et al. (2012) that understanding and graphing the y-intercept involves concentrating on one variable as the other is zero. Therefore, this notion given by Hattikudur was achieved by learner P20 as the learner was able to set $x = 0$ as also noted by Hattikudur as the necessary step to begin with. While determining the y-intercept, learner P20 managed to exhibit the 1.2 indicators of understanding. The learner started by letting $x = 0$, which was an excellent start to the question. Moreover, the learner substituted $x = 0$ into the function as $y = -3(0-4)^2 + 100 = 52$. Thus, the operational skill of letting $x = 0$, and then managing to substitute into the function as $y = -3(0-4)^2 + 100$ implied that the learner posits strategic competence in determining the y-intercept. Groves (2012) noted that strategic competence is an operational skill in mathematical computing problems. Additionally, the learner concluded the statement by giving the intercepts of the turning point as $(0; 52)$. This was a good indication that the learner has developed procedural flexibility in determining the y-intercept. Rittle-Johnson (2017) noted that learners possessing procedural flexibility had attained instrumental and relational understanding. This procedural flexibility is seen from the perspective that the learner has a schema for functions and equations instead of ending his answer at $y = 52$, which limits it to the equation. This means that the learner posed adaptive reasoning (Groves, 2012) of determining the y-intercept as he could show the logical thought of intercept by writing $(0; 52)$. This procedural flexibility was noted in learner P20 as the learner used correct procedures to compute the y-intercept, which exhibits instrumental understanding and advanced relational understanding by writing in intercept format.

Conversely, the learner struggled to determine the vertex, although the function was given in the vertex form. The learner started by transitioning from the vertex form into the standard form. This skill inhibited the conception that Parent (2015) held that the vertex form reveals the vertex of the function. However, the learner managed to get the standard form of the function as $y = -3x^2 + 24x + 52$. The learner was supposed to have noted that $c = 52$ is not the y-coordinate of the vertex but the y-intercept. Thus, notion disputed the understanding that the learner posed with vertex form. The findings from learner P20 on determining the vertex replicated what Díaz et al. (2020) found. Díaz et al. found that learners posed undeveloped instrumental understanding and failed to use the knowledge if required. Moreover, the one-on-one discussion was transcribed. The transcribed discussion between learner-teacher is captured in Table 4.19 about the understanding of the y-intercept and vertex.

Case 3 below presents the one-on-one discussion between P20 and T-Math about the vertex

Table 4.19: The one-on-one discussion between learner-teacher

1. **T-Math**: What do you understand about the concept of the vertex?
2. **P20**: mmmhhhh…. A vertex is a…. is not the c, I mean the vertex… *(While trying to think)*. It is the point at the maximum of the graph.
3. **T-Math**: Okay…. You don't want to expand your concept about the vertex?
4. **P20**: Yes… a vertex is a maximum of the graph.
5. **T-Math**: Okay. Then how is the vertex different from the y-intercept?
6. **P20**: The vertex is the maximum, and the y-intercept is the y-intercept *(While nodding his head)*.
7. **T-Math**: Okay. Then how do you find the y-intercept of any quadratic function?
8. **P20**: You need to find y first and substitute from the standard form.
9. **T-Math**: What do you mean? Can you demonstrate what you are saying using this function $g(x) = -3(x-4)^2 + 100$?
10. **P20**: Yes… *(He quickly took the paper which was on the Table and started writing)*

I gave him the space to write while following what he was writing see Figure 43.

11. **P20**: Here, Sir….
12. **T-Math**: Okay. If I would say now determine the vertex, what will you do?
13. **P20**: mmmhhhh…. Isn't that you supposed to have the standard first? Yes, the standard form first, and then you can find the vertex.
14. **T-Math**: Okay.

From the transcribed one-on-one discussion, it can be noted that P20 is confused about the concept of vertex and the y-intercept. While trying to explain the concept of the vertex, he only stated one side of the definition: the maximum. As Childers and Vidakovic (2014) noted, the learner also thought that the vertex was referring to the maximum value of the graph. The word "maximum" is correct; it shows a limited understanding of the vertex and is not grounded. The learner ignored that the vertex can also be the minimum of the function.

Consequently, limiting the concept to one view demeans the relational meaning of the concept (Childers & Vidakovic, 2014). This notion of viewing the vertex as the maximum indicates that the learner's understanding of the parameter is fragmented. Moreover, the learner is over-reliant on the standard form and does not acknowledge that the other form can be used. This overreliance impedes the full grasp of the concept, as Kotsopoulos (2007) noted. Therefore, this notion of being over-dependent on one form inhibits the full development of 1.1, 1.2, 1.6, 2.1, 2.2, 2.3 and 2.6 indicators for understanding. This is why his work in Figure 4.20 was stuck as the learner tried to transit from vertex form into a standard form to determine the vertex.

The learner failed to write down the vertex correctly, although he managed to determine the y-intercept. Therefore, it cannot develop fully into the action conception, hindering the development of the other levels of understanding. Consequently, the learners' understanding of the vertex and y-intercept seem to cause conceptual obstacle among learners persistently.

However, navigating through the activities phase and classroom led to the realisation that learners' understanding of quadratic functions is fragmented. The learners' knowledge of being fragmented was due to their conceptual obstacle of range, intercepts, vertex, and transformation of quadratic functions. Similarly, these conceptual obstacles conform to Eraslan (2005) findings that learners have difficulty grasping the intercepts, vertex, and transformation. Eraslan found that learners struggled to explain the mathematical reason behind the horizontal and vertical shifts. The same findings were reached in a related study by Zazkis et al. (2003), which noted that learners struggled to give mathematical reasons to explain the transformation of $y = x^2$ into $y = (x - 3)^2$ or $y = x^2 - 3$. The same question was posed in a study by Eraslan, which was found that instead of the participants giving the mathematical reasons, the learners wanted to transit the function $y = (x - 3)^2$ from the vertex form to the standard form. Thus, such a method chosen by the learner provided enough reasons that the learner was over-reliant on the standard form.

The analysis and discussion from the activities phase informed classroom discussions and the unstructured interview session. Arnon et al. (2014) noted that the activities phase informs the classroom discussion phase. From the activities phase, learners seem to pose severe conceptual obstacles in grasping the full attainment of the quadratic function. Analysis and discussions revealed that the learners' understanding of the range, intercepts, vertex, and function transformations posed conceptual obstacles. These conceptual obstacles inhibited learning quadratic functions entirely. As Livy and Vale (2011) noted, learners must develop their mathematical structures and capacity to deconstruct concepts. These skills allow them to make connections within a mathematical concept. Therefore, if the links are made correctly, the learner will not confuse the vertex and the y-intercept of quadratic functions. As noted by Nielsen (2015), learners thought that the y-intercept was the vertex of the function, which impeded the attainment of the two concepts fully.

Consequently, analysing the findings from the study by Nielsen from the notions of Livy and Vale would mean that these learners posed underdeveloped knowledge of mathematical connections and an inability to deconstruct concepts. Consequently, this led to learners' lack of procedural flexibility. These notions, laid by Livy and Vale, are replicated in this study as the learners treated the understanding of vertex and y-intercept in isolation instead of as a coherent whole to build a more extensive mental structure, i.e., quadratic functions. As a result, I wanted to strengthen learners' understanding of quadratic functions through the exercise phase with the findings from activities and classroom discussions.

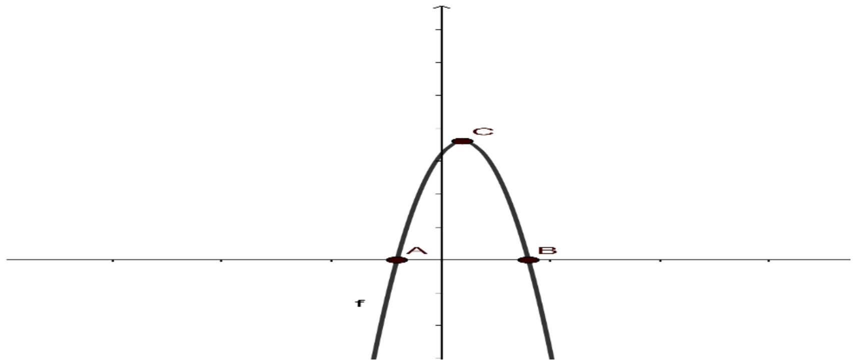
4.2.3. Exercises

In this phase, an exercise (test) was used to explore the impact of the activities and classroom discussions after the participants interacted with the learning tasks. Learners were given an exercise (test) to reinforce the activities and classroom discussion phases. Borji et al. (2018) state that the exercise task has to be fairly standardised to strengthen the activities and classroom discussions. Additionally, the task supports the continued development of mental constructions proposed by the precedent genetic decomposition (Borji et al., 2018). Subsequently, the exercise continued development by reinforcing learners’ conceptual understanding of intercepts, vertex, range and transformations. As a result, the exercise had four questions that nurtured knowledge of the abovementioned concepts. Therefore, the exercise is captured in Table 4.20 below.

Table 4.20: Reinforcing the activities and classroom discussion phases

Exercise (Test)

Sketched below is the graph of $f(x) = -2x^2 + 4x + 16$. A and B are x-intercepts of f. C is the turning point of f.



1. Calculate the coordinates of A and B.
2. Determine the coordinates of C, the turning point of f.
3. Write down the range of f.
4. The graph of $h(x) = f(x+p) + q$ has a maximum value of 15 at $x = 2$. Determine the values of p and q.

The structure of the exercise phase was arranged to probe deeper into learners' conceptual understanding of quadratic function concepts in intercept, vertex, range and transformations. Arnon et al. (2014) state that the exercise phase reinforces the concepts learned in the activities and classroom discussions. Therefore, in this study, it was necessary to give learners an exercise (test) to strengthen the quadratic function concept, which posed severe conceptual obstacles in the activities and classroom discussions. Subsequently, during the exercise phase, learners were given a test to write in the classroom, unlike the other studies that used the ACE teaching cycle. Since in most studies, the last phase is done as a homework activity (Borji et al., 2018; Voskoglou, 2013).

Conversely, in this study, the exercise phase was done as a test and in the classroom. Therefore, the test advanced to attain the 1.1, 1.4, 1.5, 1.6, 2.1, 2.2, 2.5, 3.1, 3.2, 3.3 and 3.4 indicators of understanding. To thoroughly get the level of learners' conceptual understanding of quadratic functions, I present a synopsis of learners' average marks for the exercise in Table 4.21.

Table 4.21: Learners' average marks per concept

Concept	Total marks	Average marks
Intercepts	3	2,2
Graph orientation (Vertex and range)	3	1,2
Transformation	3	0,8

The Table shown above discloses concepts in which learners posed severe conceptual obstacles the most. However, to establish a good comparison between Task 0 and the exercise (test), I have conceptualised vertex and range to graphical orientation; as in Task 0, it was explored in such a format. Given the synopsis of learners' average marks, it can be noted that learners continued to pose conceptual obstacles of vertex, range and transformations compared to understanding intercepts. Unlike in the activities phase, they posed conceptual obstacles with intercept, transformation, and graph orientation. Conversely, learners posed various understandings of these concepts in the exercise phase. For example, they posed (1) fragmented knowledge of intercepts and vertex; (2) underdeveloped knowledge of transformations; and (3) undeveloped knowledge of range concept. A related study by Hattikudur et al. (2012) found that conceptual obstacles differed in context when dealing with functions posed by the participants. This notion laid out by Hattikudur et al. is evident in this study as the learners' average marks are not zero in each of the concepts under exploration.

In contrast to learners' understanding of Task 0, the exercise task improved their knowledge of intercepts and graph orientation. Thus, the participants seem to have attained the action, process, object, and schema indicators of understanding when I use the synopsis of the average marks. Nevertheless, the improvement and the concept of range and transformation remained challenging for learners. To explore their understanding of quadratic functions well, I have drawn a graph in Figure 4.19 to demonstrate the results of Task 0 versus the exercise. For me to be able to draw the bar graph, I had to first convert the marks of Task 0 into 10 to correspond with the test.

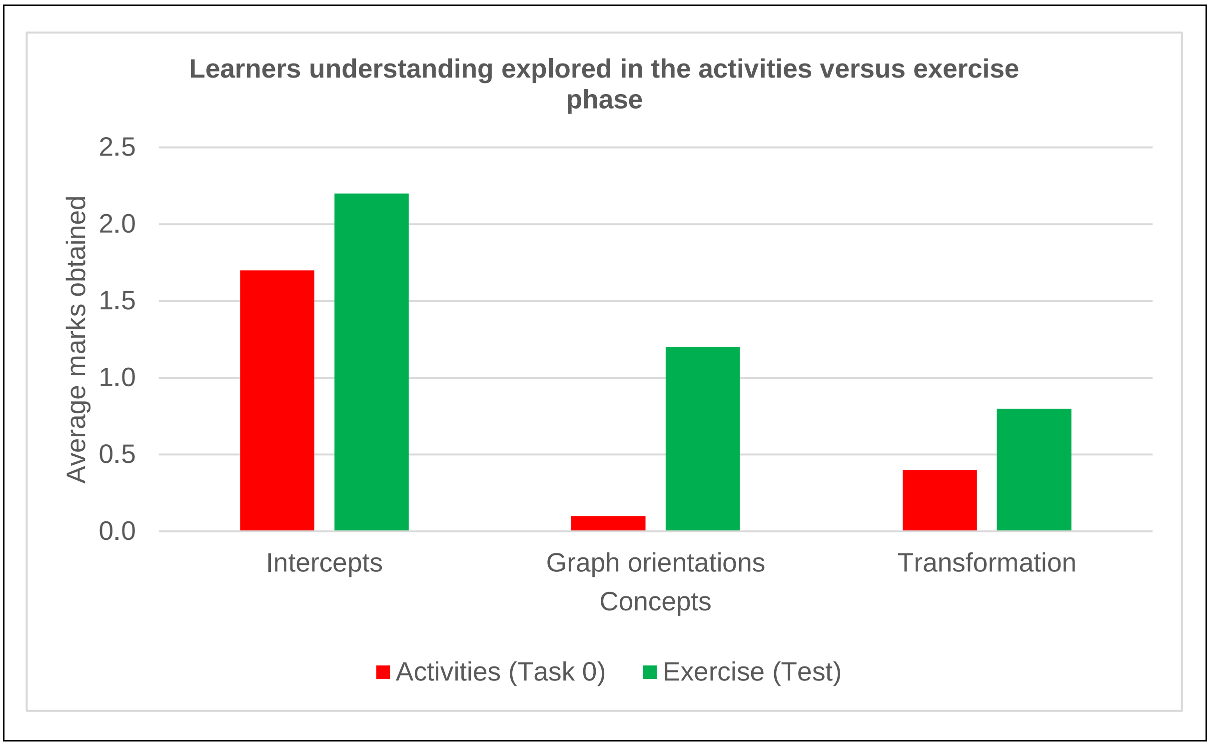


Figure 4.19: Learner's performance in Task 0 versus the exercise

The above graph demonstrates the average performance between Task 0 and the test. The results demonstrate an improvement in learners' conceptual understanding of the quadratic function concept, as shown in Figure 4.19. The comparison of learners' conceptual understanding of intercepts revealed an improvement in the test compared to Task 0. Learners interacted with the y-intercept in Task 0 and x-intercepts in the test. As noted by Hattikudur et al. (2012), determining the y-intercept is an easy process, as it involves focusing on one value while the other is zero. Yet, it is evident from Figure 4.19 that learners failed to grasp the understanding of the y-intercept but managed to show an improvement when interacting with the x-intercepts. The question of determining the x-intercepts is not flexible as determining the y-intercept. This is so because x-intercepts involve the comprehension of various methods such as factorisation, simplification, operation skills and finally arriving at a solution. Moreover, the participants struggled with understanding the grasp orientation and transformations.

Therefore, given this, I wanted to probe deeper into learners' conceptual understanding of quadratic functions in order to track the impact of the ACE cycle on the development of their understanding. Subsequently, to explore their understanding, I sampled learners P20, T24, W27, X28 and Y29 in order to explore their knowledge of quadratic. The sampling of these scripts was guided by the work of McMillan and Schumacher (2001). The two authors explicitly state that sampling should be done to choose a few grain-sized participants to studied to yield more insights about the concept. Hence, being guided by McMillan and Schumacher, I wanted to delve deeper into these sampled learners' conceptual understanding of intercepts, vertex, range, and transformation.

In exploring learners' conceptual understanding of quadratic functions from the test task, it is important to look at participants' responses to the test as attached in Table 4.20. To begin the exploration, I start by probing deeper into learners' conceptual understanding of intercepts. Thus, I sampled learners P20 and T24 to probe their knowledge of determining the x-intercept. The learners were given a function in a standard form, i.e., $f(x) = -2x^2 + 4x + 16$ and were required to determine the x-intercepts of the function. Literature on quadratic functions reveals that learners posed various conceptual obstacles such as transcription, procedural, inconsistent factorisation, and overreliance on the procedural embodiment. For example, Godden et al. (2013) noted that their participants posed transcription conceptual obstacles as learners were given a quadratic in form $3x^2 = 2(x+2)$ and was required to determine the x-intercepts. Conversely, learners transcribed $3x^2 = 2(x+2)$ as $3x^2 = 2(x+3)$, which led to the failure to determine the x-intercepts. Thus, ignorance of these learners from the study by Godden et al. meant their procedural flexibility was undeveloped. In a related study by Memnun et al. (2015), learners posed conceptual obstacles of comprehending the meaning of $f(x)$ and y as it set to give the function. Similarly, Didiş and Erbas (2015) found that learners could not factorise functions where $a \neq 1$, which repeatedly requires them to understand factors in functions thoroughly. As such, some used the square root to determine the x-intercepts, which led to one intercept instead of two intercepts. For example, $m^2 = 9$ was computed $m = \sqrt{9} = 3$, which demeans the essence of quadratic functions. Thus, from these studies, I wanted to delve deeper into learners P20 and T24’s understanding of intercept as it is seen as a muddled concept in quadratic functions. Therefore, the expected solution for learners to understand the x-intercepts is shown in Table 4.22.

Table 4.22: Expected learners' solutions to conceptual understanding the x-intercepts

$f(x) = -2x^2 + 4x + 16$	Indicators *1.1, 1.3*
x-intercepts	
Let $y = 0$	Indicator *1.3.1*
$-2x^2 + 4x + 16 = 0$	Indicator *1.3.2*
$x^2 - 2x - 8 = 0$	
$(x-4)(x+2) = 0$	
$\therefore x = 4$ and $\therefore x = -2$	Indicator *1.3.3*
$\therefore A(-2; 0)$ and $B(4; 0)$	Indicator *1.3.4*

Learners responded to the question of determining the intercept, and the participants understood the concept. Thus, to begin delving deeper with learners P20 and T24's understanding of intercept, I investigated their answers, which give a synopsis of their comprehension as captured in Figure 4.20.

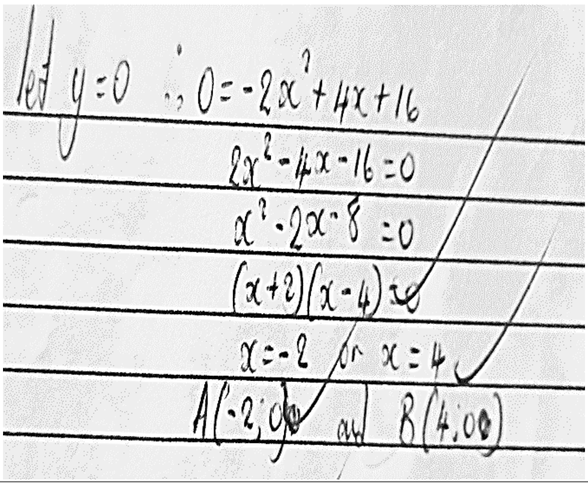
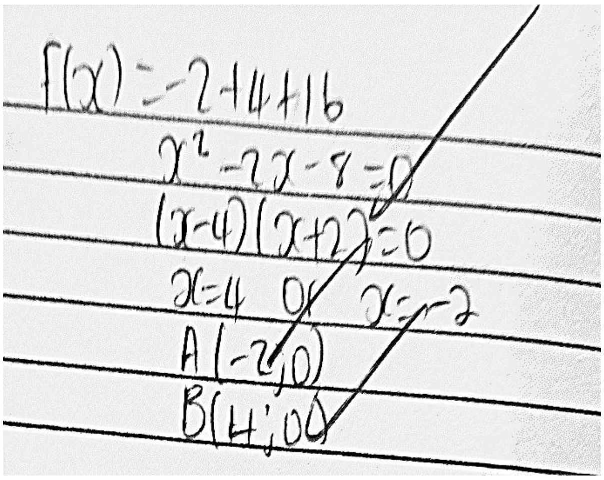
Learner T24	**Learner P20**
	

Figure 4.20: Learners' conceptual understanding of intercepts

Exploring learners' conceptual understanding, I looked at them through the lens of genetic decomposition and related literature on quadratics and understanding. The learners managed to determine the x-intercepts of the functions but posed some conceptual obstacles in their workings. Nevertheless, their posed conceptual obstacle, from the final solutions, it is evident that they nurtured the attainment of 1.1 and 1.3 indicators for understanding. For example, learner T24 started the question with a good step as the participant set $y = 0$ to determine the intercepts. Thus, such skill exhibited 1.1 and 1.3 indicators of understanding as the learner demonstrated knowledge of the forms of quadratic functions, which posit 1.1 indicators of understanding. Subsequently, such a position of knowledge implied that the learner comprehends the forms of quadratic functions, which meant that they had a prior conceptual understanding of the forms of these functions (Kilpatrick et al., 2002). Moreover, learner T24 substituted $y = 0$ into the function to determine the x-intercepts of the function as he efficiently and accurately computed the equation by factorisation and later simplified it to obtain the x-intercept. Therefore, while doing these steps of flexibly conducting procedures to determine the x-intercepts exhibited procedural fluency. Consequently, learner T24, coupled with conceptual understanding and procedural fluency in determining the x-intercepts nurtured and attained strategic competence as he could formulate that y should be zero to determine the x-intercepts, and showed an ability to solve for x to determine the x-intercepts (Groves, 2012). Getting more profound with the answer of learner T24, after he made parameter $a > 0$ by multiplying throughout the equation by a negative one to obtain $2x^2 - 4x - 16 = 0$. Such competence implies that the learner is efficient about the procedure and can carry it appropriately to accurately determine the x-intercepts, which nurtures advanced conceptual understanding. Kilpatrick et al. (2002) assert that conceptual understanding is an ability to comprehend mathematical concepts (intercepts and the forms of quadratic functions), operations (divided, substituted, and simplified), and relations (can note the connection between functions and equations). Additionally, the learner divided the equation by 2 to obtain $x^2 - 2x - 8 = 0$; thus, this is what Didiş and Erbas (2015) noted in their study that participants usually found it difficult to work with quadratics where $a \neq 1$. Therefore, the learner opted for working with $a = 1$ instead of $a = 2$. However, the divergent of opting for $a = 1$ instead of $a = 2$ does not demean the learner T24's procedural flexibility in determining the x-intercept. The learner showed some schema of ability to make the value of a positive by multiplying and dividing throughout by positive 2. Therefore, as put by Rittle-Johnson (2017) that procedural flexibility is a long ongoing process that requires the integration of making connections as seen from the learner's work that he substituted, multiplied, divided, simplified, and later substituted the values of the x int the function to obtain the corresponding values of y. Therefore, having presented such based-on learner T24 working on determining the x-intercepts means that the participant exhibited a complete grasp of procedural flexibility and attained the 1.1 and 1.3 indicators of understanding.

In contrast to learner T24, I delved into learner P20's understanding of intercepts of the quadratic function. The learner started by omitting the variables of the function where the participant wrote $f(x) = -2 + 4 + 16$. This option posed transcription conceptual obstacles. As noted by Godden et al. (2013), participants in their study posed transcription conceptual obstacle as they wrote $3x^2 = 2(x + 2)$ as $3x^2 = 2(x + 3)$. These conceptual obstacle yields incorrect solution as the participant will carry out a procedure using an incorrect function, impeding the learner's procedural fluency. Consequently, learner P20’s writing of $f(x) = -2 + 4 + 16$ indicates that the participant posits fragmented knowledge of arithmetic and algebraic conceptions, which impedes the full development of 1.1 indicators of understanding. Such fragmentations inhibit learners' knowledge of noting the relations between arithmetic and algebraic concepts. Li (2010) indicated that learners posed undeveloped knowledge of variables and opted to demean the notion that a variable is a placeholder. Subsequently, the learner who disputes the variable x disputes the relevance of functions, which inhibits the grasp of the conceptual meaning of a variable and a constant. Nonetheless, learner P20 did not proceed to work with $f(x) = -2 + 4 + 16$ as in the second step; the participant wrote $x^2 - 2x - 8 = 0$. From this step, the learner correctly factorised the equation and applied the null factor method by writing that $(x - 4)(x + 2) = 0$, and managed to determine the x-intercepts. However, learner P20’s understanding of intercepts is fragmented as he neglected the variable in the first step but managed to compute to have the intercepts. Therefore, as the learner managed to factorise and apply the null factor method, this meant that the participant exhibited a limited procedural fluency as he was not efficient enough to work a function and demean it to $f(x) = -2 + 4 + 16$. Consequently, the limitation of not being fluent in procedures inhibits the full attainment of the 1.1 and 1.3 indicators of understanding.

The second question focused on the vertex of the function. The expected learners' solution to their understanding of the vertex is captured in Table 4.23.

Table 4.23: Expected solution on learners' conceptual understanding of the turning point

$f(x) = -2x^2 + 4x + 16$	Indicators *1.1, 1.5, 2.1, 2.2, 2.6, 2.7, 3.1, 3.2, 3.3, 3.4*
$a = -2$, $b = 4$	
$$x = -\frac{b}{2a}$$	Indicator *1.5*
$$x = -\frac{4}{2(-2)} = 1$$	Indicator *1.5.1*
Substitute $x = 1$ into $f(x)$	Indicator *1.5.2*
$f(1) = -2(1)^2 + 4(1) + 16 = 18$	Indicator *1.5.3*
$\therefore TP\ (1; 18)$	

In this question, I again sampled the work of P20 and T24 to track their understanding of the turning point concept. The learners' work is captured in Figure 4.21.

Learner P20	**Learner T24**

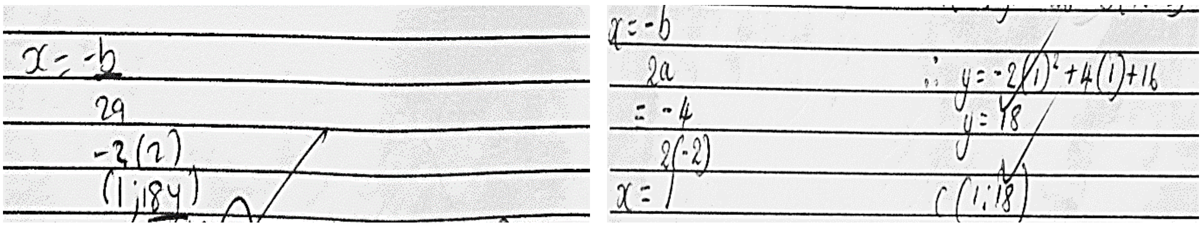


Figure 4.21: Learners' conceptual understanding of vertex concept

The learners interacted with the vertex concept; some posed understanding, and others posited conceptual obstacle with the concept. For example, learner P20 started by writing the correct formula to determine the vertex. Such a step exhibited instrumental understanding as put by Skemp (1976), and subsequently exhibited 1.1, 1.5 and 1.6 indicators of understanding. The learner correctly noted that the formula $x = -\frac{b}{2a}$ (1.5 indicators) needs to be used as the function is in a standard form (1.1 indicators). Contrarily, in the vignette learner, P20 failed to exhibit procedural fluency and strategic competence (Kilpatrick et al., 2002). The learner was unable to substitute into the formula; this meant that his prior conceptual understanding of

$x = -\frac{b}{2a}$ is fragmented, which impedes the grasp of the vertex of the function. Moreover, the learner could not carry the procedure efficiently and flexibly to accurately determine the solution of the x-coordinate of the vertex. Instead, learner P20 substituted and wrote $-2(2)$, which is no longer in a fraction form as it was given in the formula, that is, $x = -\frac{b}{2a}$. Subsequently, the learner wrote the vertex as $(1; 18y)$; this meant that the participant lacked a thorough knowledge of the variable, as noted by Li (2010). This led to learners' understanding of vertex to be underdeveloped knowledge of vertex as he could not correctly substitute into the formula and failed to note the vertex is $(1; 18)$ instead of $(1; 18y)$. Therefore, given that the learner was unable to accurately substitute and appropriately carry out the procedure of simplifying the solution and being efficient enough to accurately determine the x-coordinate of the function implied that he/she posed underdeveloped procedural fluency, and his conceptual understanding of vertex was fragmented. Therefore, given this case, it meant that learner P20's understanding of vertex is at a fragmented procedural-conceptual understanding stage, which impedes the full development of the iterative bidirectional nature of understanding. Subsequently, the absence of the bidirectional understanding of vertex inhibits the attainment of 1.1, 1.4, 1.5 and 1.6 indicators of understanding.

In contrast to learner P20, who did not utilise the formula accurately, learner T24 determined the vertex of the function. For example, learner T24 wrote the formula $x = -\frac{b}{2a}$ and then correctly substituted into the formula as $x = -\frac{4}{2(-2)} = 1$. The learner was efficient enough to correctly substitute the values of a and b into the formula. Moreover, he accurately carried out the procedure appropriately and flexibly to reach the correct answer of the x-coordinate of the vertex. Thus, skills imply that the learner posits procedural fluency in determining the x-coordinate of the vertex. This meant that he held a prior conceptual understanding of the axis of symmetry as he could correctly utilise the formula $x = -\frac{b}{2a}$. Consequently, the presence of these types of understanding that is procedural-conceptual nurtures the development of strategic competence of the learner as he could solve for the y-coordinate of the vertex by substituting $x = 1$ into the function to obtain $y = 18$, and he appropriately wrote the final answer in an intercept format as $(1; 18)$. Thus, such nurtures the strategic competence of the learner to determine the x-coordinate of the vertex. Kilpatrick et al. (2002) state that strategic competence is the ability to formulate and correctly compute to determine the answer. Consequently, learner P20 could correctly execute the computation to reach the final solution of x as $x = 1$. Additionally, learner T24 went on to substitute the value of $x = 1$ into the function to obtain the y-coordinate of the vertex. The learner substituted into the function as $y = -2(1)^2 + 4(1) + 16 = 18$. As a result, the learner exhibited adaptive reasoning, which is the logical thought to reflect and justify (Groves, 2012) the stage of the mathematical proficiencies of a quadratic function. The learner could use his logical thought to reflect that to get the y-coordinate of the vertex $x = 1$ has to be substituted into the function. The learner logical determined the x-coordinate first, and secondly, he substituted the value of x into the vertex to efficiently obtain the corresponding y-coordinate. Subsequently, learner T24 posed flexibility as he could cite the correct formula, substitute it to get the x value of the vertex, and substitute the value of x to obtain the y value of the vertex as $(1; 18)$. Therefore, learner T24 posed developed knowledge of determining the vertex as he could portray 1.1, 1.5, 2.1, 2.2, 2.6, 2.7, 3.1, 3.2, 3.3, 3.4 indicators of understanding.

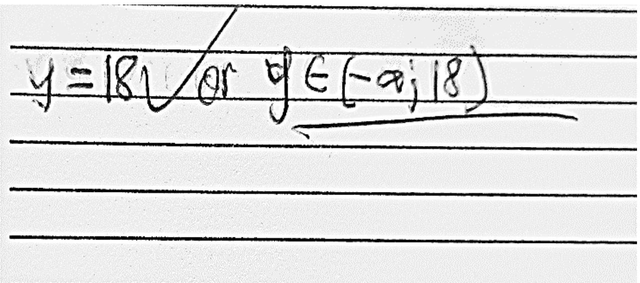
Questions three and four required the learners to write the range, transform the function, and apply transformations to obtain a new function. Almost 70% of the learners struggled with the writing of the range, and others did not attempt to write the fourth question. The expected solution to these two questions is captured in Table 4.24 below.

Table 4.24: Expected solution of the range and transformation

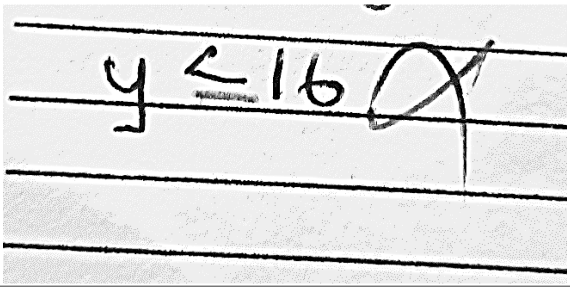
Range: $y \leq 18$ or $y \in (-\infty; 18]$	Indicators 1.1, 2.1, 2.2, 2.5 Indicator 2.5.1
The graph of $h(x) = f(x + p) + q$ has a maximum value of 15 at $x = 2$. Determine the values of p and q The turning point of $f(x)$: $(1; 18)$ The turning point of $h(x)$: $(2; 15)$ $\therefore p = -1$ and $q = -3$	Indicators *1.1, 1.4, 1.5, 1.6, 2.1, 2.2, 2.3, 2.6, 2.7, 2.10, 3.1, 3.2, 3.3, 3.4*

Amongst the learners that struggled with the two questions include learners K15, P20, T24 and W27. Thus, the answers are captured in Figure 4.22.

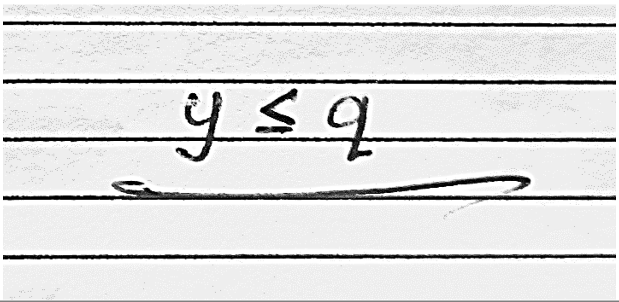
Learner K15



Learner P20



Learner T24



Learner W27

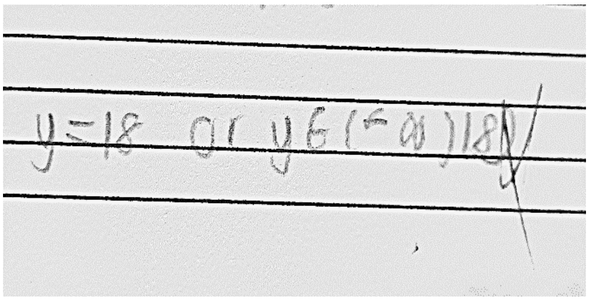


Figure 4.22: Learners' conceptual obstacle of range concept

All sampled learners did not correctly respond to the question, and their solutions were the same. Consequently, this meant that these learners were unable to attain 1.1, 1.4, 1.5, 1.6, 2.1, 2.2, 2.3, 2.6, 2.7, 2.10, 3.1, 3.2, 3.3 and 3.4 indicators of understanding. For example, learners K15 and W27 had an idea that the range involves the y-intercept of the vertex but were not efficient enough to accurately use the correct notation to solve the mathematical problem of determining the range. Additionally, learner K15 wrote $y = 18$ or $y \in (-\infty; 18)$. This is enough to prove that the learner posed some underdeveloped knowledge of the range, as she wrote that $y = 18$, which no longer gives the range of the function but refers to a stationary point on the graph. Consequently, the learner posits a conceptual obstacle of what is the range and linear function. Subsequently, this suggested that the learner lacked adaptive reasoning of what range is in functions. Kilpatrick et al. (2002) stated that adaptive reasoning allows learners to reflect and justify their solutions. Therefore, if the learner were posed with adaptive reasoning, they would have reflected that $y = 18$ is a linear function; however, her silence on that solution implied that he posits undeveloped conceptual knowledge of a function. Moreover, learner K15 failed to note the interference of linear functions in her answer but proceeded to write that the range of the function is also $y \in (-\infty; 18)$. This answer posed that the learner knew what range is but failed to provide a mathematical explanation of how the range is found; instead, she used the word "or" while giving the answers. Subsequently, learner K15 was underdeveloped as she could use the correct notation to give the range and thought that the range was a linear function or asymptote. In contrast, learner P20 determined the range as $y \leq 16$. The learner equated the range to the y-intercept of the function. As noted by Parent (2015), learners posed a conceptual obstacle of vertex and y-intercept. Parent's notion was replicated in this study as the participant knew that range deals with the value of y but failed to note which y it deals with. Such conceptual obstacle demonstrates an undeveloped knowledge of range as the learner focused on y without considering which y value should be considered. These conceptual obstacles inhibit the grasp of 1.1 and 2.5 indicators of understanding. Subsequently, this posed that the learner still has underdeveloped knowledge of vertex and y intercept, which impedes the attainment of the range concept.

Consequently, through the exercise, it can be noted that range and transformation persist in impeding learners' conceptual understanding of quadratic functions. Some learners posit undeveloped, underdeveloped, and fragmented knowledge of the concepts. For example, learners assumed that range is a linear function by using an equal sign instead of inequalities, that is, $<, >, \leq,$ or $\geq$. Additionally, some failed to use the correct brackets to give the range of the function. For example, the learner misused the brackets of exclusion only without considering that the range of a function is a set of all values that the function takes; hence it cannot be limited to one value. Moreover, some posed conceptual obstacles with transformation, which was inhibited by learners' understanding of the forms of quadratics. In the next section, findings from the ACE cycle are synthesised to give the principal findings of the study.

4.3. SYNTHESIS OF RESULTS AND FINDINGS

4.3.1. Principal findings emerging from the ACE cycle

In this section, I synthesise the research results of the study to offer the principal findings, therefore conceptualising findings from the activities, classroom discussions, and exercises, not in isolation but as a whole entity. This is because the ACE cycle is intertwined and interwoven, meaning each phase informs the proceeding sections (Arnon et al., 2014). Subsequently, to achieve the notion posed by Arnon et al., I have synthesised the principal findings emanating from the activities, classroom discussion and exercises, resulting in the main findings of the study. Thus, the use of APOS and related literature on quadratic functions and understanding led to the realisation of the purpose I set in the beginning: “to explore the role of the ACE teaching cycle in improving Grade 12 learners’ conceptual understanding of quadratic functions.” Subsequently, I reached some principal findings by implementing the ACE cycle.

Firstly, learners seem to grapple at the action level relating to the knowledge of the properties of quadratic functions. These conceptual obstacles include, among others, underdeveloped understanding of the forms of quadratics. As a result, an underdeveloped understanding of the forms led to learners being over-reliant on the standard form only. However, such dependency on the standard form fosters the learners to hold a primitive definition of a quadratic function and not be efficient enough with the concept. As posited by Parent (2015), quadratic functions are primarily defined as $f(x) = ax^2 + bx + c$ when $a \neq 0$. Consequently, such a definition limits learners' conceptual understanding of quadratic functions to standard form only. Subsequently, this limitation inhibits the attainment of adaptive reasoning (Groves, 2012; Kilpatrick et al., 2002) of quadratic functions. Adaptive reasoning of quadratic functions affords learners with a capacity to think logically that quadratics can be given in the standard, vertex, and factored form; to provide reflections on the concept and know that each form reveals specific critical points of the function; and to justify and offer explanations apply the transformation concept. Therefore, it is evident that being over-reliant on the standard form will limit learners to being fluent with the y-intercept only. Therefore, some learners thought that the vertex is the y-intercept of the function. Ubah and Bansilal (2018) stated that learners preferred the standard form to the vertex form when interacting with questions. Due to this overdependence on the standard form, Ubah and Bansilal noted that learners posed conceptual obstacles with the y-intercept of the standard form and the y-coordinate of the vertex of the function. Thus, these conceptual obstacles impede the iterative view of understanding the concept of the forms of quadratic functions. Therefore, the absence of such understanding limits learners not to see quadratic functions as applicable. Consequently, this hinders the notion laid by Benning and Agyei (2016) that the knowledge of quadratic functions is essential in mathematics as it forms a bridge to understanding other concepts such as differential calculus.

Moreover, learners grappled with parameters as they could neglect their (parameters) importance. They misunderstood the parameters which give the y-intercept, and the y-coordinate, of the function. Thus, such conceptual obstacles impede the grasp of the vertex of the function and the understanding of the y-intercept. As seen by Fonger et al. (2020), learners thought that the y intercept of the standard form was the same as the y-coordinate of the vertex. Additionally, the absence of an understanding of parameters inhibits procedural flexibility of quadratic functions. For example, it was noted in this study that some learners posed conceptual obstacles with the parameter a. Subsequently, the absence of knowledge of parameters impedes their understanding of correctly transitioning from the standard, factored and vertex forms.

Similarly, Ellis and Grinstead (2008) found that learners posed a conceptual obstacle with parameter a, and that they thought that parameter a was the slope of the function. Moreover, Ellis and Grinstead noted that learners grappled with parameters as they felt parameter a does not influence the graph. In a related study by Nielsen (2015), it was also observed that learners thought that the parameter a in the quadratic function gives the slope of the graph. Therefore, negligence in understanding parameters yields conceptual obstacles with the knowledge of graphing the quadratic functions. These included fragmented knowledge with horizontal shifts and an unclear understanding of the equation of the line of symmetry, which failed to draw the line of symmetry. The difficulties with horizontal shifts were evident in the work of Zazkis et al. (2003). In their study, it was noted that learners struggled to translate functions correctly (Dede & Soybas, 2011; Yulian, 2018).

Moreover, learners failed to understand the concept of range as they could not link the connection between the range and vertex of the function. These pitfalls resulted in their inability to plot the quadratic functions as their knowledge was fragmented and at a limited action conception of understanding. The failure to grasp the knowledge of properties of quadratic function poses a challenge for learners to fail to enter the process of conception, which hinders the object, and the resultant schema cannot be nurtured.

Secondly, learners posed conceptual obstacles to quadratics while interacting with quadratic functions. These conceptual obstacles include failure to simplify the function determining roots due to pitfalls in algebraic and arithmetic skills. The lack of algebraic and arithmetic skills led to challenges in applying product rules with negative numbers, failure to factorise, and pitfalls posed in completing the square. Bossé and Nandakumar (2005) also saw the difficulties relating to product rules, and found that learners could not correctly product rule, especially if $a \neq 1$. As also seen in the work of Kotsopoulos (2007), learners opted to cancel the negatives. Moreover, learners were shown to have a fragmented understanding of the brackets in algebra, and conceptual obstacle concerning the meaning of $f(x)$ *or* $f(x-7)$*,* which amounted to a limited understanding of the notation used in functions. Furthermore, learners confused the formula for the line of symmetry with the quadratic formula, posed a limited understanding of the properties of quadratic functions and linear functions, and used the word area interchangeably with the axis while referring to the line of symmetry.

Lastly, learners faced difficulty making connections between the forms of quadratic functions due to a deficiency in solving techniques. These difficulties include failure to transit from graphs to algebraic form due to overreliance on one form. Knuth (2000) shows that learners relied on one form due to the absence of knowledge of various forms of representations. The transition from one form to another seems to be muddled with difficulties in arithmetic and algebraic operational skills. Moreover, learners equated c and q*,* which hindered the development of understanding of the transition from one form to another, especially from standard to vertex form. The notion of equating c and q implied that learners did not fully grasp the role parameters in functions. As seen by Didiş et al. (2011), the learners posed a conceptual obstacle of the parameters, and assumed that if parameters $b = 0$ and $c = 0$ meant, this was not a quadratic function.

Furthermore, learners thought of the line of symmetry as the value of y, which is given by the x value of the vertex. In contrast, others incorrectly misused the brackets without understanding their meaning when writing the range. The misuse of the brackets meant that they were grappling with notation. The understanding of notation in mathematics is vital. However, Adu-Gyamfi et al. (2019) found that most learners posed fragmented knowledge of notation and could not fully simplify it. Therefore, these findings meant that they were participants operating at various levels of understanding as others posed undeveloped, underdeveloped, fragmented, and developed knowledge of quadratic functions.

4.3.2. Conceptualising the understanding of the quadratic function

The analysis and discussion of the data in the ACE cycle paved a route for conceptualising the conceptual understanding of quadratic functions. Conversely, it is noted from the literature that the concept of understanding is fluid (Dhlamini & Luneta, 2016; Groves, 2012; Hiebert & Lefevre, 1986; Kilpatrick et al., 2002; Rittle-Johnson, 2017; Skemp, 1976; Star, 2005). Some hold that knowledge is the same as understanding and can be used interchangeable (Star, 2005). Subsequently, I build from the view of Star that knowledge and understanding can be used identically. Therefore, I have used knowledge and understanding interchangeably in my study, anchored by the notion laid by Star.

Kilpatrick et al. (2002) define conceptual knowledge as an acquaintance of mathematical facts, concepts, procedures, and the connection among the concepts. Moreover, Kilpatrick et al. assert that conceptual understanding is the learners' grasp of mathematical facts, which are connected to multiple conceptions. Drawing from these two assertions by Kilpatrick et al., it can be noted that knowledge deals with the comprehension of mathematical facts while understanding deals with the grasping of mathematical facts. Furthermore, knowledge nurtures the connection of these mathematical facts. Skemp (1976) asserts that understanding can be two-fold: knowing how and why. As a result, Skemp argues that there are two types of understanding: instrumental and relational understanding. Instrumental understanding is knowing and applying the procedure, whilst relational understanding is meaningful comprehension of why the procedure works and how it connects to other procedures.

In contrast, this view posed by Skemp, Hiebert and Lefevre (1986) enriched the conception of understanding in mathematics by first asserting that instrumental is procedural and relational is conceptual. Procedural knowledge is a comprehension of procedures and the rules of manipulation, and conceptual understanding is knowledge enriched in relationships, and thus the schemas are linked to form a coherent whole (Hiebert & Lefevre, 1986). In a related study, Kilpatrick et al. (2002) extend the conceptions of understanding and procedural further by asserting that conceptual understanding is the comprehension of concepts, and procedural understanding is the skill of carrying out the procedures (Dhlamini & Luneta, 2016; Groves, 2012; Kilpatrick et al., 2002). Another study by Star (2005) advanced the attainment of deep procedural understanding. In pursuit of what nurtured deep procedural understanding, Star asserted that this understanding refers to comprehension, flexibility, and critical judgement. Therefore, such conception no longer nurtures only procedural understanding but also advances conceptual understanding. However, such a conception is challenging as it only stresses three tenets of understanding: comprehension, flexibility and critical, which neglects the view that understanding develops in an iterative process. Subsequently, Rittle-Johnson (2017) extends what Star termed as deep procedural understanding, thereby asserting that the attainment of conceptual understanding nurtures procedural and contrariwise. Therefore, Rittle-Johnson maintains that knowing more than one procedure and carrying them flexibly, accurately, efficiently, and appropriately refers to procedural flexibility.

Consequently, I draw from Rittle-Johnson’s work to conceptualise what it means to understand quadratic functions flexibly. The learners' conceptual understanding of quadratic functions unearths what they are and what makes them. Quadratic functions are polynomials of the form $f(x) = ax^2 + bx + c$ where $a \neq 0$ and the values of a, b and c are integers. As a result, a quadratic function can also be expressed in a vertex form as $f(x) = a(x + p)^2 + q$ and in a factored form as $f(x) = a(x + x_1)(x + x_2)$. Subsequently, these forms of quadratic functions reveal specific critical points regarding the function. For example, the standard form indicates the y-intercept of the function. Hence, the learners' understanding of y-intercept requires them to: (1) know the forms of quadratic functions; (2) be able to flexibly transit from one form to another efficiently and accurately to appropriately carry out the procedure to solve for the value of the y-intercept; (3) and eventually, demonstrating the conceptual meaning of the y-intercept. Additionally, the vertex form reveals the turning point of the graph. For example, the learner's conceptual understanding of the vertex permits him/her to note the connections between the turning point and the axis of symmetry. The vertex is given as $(p; q)$ and the value of p gives the axis of symmetry as $x = p$. Moreover, the value of p is determined as follows in the standard form $x = -\frac{b}{2a}$. Therefore, understanding the vertex and axis of symmetry is iterative as each complements the other. Thus, this notion fosters the connection of the concepts in quadratic function. Furthermore, the factored form reveals the x-intercepts of the function. Therefore, for one to compute the x-intercept, it means that they should tap into their prior conceptions of quadratic equations. Thus, the conceptualisation of quadratic functions implies that the learner posits procedural flexibility of quadratics as they can flexibly, accurately, efficiently, and appropriately utilise the forms to comprehend the concepts of vertex, y-intercept, x-intercepts, an axis of symmetry, and transformation. Thus, the quadratic function understanding is the flexibility of the learner to comprehend their forms as they posit critical points and appreciate them as coherently linked schemas of interrelated concepts.

Therefore, to conceptualise what it means to understand, I hypothesise a model that captures conceptual understanding of quadratic functions. The model is anchored by literature on understanding. Thus, it is a hypothesis and still permits revision.

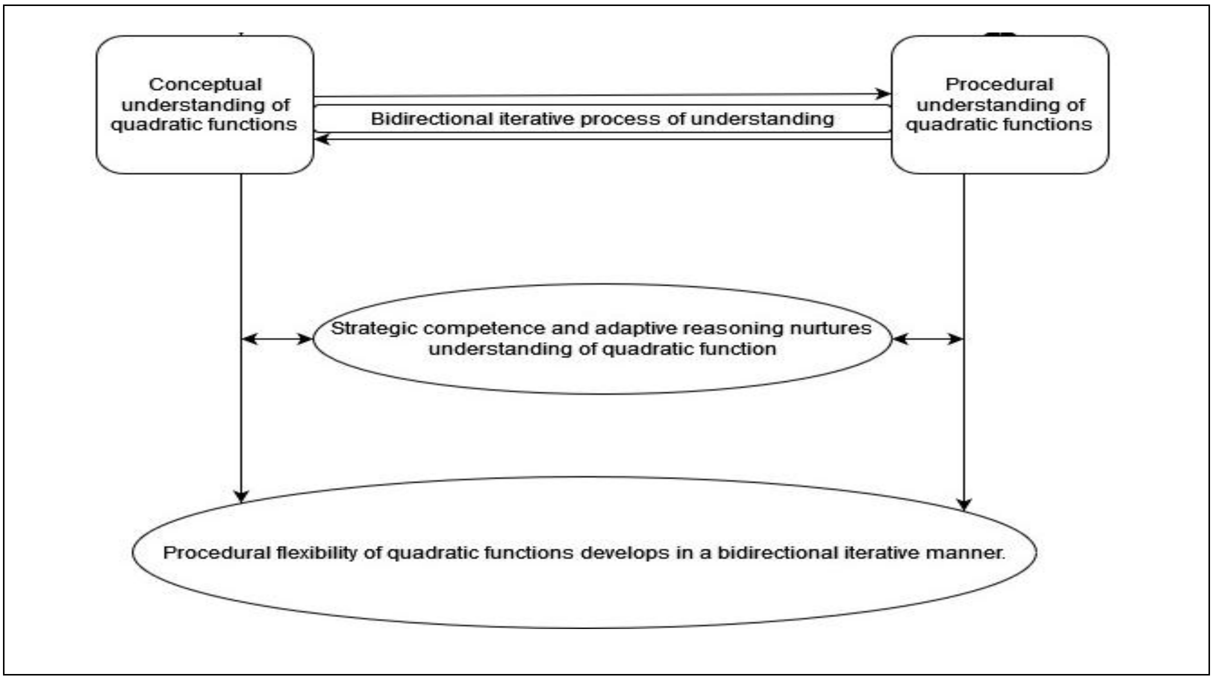


Figure 4.23: The iterative bidirectional model of conceptual understanding

I conceptualise that procedural flexibility captures both types of understanding, i.e., instrumental, and relational, in an iterative manner. Additionally, in comprehending quadratic functions, I hold that a concept first nurtures procedural knowledge and vice versa. Moreover, procedural knowledge is more than formulas, but it is a cognitive process that a learner undergoes to nurture conceptual knowledge. Therefore, procedural knowledge entails the efficiency and ability to carry out procedures to determine solutions accurately and appropriately. Thus, these flexibility skills nurture conceptual understanding as they complement strategic competence and adaptive reasoning in quadratic functions. Subsequently, through this conceptualising of understanding, I present the model below as a hypothesis of what it means to understand quadratic functions (see Figure 4.23).

4.4. SUMMARY OF THE CHAPTER

This chapter dealt with the analysis of the collected data. To ensure the validity of the data, triangulation was adhered to. The process of data analysis was guided by the theoretical framework of the APOS theory and related literature on understanding. The chapter has three sections: data analysis and discussion, synthesis of principal findings and the summary of the chapter. The analysis and discussion section informed the synthesis of the principal findings through the activities, classroom discussions and the exercise phase. The next chapter presents the conclusions and recommendations of the study.

CHAPTER FIVE: RECOMMENDATIONS AND CONCLUSION

5.1. INTRODUCTION

Literature reveals that learners' conceptual understanding of quadratic functions is undeveloped, underdeveloped, and fragmented, which impedes the full grasp of the concept (Parent, 2015; Nielsen, 2015; Eraslan, 2005; Ellis & Grinstead, 2008). For example, Parent (2015) noted that a result of overdependence on the standard form over the vertex and factored form inhibited learners' conceptual understanding of the y-intercept of the function versus the y-coordinate of the vertex. In a related study, Nielsen (2015) noted the interference of the linear function concepts. Similarly, Ellis and Grinstead (2008) observed that learners thought that parameter a refers to the slope of the function. Subsequently, unfocused attention to remedy these conceptual obstacles impedes learners' comprehension of the concept.

Consequently, the study sought to explore how learners' conceptual understanding of quadratic functions could be improved through the ACE teaching cycle. The study will potentially add knowledge to the literature on the learners' conceptual understanding of quadratic functions. Moreover, the study paves the way for further exploration of how APOS can develop learners' conceptual understanding of quadratic functions. To achieve this, chapter five is divided into six sections. I begin by presenting the research design and methods that the study utilised to pursue the purpose. Additionally, the findings of the study are interpreted, and subsequently, the recommendations, contributions and limitations of the study are presented in subsequent sections. Lastly, the conclusion is drawn, and the summary of the chapter is made.

5.2. RESEARCH DESIGN AND METHOD

In this study, I adopted an interpretive approach to qualitative research and an exploratory case study design. Therefore, I employed the constructs of ACE teaching cycle to collect qualitative data by conforming to Merriam's (1998) case study design. Thus, qualitative data were collected using activities and exercises that conform to documents held by Merriam. Moreover, I used classroom discussions which conform to Merriam’s unstructured interviews. Subsequently, I have set boundaries of the study as Grade 12 learners’ conceptual understanding of quadratic functions concepts, in line with the case study design as put by Merriam.

Additionally, I developed a genetic decomposition which informed the ACE cycle. Subsequently, I collected qualitative data using Task 0 and additional learning tasks in the activities phase. Moreover, the data collected from the activities phase guided the classroom discussions. The activity and classroom discussion data successfully informed the exercise phase. I used a test in the exercise to reinforce the activities and classroom discussions phase. Therefore, the ACE cycle was used as a research design to inform the case study design.

5.3. INTERPRETATION OF RESEARCH FINDINGS

In this section, I will conceptualise the findings of the study to answer the research question posed at the beginning. Therefore, to respond to this question, I have divided this section into two subsections, namely (1) the principal findings of the study and (2) the research question.

5.3.1. The principal findings of the study

Data were collected using the constructs of the ACE cycle, and analysed and discussed using literature on understanding, APOS, and quadratic functions to provide the principal findings of the study. In this study, learners interacted with quadratic function questions, and I wanted to explore their knowledge based on the forms of quadratic functions, the axis of symmetry, the domain and range of quadratic functions, the y-intercept, the transformation of the function and the graph orientation. Subsequently, some principal findings are reached after the implementation of the ACE cycle. Firstly, learners seem to over rely on the standard form over the vertex and factored form. For example, the learners used the vertex form $f(x) = a(x-p)^2 + q$, to write down the y-intercept as $y = q$. Such conceptual obstacle meant that these learners posit an undeveloped conception of the forms of quadratic functions. Secondly, they (learners) posed conceptual obstacle of the quadratic formula, i.e., $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ and the axis of symmetry formula for the x-

coordinate, i.e., $x = -\frac{b}{2a}$. For example, learners are given a function in a standard form, they mistakenly use the values of a, b, and c to determine the vertex whilst the turning point is determined by only the values of a and b. Hence, misconceived understanding of which value gives the x-coordinate of the vertex does not only impede the understanding of the turning point but also inhibits the comprehension of the axis of symmetry. Thirdly, some could not correctly substitute and simplify the formula $x = -\frac{b}{2a}$. For example, they wrote the correct formula $x = -\frac{b}{2a}$ but they substituted one value only as $x = -2(2)$ and ignored the value of b. Such ignorance explains that these learners were not entirely exhibiting procedural flexibility with the equation of the line of symmetry. Fourthly, the learners misunderstood the y-intercept of the function and the y-coordinate of the vertex, especially when the function was in vertex form where they equated c and q. Subsequently, the misconceived concept of the vertex impeded their knowledge of the range of the quadratic function. Lastly, learners posed an underdeveloped conception of linear functions concept as they thought that parameter a in the standard form $f(x) = ax^2 + bx + c$ is the gradient.

5.3.2. The research question of the study

This study initially intended to explore the role of the ACE teaching cycle in improving Grade 12 learners' conceptual understanding of quadratic functions. To pursue the purpose of the study, the study answered the following research question:

- How does the ACE teaching cycle improve learners' conceptual understanding of quadratic functions?

Before I respond to the question posed at the beginning of this study, I would first explain what I mean by conceptual understanding quadratic functions. The quadratic understanding function implies that the learner posits procedural flexibility of quadratics as they can flexibly, accurately, efficiently, and appropriately utilise the forms to comprehend the concepts of vertex, y-intercept, x-intercepts, an axis of symmetry, and transformation. Moreover, it is the flexibility of the learner to understand the forms of quadratic functions as they posit critical points and appreciate them as coherently linked schemas of interrelated concepts. As a result of this conceptualising of what it means to understand quadratic functions, I can answer the research questions in line with the principal findings. The question posed is twofold. Firstly, it requires me to unearth the conceptual obstacles that learners posed; and secondly, it focuses of how the ACE teaching cycle improved the learners' conceptual understanding. Therefore, learners posed conceptual obstacles during the three phases of the ACE cycle. For example, some learners seemed to rely mainly on the standard form over the vertex and factored form, which impedes understanding the forms of quadratics. Additionally, the learners misunderstood the quadratic formula as they misused it to determine the x-coordinate of the vertex. In contrast, some wrote the correct formula for the axis of symmetry but were not efficient enough to accurately substitute into the formula $x = -\frac{b}{2a}$. Moreover, learners are confused about the y-intercept of the function and the y-coordinate of the vertex. Lastly, they posed an undeveloped analogy of linear and quadratics as they misconceived the meaning of the value of a in the linear and quadratic functions.

Consequently, the ACE teaching cycle improved learners' conceptual understanding of quadratic functions. The ACE cycle unearthed their conceptual obstacles in the activities, classroom discussions, and exercise phases as the process allow learners to reflect on the conceptual obstacles through classroom discussion as experienced from the activities phase. Subsequently, the ACE teaching cycle effectively remedied their conceptual obstacle by exposing them to constructive learning tasks through activities, classroom discussions and exercises. Therefore, learners' understanding revealed some knowledge traits in the exercise phase. For example, they were efficient and flexible in determining the intercepts and vertex of quadratic functions.

Additionally, learners in the exercise improved their comprehension of the intercept and vertex concept by flexibly and efficiently being competent to substitute $x = 0$ into functions. Thus, the ability to work with quadratic functions flexibly and efficiently yields the skill of competency in carrying out the procedure. Subsequently, this meant that the learner had developed their procedural flexibility to carry out procedures accurately and appropriately. Such competencies reflected improved learners' procedural flexibility of quadratic functions. Additionally, the competencies implied that learners could iterate in their understanding as they could use procedures to develop concepts and vice versa.

5.4. RECOMMENDATIONS

This study has implications for the learning of quadratic functions, as it unpacks the concept to the point that one should not confine it to memorising it and getting correct answers but as an essential source of knowledge. Additionally, the study offers affordances to teachers to enhance learners' conceptual understanding of the concept entirely and provides them with a trajectory to improve their understanding of quadratic functions through the ACE teaching cycle. Therefore, I recommend that future studies be broadened on learners understanding of the vertex and the y-intercept of quadratic functions.

5.5. CONTRIBUTIONS OF THE STUDY

The study potentially adds knowledge to the body of mathematics literature on learners' conceptual understanding of quadratic functions. Additionally, the study has broadened the scope of research on how learners understand the quadratic function within the forms of quadratic functions, the axis of symmetry, the domain and range of quadratic function, the y-intercept, the transformation of the function, and the graph orientation. Moreover, the study adds to the literature that conceptual understanding quadratic functions entails the flexibility of the learner to comprehend the forms efficiently and appropriately as they posit critical points and appreciate them as coherently linked schemas of interrelated concepts. For example, if a learner is given a function in the standard form $f(x) = ax^2 + bx + c$, to explain the learner's understanding of the form implies that the learner can efficiently determine the vertex as $\left(-\frac{b}{2a}; f\left(-\frac{b}{2a}\right)\right)$ and appropriately acknowledge that $-\frac{b}{2a}$ gives the line of symmetry of the function. Thus, the vertex advances in understanding coherent linked concepts such as turning point and the axis of symmetry.

5.6. LIMITATIONS OF THE STUDY

Literature reveals two notions of quadratic functions: the algebraic and geometric parts. Therefore, I did not look at the geometric understanding of quadratic functions. Contrarily, I limited my study to algebraic knowledge of quadratic functions because of the silent issues related to what happens in the mathematics classroom while learning the concept. Thus, given the context that learners usually interact with the algebraic part of quadratic functions, I did not want to change the context but to unearth the silent issues in understanding the concept. Therefore, anchored by the context, I limited the exploration to the algebraic part of quadratic functions. Consequently, the data unearthed from the algebraic part are worth researching.

5.7. CONCLUSION

The study revealed that learners operated in various levels of the APOS construct on specific concepts of a quadratic function. The implementation of the ACE teaching cycle unearthed conceptual obstacles that learners posed. Firstly, learners posed an underdeveloped knowledge of the forms of quadratic functions. Consequently, such underdeveloped understanding posed overreliance on the standard form only. Subsequently, dependency on the standard form inhibited the learners' procedural flexibility to transit from one form efficiently and appropriately to another. This led to failure in attaining an action-process conception of the forms of the quadratic functions, which impedes the full grasp of a schema for understanding the forms. Secondly, the learners misunderstood the y-intercept of the function and the y-coordinate of the vertex. Lastly, learners posed challenges with the parameter a in the standard form, i.e., $f(x) = ax^2 + bx + c$, which was caused by an undeveloped knowledge of linear functions. As a result of these conceptual obstacles unearthed and remedied by the ACE teaching cycle, it was evident that most learners posed a fragmented understanding of quadratic functions. Generally, most learners were operating at the action-process level of understanding quadratic functions. For example, they could not flexibly and efficiently transit from one form to another, which is the basis of conceptualising the understanding of quadratic functions. Subsequently, only six learners managed to interiorise the action level into the process and limited object-level understanding of quadratic functions. These learners could transition from one form to another but were inhibited from exhibiting strategic competency entirely in the learning tasks. Inhibiting learners' strategic competence impedes the full grasp of quadratic functions. Therefore, none of the learners reached the schema level of understanding of the quadratic functions. This was caused by the fragmented knowledge of algebra that they posed. Consequently, the ACE cycle did not fully improve learners' conceptual understanding of quadratic functions as the learners could not entirely attain a coherent mental structure of the actions, processes, objects, and schemas.

5.8. SUMMARY OF THE CHAPTER

This chapter dealt with the recommendations and conclusions. In this chapter, I presented the design issues which guided the study. Additionally, the principal findings were outlined, and I answered the research question. Moreover, I offered the recommendations and limitations of the study. Lastly, I considered learners' levels of understanding based on the APOS theory.

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APPENDIX A: LETTER TO DEPARTMENT OF EDUCATION

P O Box 836
PHALABORWA
1390
17 November 2021

The Head of Department
Private Bag X9489
Polokwane
0700

Dear Sir/Madam

REQUEST FOR PERMISSION TO CONDUCT CASE STUDY RESEARCH

1. The above matter bears reference.
2. I, **HLANGWANI W**, request the Department of Basic Education to allow me to conduct Research at XXXXX(Pseudonym) High School on the title: **AN EXPLORATION OFACE TEACHING CYCLE IN IMPROVING GRADE 12 LEARNERS' UNDERSTANDING OF QUADRATIC FUNCTIONS**.
3. The research will be conducted during the process of teaching and learning Mathematics involving Grade 12 learners as participants of the study.
4. Everything that will be done during the research process will involve behaviour within the scope of participants' normal daily activities. As such, learners' work will be used as data for the research findings and recommendations.
5. Hope you will find the above in order.

Kind regards:

Hlangwani W

Contact no: 078 340 1542

E-mail: hlagwani1w@gmail.com

APPENDIX B: LEARNER'S CONSENT FORM

Learners' consent form, for learner participation in research in grade 12 topic of Quadratic functions:

I,_______________________________________ (name and surname), as the participants of study and the and as the participants, I have fully read carefully and understood the content of the letter concerning the participation in the research on: An exploration of ACE teaching cycle in improving Grade 12 learners' understanding of quadratic functions.

I agree or disagree (mark one) to participate in the research project. (Yes or No) tick one.

Parent's signature: ________________________

Signed at ________________________________on this day of _________________ 2022.

Learner's signature: _____________________________

Signed at ________________________________on this day of _________________ 2022.

Researcher: Hlangwani W

APPENDIX C: LETTER TO THE PRINCIPAL

Stand no. 660
MHADAWA
Letaba
0870
27 January 2022

The Principal
P. O Box 836
Phalaborwa
1390

Dear Principal

A REQUEST FOR PERMISSION TO DO RESEARCH AT THE SCHOOL IN NAMAKGALE

As I have brought to your awareness that I am currently doing Master of Education (in Mathematics Education) at the University of Limpopo. I am conducting research on: **An exploration of ACE teaching cycle in improving Grade 12 learners' understanding of quadratic functions**.

As such, the objectives of this study is to:

- Guide the teacher about his learners' performance in quadratic functions activities.
- The study will help the teacher to know which relevant approach and the method that can be used in teaching quadratic functions for conceptual understanding.
- Furthermore, it will give the teacher an idea of what type of knowledge do these learners possess between procedural knowledge and conceptual knowledge.
- Lastly, this study will make learners to realise their mistakes that they make and why they make those mistakes when solving differential calculus problems.

I viewed grade 12 curriculum and choose it as important grade because is the exit point to university where learners are either motivated or discouraged to continue doing mathematics. The research will start from 2022 academic year and will take 2 weeks.

Should you have any questions relating to this research, you are welcome to get in touch with my supervisor:

Name: Dr Dhlamini Z.B

School of Education (DMSTE)

University of Limpopo

Yours sincerely

Hlangwani W

APPENDIX D: LEARNING UNIT

NAME OF THE PARTICIPANT: ____

This Topic...

This topic introduces quadratic functions, their graphs and their important characteristics.Quadratic functions are widely used in mathematics and statistics. They are found in applied and theoretical mathematics, and are used to model non-linear relationshipsbetween variables in statistics. The module covers the algebra and graphing skills needed for analyzing and using quadratic functions.

Task 0

1. Determine the y-intercept for the following equation: $y = -3(x-4)^2 + 100$

2. Clearly explain in words all the transformations that must be applied to $y = x^2$ to obtain the graph of the function below $y = -\frac{1}{4}(x+6)^2 + 12$

3.Sketch each quadratic function and fill in the blanks below:

3.1. $y = (x-2)^2 + 3$

Vertex; Axis of symmetry; x-intercepts; y-intercept

3.2. $y = -(x+5)^2 - 2$

Vertex; Axis of symmetry; Max/Min value; Range

3.4. $y = 0{,}5(x-4)^2 + 5$

Vertex; Axis of symmetry; Step pattern; Domain

PART A

1. THE FORMS OF QUADRATIC FUNCTIONS

Task 1

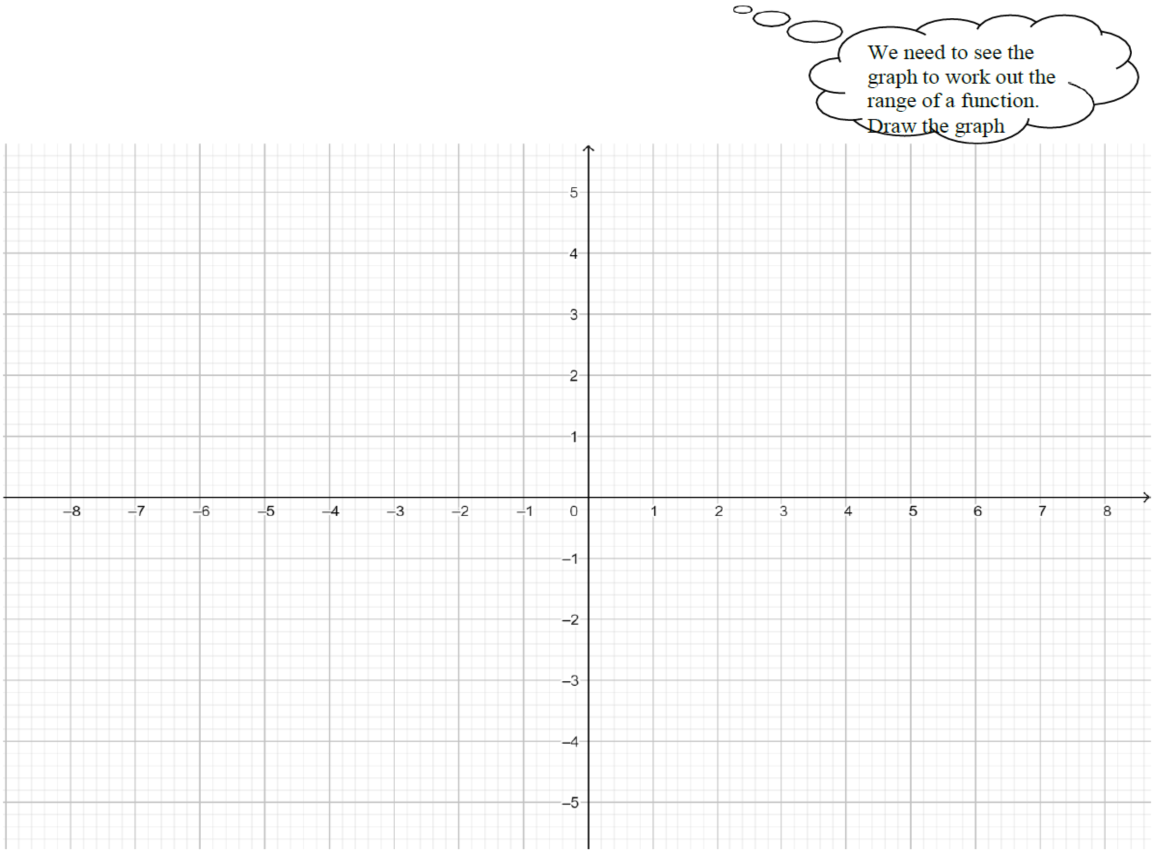
1. Check that $y = 1 - (x-1)^2$ is a parabola by rewriting it in the standard form $y = ax^2 + bx + c$.

2. Complete the following table, then use it to draw the parabolas $y = x^2$ and $y = 1-(x-1)^2$.

x	-3	-2	-1	0	1	2	3
$f(x) = x^2$							
$g(x) = 1-(x-1)^2$							

3. What is the line of symmetry for each parabola?

4. What are the ranges of the quadratic functions x^2 and $1-(x-1)^2$?



PART B

2. SKETCHING THE QUADRATIC FUNCTION

Task 2

Rewrite the following parabolas in the form $y = ax^2 + bx + c$, and state whether they are concave up or concave down.

(a) $y = (x - 3)^2 + 4$

____

(b) $y = 3 - (x + 2)^2$

____

(b) $y = 10(x - 2)^2 - 15$

____

(c) $y - x^2 + 2x + 3 = 0$

____

(d) $s = (t - 3)^2 + 2t + 1$

____

Task 3

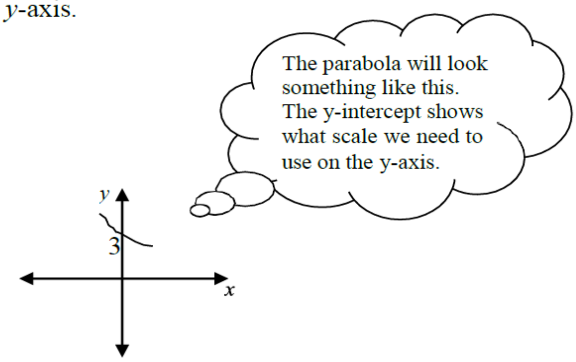
The intercepts of a parabola

Before we can sketch a parabola, we need to draw scales on the x- and y-axes. This can be done once the x- and y-intercepts are known.

The y-intercept is where the graph meets the y-axis.

Task 3

Find the y-intercept of $y = x^2 - 4x + 3$.



Answer the question:

Let $x = 0$, then

$y = x^2 - 4x + 3$

$=$ ____________

$=$ ____________

The y-intercept is (___;___).

Task 4

Find the y-intercept of $y = (2 - x)(1 - x)$.

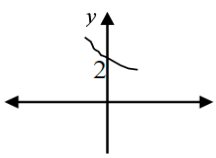
Answer

let $x = 0$, then

$y = (2 - x)(1 - x)$

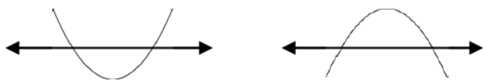
$=$ ____________

$=$ ____________

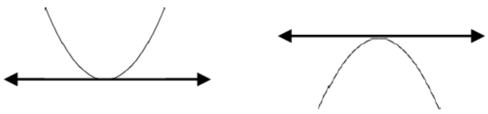


The y-intercept is (__;__).

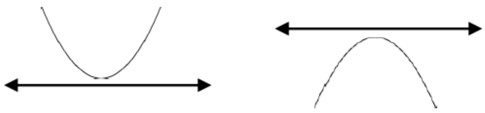
The x-intercepts are where the graph meets the x-axis. There will be either

- two x-intercepts: 

The graphs cross the x-axis.

- exactly one intercept: 

The graphs just touch the x-axis.

- no intercepts: 

The graphs don't meet the x-axis.

Task 5

Find (i) the x-intercepts and (ii) the vertex of $y=(x-1)(x-3)$.

(i) The x-intercepts.

Let $y=0$, then

$(x-1)(x-3)=0$.

So either

When a product of two numbers is zero, either one number or the other must be zero.

$x-1=0$

$x=$____,

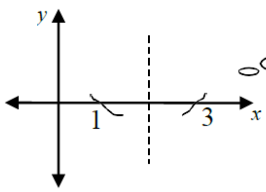
or

$x-3=0$

$x=$____.

The x-intercepts are (_;_) and (_;_).

The x-intercepts show what scale should be used on the x- axis.



First find the line of symmetry.

(ii) The line of symmetry and vertex.

The line of symmetry passes through the midpoint of (1, 0) and (3, 0), so it must have equation $x = 2$.

To find the vertex, put $x = 2$.

Put $x = 2$ as the vertex is on the line of symmetry.

$y=(x-1)(x-3)$

$=$ ____________

$=$ ____________

The vertex is (2, –1)

Task 6

Find the x-intercepts and vertex of $y = 2 + x - x^2$.

Answer

(i) The x-intercepts.

Put $y = 0$, then

$$2 + x - x^2 = 0$$

$$x^2 - x - 2 = 0$$

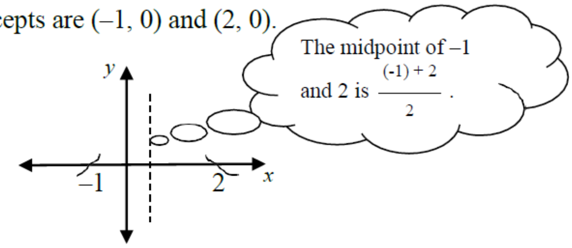
Solve by factorising.

(______)(_____) = 0.

So either:

x=____ *or* x=____

The x-intercepts are (–1, 0) and (2, 0).



(ii) The line of symmetry and vertex.

The equation of the line of symmetry is x=1/2

To find the vertex, put $x = \frac{1}{2}$.

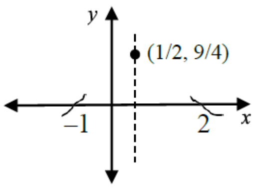
The vertex is on the line of symmetry.

$$y = 2 + x - x^2$$

= ____________

= ____________

The vertex is (__;___)



If you cannot solve a quadratic equation by factorisation, then try completing the square or the quadratic formula. The line of symmetry can also be found from the formula below.

The parabola $y = ax^2 + bx + c$ has line of symmetry $x = -\frac{b}{2a}$.

Sketching a parabola

A sketch of a parabola should show the **intercepts**, the **line of symmetry** and the **vertex**.

Task 7

Sketch the parabola $y = x^2 - 4x + 3$.

Answer

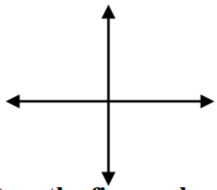
(a) Shape.

The parabola is concave **UP / DOWN** (cancel the wrong one) as the coefficient of x is **GREATER THAN 0** or **LESS THAN 0** (cancel the wrong one).

(b) Intercepts.

Put $x = 0$, then

$$y = x^2 - 4x + 3$$
$$= \_\_\_\_\_\_\_\_\_\_\_\_$$
$$= \_\_\_\_\_\_\_\_\_\_\_\_$$

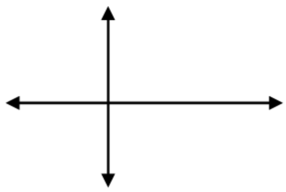


The y-intercept is (__;__). **Demonstrate your intercept on the figure above.**

Put $y = 0$, then

$$x^2 - 4x + 3 = 0$$
$$(\_\_\_\_\_)(\_\_\_\_\_) = 0.$$

So either x_____or x_____



The x-intercepts are (____) and (____). **Demonstrate your intercept on the figure above.**

(c) Line of symmetry and vertex.

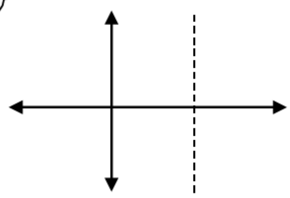
The line of symmetry is ***x***=_____, as it passes through the midpoint of (1, 0) and (3, 0).

Check: $x = -\dfrac{b}{2a}$

Check using the formula for the line of symmetry.

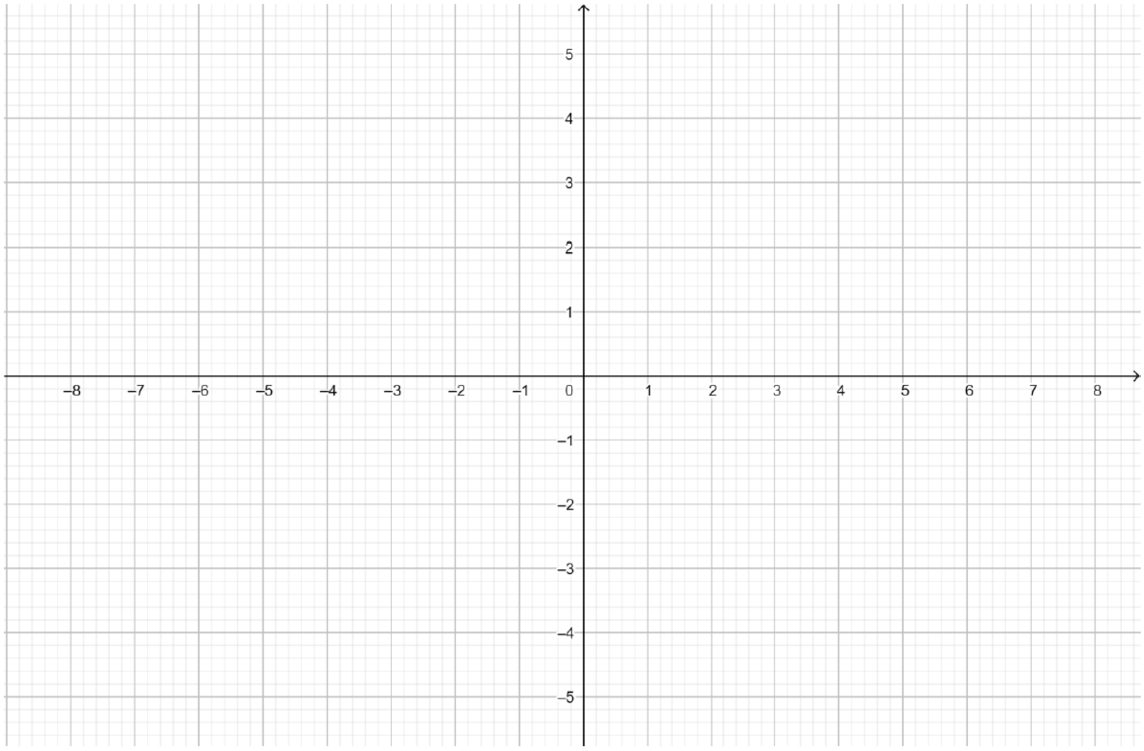
Put $x = 2$, then

$$y = x^2 - 4x + 3$$
$$= \_\_\_\_\_\_\_\_\_\_\_\_$$
$$= \_\_\_\_\_\_\_\_\_\_\_\_$$



The vertex is (2, –1).

(d) Sketch.

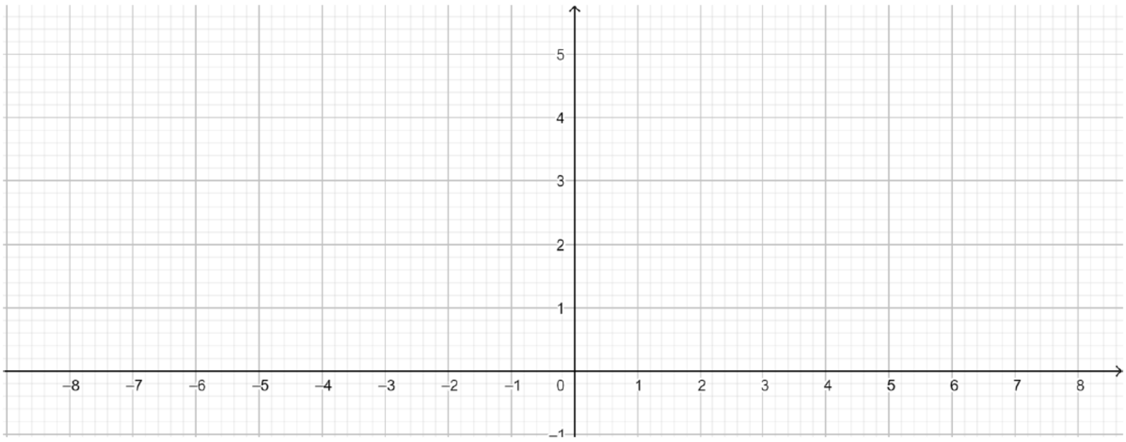


Note. Sometimes the method above does not produce enough points for a sketch. If this happens, then you should calculate more points by substitution and by using symmetry.

Task 8

The parabola $y = x^2$ has y-intercept (0, 0), x-intercept (0, 0) and line of symmetry $x = 0$.

Four more points were calculated for the sketch below: (1, 1), (–1, 1), (2, 4), (–2, 4).

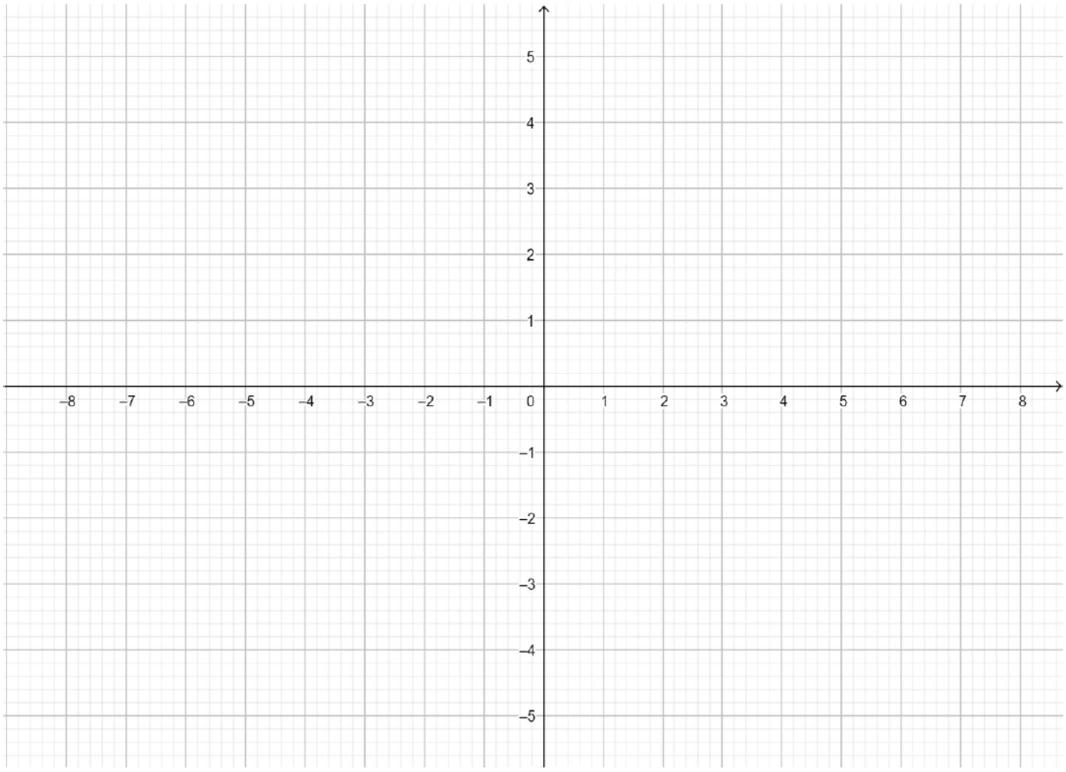


Task 9

1. Sketch the following parabolas, showing their intercepts, line of symmetry and vertex. In each case state the domain and range of the quadratic function.

(a) $f(x) = x^2 - 4x$ (b) $g(x) = x^2 - 5x + 6$ (c) $h(x) = x^2 + 2x - 8$

(d) $s(x) = 6 - x - x^2$ (e) $t(x) = x^2 - 4x + 4$ (f) $v(x) = x^2 + 2x - 3$



PART C

3. TRANSFORMATIONS

Task 10

Sketch the parabola $y = x^2 - 4x + 5$.

Answer

Write the equation in the form $y = (x - p)^2 + q$ by completing the square:

$y = x^2 - 4x + 5$

____

____

____

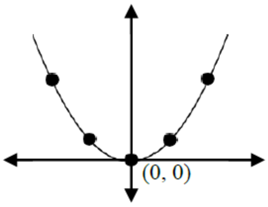
____

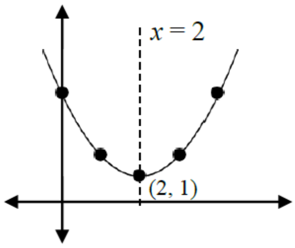
____

____

____

So the parabola is obtained from $y = x^2$ by shifting it by 2 units to the right and 1 unit upwards.



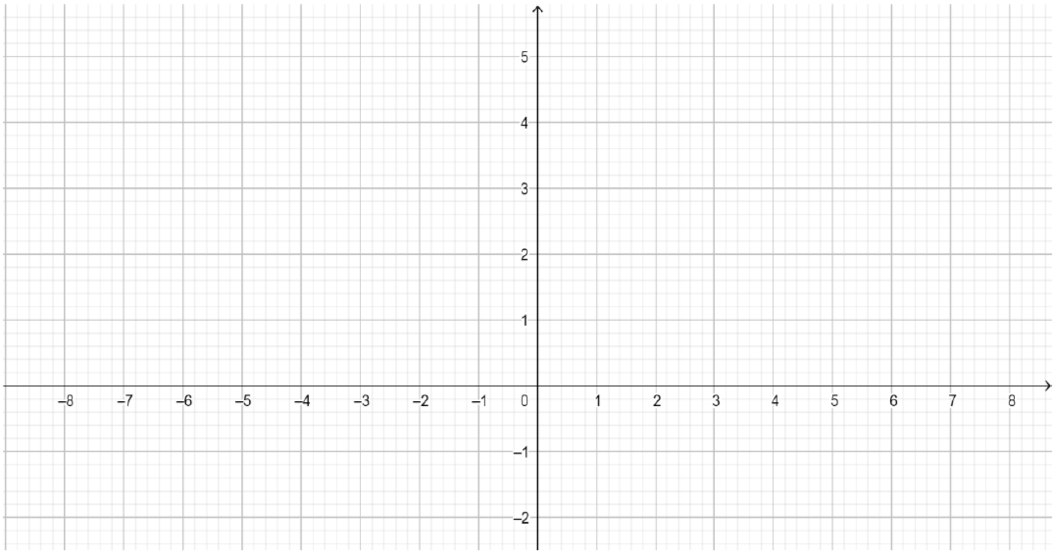


This example shows how to sketch a parabola with no x-intercepts. It also shows that the shape of the parabola $y = x^2 - 4x + 5$ is *exactly* the same as the shape of the parabola $y = x^2$!

Task 11

1. Sketch the parabola $y = x^2$ on the given Cartesian plain, then complete the square in the following functions in order to sketch their graphs using the translations.

(a) $y = x^2 - 6x + 8$ (b) $y = x^2 - 6x + 9$ (c) $y = x^2 - 6x + 10$

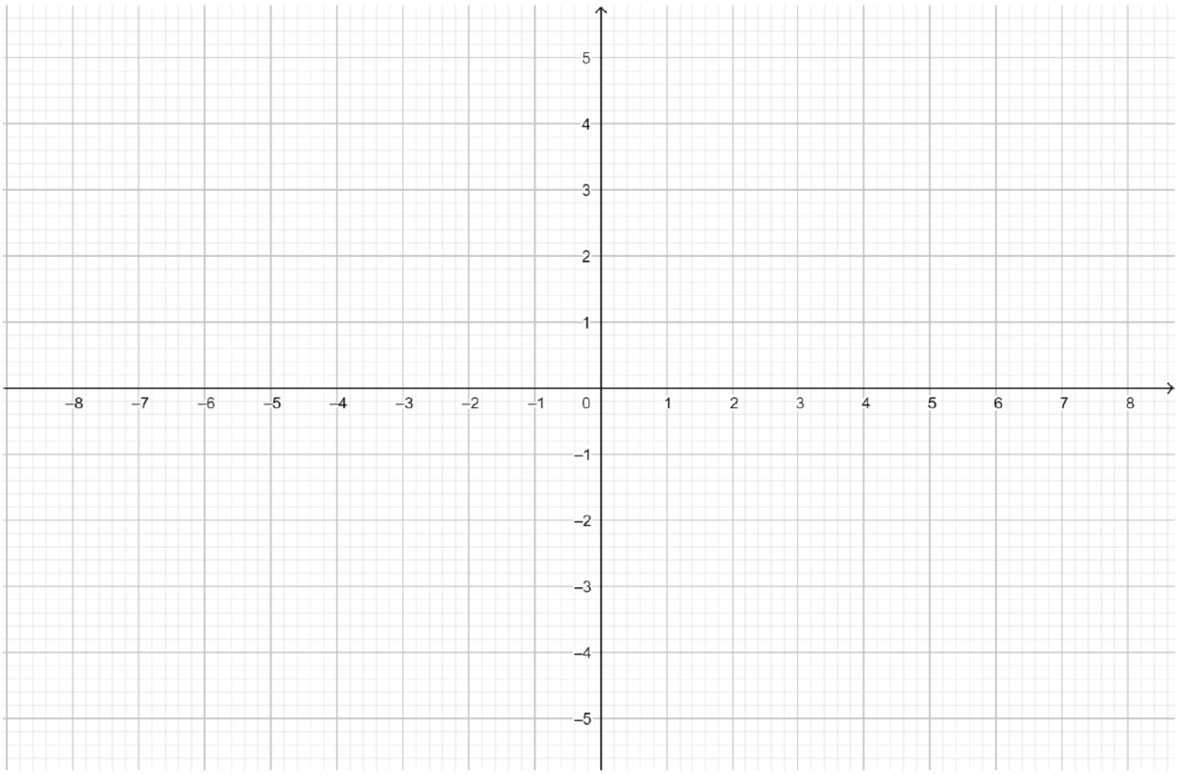


2. A parabola has the same shape as $y = x^2$ but with vertex at (2, 3). What is its equation?

Task 12

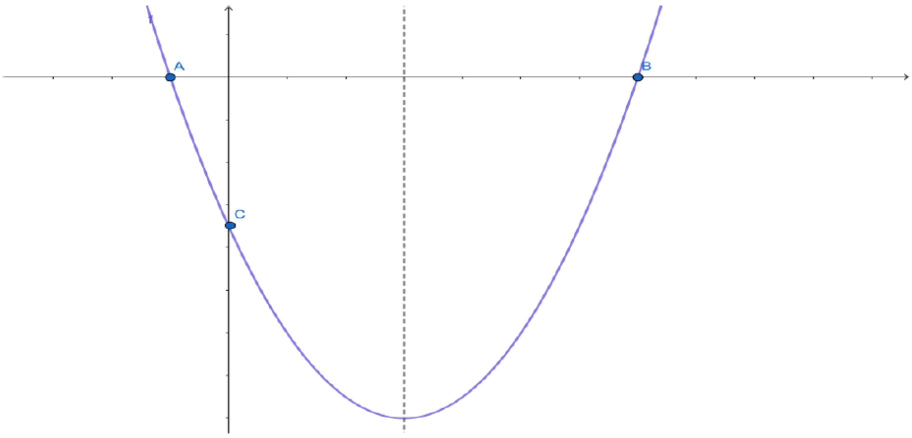
Sketch the parabola $f(x) = x^2$, then use translations and reflections to sketch

(a) $g(x) = 6 + 2x - x^2$ (b) $h(x) = -6 - 2x + x^2$



Task 13

A parabola intersects the x-axis at B and C and the y-axis at E. The axis of symmetry of the parabola has equation $x = 3$. The function is defined by $x(x - 6) = 2y + 7$.



(a) Show that the coordinates of B are $(7; 0)$.

(b) Calculate the x-coordinate A.

(c) Determine the equation of the parabola in terms of $f(x) = y = a(x - p)^2 + q$.

(d) Write down the equation of h if the reflection of f in the x-axis.

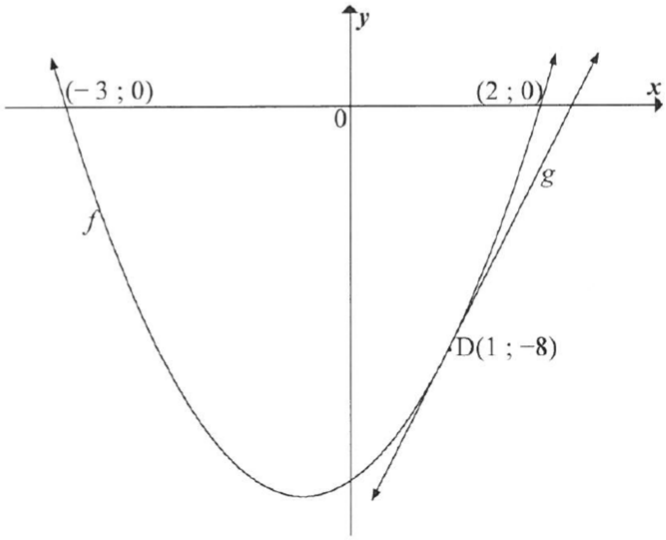
(e) Write down the maximum value of $t(x)$ if $t(x) = 1 - f(x)$.

Task 14: HOMEWORK

The graphs of $f(x) = ax^2 + bx + c$; $a \neq 0$ and $g(x) = mx + k$ are drawn below.

D(1 ; – 8) is a common point on f and g.

- f intersects the x-axis at $(-3 ; 0)$ and $(2 ; 0)$.
- g is the tangent to f at D.



(a) For which value(s) of x is $f(x) \leq 0$? (2)

(b) Determine the values of a, b and c. (5)

(c) Determine the coordinates of the turning point of f. (3)

(d) Write down the equation of the axis of symmetry of h if $h(x) = f(x-7)+2$. (2)

ANSWERS:

(a) ____

____

____

____

(b) ____

____

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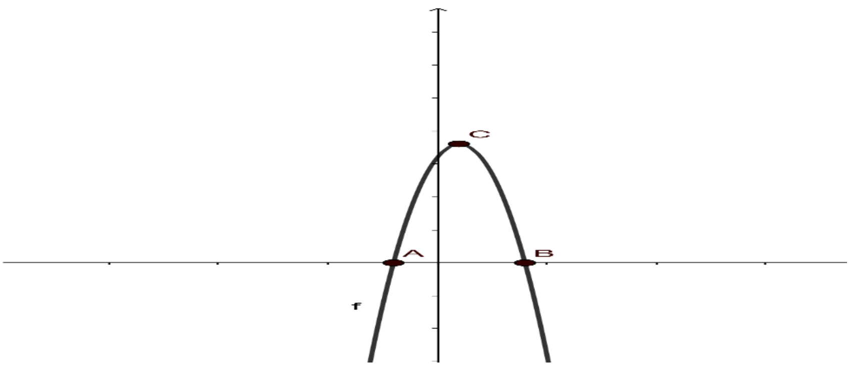
(c) ____

(d) ____

Exercise (Test)

Sketched below is the graph of $f(x) = -2x^2 + 4x + 16$. A and B are x-intercepts of f. C is the turning point of f.



1. Calculate the coordinates of A and B.

2. Determine the coordinates of C, the turning point of f.

3. Write down the range of f.

4. The graph of $h(x) = f(x + p) + q$ has a maximum value of 15 at $x = 2$. Determine the values of p and q.

APPENDIX E: EMERGED CODES AND THEMES

Main concepts

Category	Descriptive themes	Codes
Quadratic function concept	Properties of quadratic functions and graphing methods	• The forms quadratic function • The axis of symmetry • The domain and range of the quadratic function • The y-intercept • The transformation involving the quadratic function • Can graph the function given the vertex form transformation • Uses the intercept method • Can correctly plot the y-intercept
	Quadratic functions connections through solving techniques	• Can connect the graph to the equation • Can connect the Table with a graph • Factoring for finding the roots • Completing the square to move from standard form to vertex form • Line of symmetry formula • Incorrect use of the quadratic formula • Tries but not able to solve

	Conceptual obstacles	• Preferred working with the standard form compared to the other forms, which are the vertex and factored forms. • Conceptual obstacles regarding the y-intercept of the function and the y-intercept of the vertex. • The confusion of the parameters of the quadratic function, especially the a. • Unable to represent the range of the function using different notations

APPENDIX F: ETHICAL CLEARANCE CERTIFICATE

University of Limpopo
Department of Research Administration and Development
Private Bag X1106, Sovenga, 0727, South Africa
Tel: (015) 268 3935, Fax: (015) 268 2306, Email: anastasia.ngobe@ul.ac.za

TURFLOOP RESEARCH ETHICS COMMITTEE

ETHICS CLEARANCE CERTIFICATE

MEETING: **13 October 2021**

PROJECT NUMBER: **TREC/283/2021: PG**

PROJECT:

Title:	An exploration of ACE teaching cycle in improving Grade 12 learners' understanding of quadratic functions
Researcher:	W Hlangwini
Supervisor:	Dr Dhlamini Z.B
Co-Supervisor/s:	Prof K.M. Chuene
School:	Education
Degree:	Master of Education in Mathematics Education

PROF P MASOKO
CHAIRPERSON: TURFLOOP RESEARCH ETHICS COMMITTEE

The Turfloop Research Ethics Committee (TREC) is registered with the National Health Research Ethics Council, Registration Number: **REC-0310111-031**

Note:

i) **This Ethics Clearance Certificate will be valid for one (1) year, as from the abovementioned date. Application for annual renewal (or annual review) need to be received by TREC one month before lapse of this period.**

ii) **Should any departure be contemplated from the research procedure as approved, the researcher(s) must re-submit the protocol to the committee, together with the Application for Amendment form.**

iii) **PLEASE QUOTE THE PROTOCOL NUMBER IN ALL ENQUIRIES.**

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APPENDIX G: APPROVAL FROM DEPARTMENT OF EDUCATION

LIMPOPO
PROVINCIAL GOVERNMENT
REPUBLIC OF SOUTH AFRICA

DEPARTMENT OF
EDUCATION

CONFIDENTIAL

Ref: 2/2/2 Enq: Makola MC Tel No: 015 290 9448 E-mail:MakolaMC@edu.limpopo.gov.za

Hlangwani W
P BOX 836
PHALABORWA
1390

RE: REQUEST FOR PERMISSION TO CONDUCT RESEARCH

1. The above bears reference.
2. The Department wishes to inform you that your request to conduct research has been approved. Topic of the research proposal: **"AN EXPLORATION OF ACE TEACHING CYCLE IN IMPROVING GRADE 12 LEARNER'S UNDERSTANDING OF QUADRATIC FUNCTIONS "**
3. The following conditions should be considered:

3.1 The research should not have any financial implications for Limpopo Department of Education.

3.2 Arrangements should be made with the Circuit Office and the School concerned.

3.3 The conduct of research should not in anyhow disrupt the academic programs at the schools.

3.4 The research should not be conducted during the time of Examinations especially the fourth term.

3.5 During the study, applicable research ethics should be adhered to; in particular the principle of voluntary participation (the people involved should be respected).

3.6 Upon completion of research study, the researcher shall share the final product of the research with the Department.

REQUEST FOR PERMISSION TO CONDUCT RESEARCH : HLANGWANI W Page 1

Cnr 113 Biccard & 24 Excelsior Street, POLOKWANE, 0700, Private Bag X 9489, Polokwane, 0700
Tel:015 290 7600/ 7702 Fax 086 218 0560

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4 Furthermore, you are expected to produce this letter at Schools/ Offices where you intend conducting your research as an evidence that you are permitted to conduct the research.

5 The department appreciates the contribution that you wish to make and wishes you success in your investigation.

Best wishes.

27/01/2022

Mashaba KM **Date**

DDG: CORPORATE SERVICES

Cnr 113 Biccard & 24 Excelsior Street, POLOKWANE, 0700, Private Bag X 9489, Polokwane, 0700
Tel:015 290 7600/ 7702 Fax 086 218 0560

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CONFIDENTIAL

LIMPOPO
PROVINCIAL GOVERNMENT
REPUBLIC OF SOUTH AFRICA

OFFICE OF THE PREMIER

Office of the Premier

Research and Development Directorate

Private Bag X9483, Polokwane, 0700, South Africa

Tel: (015) 230 9910, Email: mokobij@premier.limpopo.gov.za

LIMPOPO PROVINCIAL RESEARCH ETHICS

COMMITTEE CLEARANCE CERTIFICATE

Online Review Date: 07th – 14th December 2021

Project Number: LPREC/120/2021: PG

Subject: An Exploration of ACE Teaching Cycle in Improving Grade 12 Learners' Understanding of Quadratic Functions

Researcher: Hlangwani W

Dr Thembinkosi Mabila

Chairperson: Limpopo Provincial Research Ethics Committee

The Limpopo Provincial Research Ethics Committee (LPREC) is registered with National Health Research Council (NHREC) Registration Number **REC-111513-038.**

Note:

i. **This study is categorized as a Low Risk Level in accordance with risk level descriptors as enshrined in LPREC Standard Operating Procedures (SOPs)**

ii. **Should there be any amendment to the approved research proposal; the researcher(s) must re-submit the proposal to the ethics committee for review prior data collection.**

iii. **The researcher(s) must provide annual reporting to the committee as well as the relevant department and also provide the department with the final report/thesis.**

iv. **The ethical clearance certificate is valid for 12 months. Should the need to extend the period for data collection arise then the researcher should renew the certificate through LPREC secretariat. PLEASE QUOTE THE PROJECT NUMBER IN ALL ENQUIRIES.**

CONFIDENTIAL

LIMPOPO
PROVINCIAL GOVERNMENT
REPUBLIC OF SOUTH AFRICA

OFFICE OF THE PREMIER

TO: DR MC MAKOLA

FROM: DR T MABILA

CHAIRPERSON: LIMPOPO PROVINCIAL RESEARCH COMMITTEE (LPRC)

ONLINE REVIEW DATE: 07th-14th DECEMBER 2021

SUBJECT: AN EXPLORATION OF ACE TEACHING CYCLE IN IMPROVING GRADE 12 LEARNERS' UNDERSTANDING OF QUADRATIC FUNCTIONS

RESEARCHER: HLANGWANI W

Dear Colleague

The above researcher's research proposal served at the Limpopo Provincial Research Committee (LPRC). The committee is satisfied with the ethical soundness of the proposed study.

Decision: The research proposal is granted approval

Regards

Acting Chairperson: Dr T Mabila

Secretariat: Ms J Mokobi

Date: 19/01/2022

CONFIDENTIAL

LIMPOPO
PROVINCIAL GOVERNMENT
REPUBLIC OF SOUTH AFRICA

OFFICE OF THE PREMIER

TO: DR MC MAKOLA

FROM: DR T MABILA

CHAIRPERSON: LIMPOPO PROVINCIAL RESEARCH ETHICS COMMITTEE (LPREC)

ONLINE REVIEW DATE: 07th-14th DECEMBER 2021

SUBJECT: AN EXPLORATION OF ACE TEACHING CYCLE IN IMPROVING GRADE 12 LEARNERS' UNDERSTANDING OF QUADRATIC FUNCTIONS

RESEARCHER: HLANGWANI W

Dear Colleague

The above researcher's research proposal served at the Limpopo Provincial Research Ethics Committee (LPREC). The ethics committee is satisfied with the ethical soundness of the proposed study.

Decision: The revised research proposal is granted full approval and ethical clearance

Regards

Chairperson: Dr T Mabila

Secretariat: Ms J Mokobi

Date: 19/01/2022

APPENDIX H: RESEARCH PROTOCOL

The instructions for data collection for the study:

- To commence the study, I sampled 30 learners to participate in the study
- The sampled participants interacted with the learning unit and were tested for two weeks.
- The learners were allocated 50 minutes daily to interact with Task 0, the learning unit, and the test.
- On day 1, the learners were made aware of their purpose in participating in the study and were given ground rules. Later after the introduction, they were given Task 0 to write and submit.
- On days 2-3, we reflected on what we on the previous day. Then after learners were given the learning unit again to interact with **PART A** of the learning unit.
- On days 4-6, we reflected on what we on the previous day. Then after learners were given the learning unit again to interact with **PART B** of the learning unit.
- On days 7-9, we reflected on what we on the previous day. Then after learners were given the learning unit again to interact with **PART C** of the learning unit.
- On the last day, i.e., day 10. We reflected on what we did from day 1-9 and after learners wrote a test.
- Note:
    - During the data collection, I may read the question upon request, but I am not to assist in giving learners answer
    - I may scaffold learners if they have a conceptual obstacle.
    - Each day learners were provided with a task and collected after interacting with it.
    - Learners can discuss together if they need to, as it is supported by the ACE teaching cycle.
- At no point am I to assist the learners in solving the problems.

APPENDIX I: APPROVAL FROM FACULTY

University of Limpopo
Faculty of Humanities
Executive Dean
Private Bag X1106, Sovenga, 0727, South Africa
Tel: (015) 268 4895, Fax: (015) 268 3425, Email:Satsope.maoto@ul.ac.za

DATE: 18 August 2021

NAME OF STUDENT: **HLANGWANI, W**
STUDENT NUMBER: [redacted]
DEPARTMENT: **MEd – Mathematics Education**
SCHOOL: **Education**

Dear Student

FACULTY APPROVAL OF PROPOSAL (PROPOSAL NO. FHDC2021/7/04)

I have pleasure in informing you that your MEd proposal served at the Faculty Higher Degrees Meeting on 28 July 2021 and your title was approved as follows:

TITLE: AN EXPLORATION OF ACE TEACHING CYCLE IN IMPROVING GRADE 12 LEARNERS' UNDERSTANDING OF QUADRATIC FUNCTIONS

Note the following:

Ethical Clearance	Tick One
In principle the study requires no ethical clearance, but will need a TREC permission letter before proceeding with the study	
Requires ethical clearance (Human) (TREC) (apply online) Proceed with the study only after receipt of ethical clearance certificate	√
Requires ethical clearance (Animal) (AREC) Proceed with the study only after receipt of ethical clearance certificate	

Yours faithfully

Prof RS Maoto,
Executive Dean: Faculty of Humanities
Director: Prof MW Maruma
Supervisor: Dr ZB Dhlamini
Co-supervisor: Dr KB Chuene

APPENDIX J: EDITORIAL LETTER

Stand 507 Caledon village, email: kubayijoe@gmail.com, cell 0794848449

01 October 2022

Dear Sir/Madam

SUBJECT: EDITING OF DISSERTATION

This is to certify that the dissertation entitled 'An exploration of ACE teaching cycle in improving Grade 12 learners' understanding of quadratic functions' by Hlangwani Wisani has been copy-edited, and that unless further tampered with, I am content with the quality of the dissertation in terms of its adherence to editorial principles of consistency, cohesion, clarity of thought and precision.

Kind regards

Prof. SJ Kubayi (DLitt et Phil)