FISCHER-CLIFFORD MATRICES AND CHARACTER TABLES OF SOME GROUP EXTENSIONS

by

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Abstract

A useful way for studying the properties of a group G is to express the elements of G in terms of matrices or permutations [27]. In 1878 Cayley showed that every group G is isomorphic to a subgroup of the symmetric group S_G, where S_G is the group of all permutations on G. A **representation** of a group G is a homomorphism $T : G \to GL(n, F)$ from G into the group $GL(n, F)$ of $n \times n$ invertible matrices over a field F. The **character** of G afforded by a representation T is the trace of the matrices $T(g)$ for each $g \in G$. The table of characters of G is called the **character table** of G [20]. Since the completion of the classification of finite simple groups in 1981 [4], current research work in group theory involves the study of the structures of simple groups. The structures and character tables of maximal subgroups of simple groups give substantive information about these groups. Most of the maximal subgroups of simple groups [8] and some of their constituent groups are of extension type (i.e. a group $\overline{G} = N.G$ such that $N \lhd \overline{G}$ and $\overline{G}/N \cong G$).

In our research we are particularly interested in faithful permutation representations of sporadic simple groups and their automorphism groups [27]. The Mathieu groups are examples of sporadic simple groups [32]. A permutation group G on X is said to be ***k*-transitive** on X if for any two k-tuples $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$ of k distinct elements of X, there exists $g \in G$ such that $x_i^g = y_i, 1 \leq i \leq k$. Apart from the symmetric groups S_n and Alternating groups A_n, the Mathieu groups are the only non-trivial faithful k-transitive permutation groups for $k = 4, 5$ [32]. Now as can be seen from the Atlas of Finite Groups [8], the Mathieu group M_{22} has a maximal subgroup of form $2^4 : S_5$. Likewise the Mathieu group M_{23} has a maximal subgroup of form $2^4{:}A_7$. As part of this project we will study the groups of forms $2^4 : S_5$ and $2^4 : A_7$. We note that some groups of form $p^{n-1} : S_n$, where p is prime, have been studied in [35], however the group $2^4 : S_5$ studied here is not one of those groups. This will give some information about the structures of these groups.

Let $m, n \in \mathbb{N}$, the set of positive integers and $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ be the set of residues modulo m, also considered as a cyclic group C_m of order m. One of the groups to be studied in this work is a subgroup of the symmetric group S_{mn} of degree $m \times n$. For example, we know from [28] that the simplectic group $SP(6,2)$, which is a maximal subgroup of the Fischer group $F_{i_{22}}$, has a subgroup of form $2^5 : S_6$. Further the group $2^5 : S_6$ has a subgroup isomorphic to the split extension $(S_3)^2 : C_2$ or the wreath product of the symmetric group S_3 of degree 3 with the cyclic group C_2 of order 2. In this project, we will also study the groups $(S_n)^m : C_m$, where m is prime and n is a positive integer. The group $(S_n)^m : C_m$ is a subgroup of the group Smn of degree $m \times n$.

Several methods for constructing the character tables of group extensions exist. However, Fischer [12] has given an effective method for constructing the character tables of some group extensions including the groups cited above. This method known as the **technique of the Fischer-Clifford matrices** makes use of Clifford Theory [7, 20]. Given a group extension $\overline{G} = H.G$ such that every irreducible character of H can be extended to its inertia group, for each conjugacy class of G we construct a matrix called a **Fischer-Clifford matrix of $\overline{G}$**. By using the Fischer-Clifford matrices of $\overline{G}$ together with the fusion maps and the character tables of inertia factor groups of $\overline{G}$, we are able to construct the character table of $\overline{G}$. We note that Fischer-Clifford matrices satisfy certain properties which may be used to construct them. The method of the Fischer-Clifford matrices has been used in many works both on split and non-split extensions [1, 10, 12, 25, 28, 30, 34, 35].

Here we will use the method of Fischer-Clifford matrices to construct the character tables of the groups $2^4 : S_5$, $2^4 : A_7$, $(S_3)^2 : C_2$, $(S_3)^3 : C_3$, $(S_4)^2 : C_2$ and $(S_4)^3 : C_3$.

Declaration

The work described in this Dissertation has been carried out under the supervision and direction of Dr Kenneth Zimba, School of Computational and Mathematical Sciences, University of Limpopo (Turfloop), Polokwane, South Africa, from March 2006 to April 2008.

The Dissertation represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other University. Where use has been made of the work of others it is duly acknowledged in the text.

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Notation

Throughout this Dissertation all groups discussed are finite. We will use the following notation from the ATLAS [8] unless stated otherwise.

$\mathbb{N}$	natural numbers
$\mathbb{Z}$	integers
$\mathbb{Z}_m = \{0, 1, 2, \ldots, m-1\}$	cyclic group of order m
F	a field
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
Ω	a set containing n elements
G, N, H, K, S, M, Q, I	groups
1_G	identity element of G
$H \leq G$	H is a subgroup of G
$N \trianglelefteq G$	N is a normal subgroup of G
$N.G$	a group extension
$N{:}G$	a split extension or semi-direct product
$N \cdot G$	a non-split extension of N by G
$o(g)$	order of an element g

d^g	action of g on d
$[g]$	a conjugacy class with representative g
d^G	an orbit of the action of G on d
$C_G(g)$	centralizer of g in G
Hg	the right coset of H
$N_G(H)$	normalizer of H in G
$Irr(G)$	the set of irreducible characters of G
I_G	identity character of G
$\chi(G/H)$	permutation character of G on H
χ_H	the restriction of a character χ to H
θ^G or $\theta \uparrow G$	the induction of a character θ to G
S_n	the symmetric group on Ω
$\mathbb{Z}_m^n$	the direct product of n copies of $\mathbb{Z}_m$
m^n	an elementary abelian group of order m^n with prime m
$GL(n, F)$	general linear group or non-singular $n \times n$ matrices over F.

List of Tables

Contents

Chapter 1

Introduction

Let $m, n \in \mathbb{N}$, the set of positive integers and $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ be the set of residues modulo m. We will also consider $\mathbb{Z}_m$ as a cyclic group C_m of order m. Let X be a set and S_X be the set of all permutations on X. Then S_X is a group under the composition of functions (**symmetric group on** X). If X is finite of order n, then $|S_X| = n!$ and we write S_n instead of S_X. All the groups studied here are permutation groups.

Definition 1.0.1 *Let G be a group. We say that G acts on X if there exists a homomorphism $\phi : G \to S_X$. Let $g \in G$ and $x \in X$. We denote $\phi(g)(x)$ by x^g or gx. The homomorphism ϕ is said to be a* ***permutation representation of*** *G. If ϕ is one-to-one, we say that G is a* ***permutation group on*** *X (in this case G can be identified as a subgroup of S_X) and ϕ is a* ***faithful*** *permutation representation of G.*

Let θ be a character of a subgroup H of a group G. Define the action of $g \in N_G(N)$ on θ by $\theta^g(h) = \theta(ghg^{-1})$, $h \in H$. Let $N_G(H)$ be the normalizer of H in G. Then $I_G(\theta) = \{g \in N_G(H) \mid \theta^g = \theta\}$ is called the **inertia group of** θ in G. If H is normal in G, then $I_G(\theta) = \{g \in G \mid \theta^g = \theta\}$ [20]. We observe that $N_G(H)$ acts on the characters of H by $g : \theta \longmapsto \theta^g$ for all $g \in N_G(H)$ and the inertia group of θ is the stabilizer of θ in $N_G(H)$. Thus $I_G(\theta) \leq N_G(H) \leq G$ and it is clear that H is a normal subgroup of $I_G(\theta)$. The quotient group $I_G(\theta)/H$, called the **inertia factor group of** θ, is a subgroup of $\overline{G}/N \cong G$.

1.1 Background

In the following sub-sections we discuss the groups that will be studied in this work.

1.1.1 Some groups of forms $2^4{:}S_5$ and $2^4{:}A_7$

In our research we are interested in faithful permutation representations of sporadic simple groups and their automorphism groups. The Mathieu groups are examples of sporadic simple groups. The group M_{22} has a maximal subgroup of form $2^4{:}S_5$ and the group M_{23} has a maximal subgroup of form $2^4{:}A_7$. Part of our work here will involve constructions of the character tables of groups of forms $2^4{:}S_5$ and $2^4{:}A_7$, which are not necessarily the maximal subgroups of M_{22} and M_{23}.

Definition 1.1.1 *Let G be a permutation group on X. We say G is k-**transitive** on X if for any two k-tuples $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$ of k distinct elements of X, there exists $g \in G$ such that $x_i^g = y_i, 1 \leq i \leq k$. If $k = 1$, we say G is **transitive**.*

Apart from the symmetric groups S_n and Alternating groups A_n, the Mathieu groups are the only non-trivial faithful k-transitive permutation groups for $k = 4, 5$ [32]. Now as seen from the Atlas of Finite Groups [8], the Mathieu group M_{22} has a maximal subgroup of form $2^4 : S_5$ of index 231 in M_{22}. Likewise the Mathieu group M_{23} has a maximal subgroup of form $2^4{:}A_7$ of index 253 in M_{23}. In this project we will construct the Fischer-Clifford matrices and character tables of groups of forms $2^4{:}S_5$ and $2^4{:}A_7$.

1.1.2 A subgroup of the symmetric group S_n

One of the groups to be studied in this work is a subgroup of the symmetric group S_n. We describe this group in this subsection.

Definition 1.1.2 *Let $m, n \in \mathbb{N}$.*

(i) An ***m-composition*** *of n is an m-tuple $(k_1, k_2, \ldots, k_m)$ such that $\sum_{j=1}^{m} k_j = n$, where $k_j \in \mathbb{N} \cup \{0\}$. The set of m-compositions of n is denoted by $A(n, m)$.*

(ii) Let $(k_1, k_2, \ldots, k_m) \in A(n, m)$. The expression

$$\binom{n}{k_1 \; k_2 \; \ldots \; k_m} = \frac{n!}{k_1! k_2! \cdots k_m!}$$

is called a ***multi-nomial coefficient****.*

Let N be the direct product of n copies of $\mathbb{Z}_m$. Let S_n be the symmetric group of degree n. The wreath product of $\mathbb{Z}_m$ with S_n is a split extension of N by S_n called the **generalized symmetric group**, denoted by $B(m, n)$ [24, 25]. We note that N is a $\mathbb{Z}_m$-module. Thus we may write the elements of N additively as $d = \sum_{s=1}^{n} l_s e_s$, where $l_s \in \mathbb{N}$ and e_s are generators. Further S_n acts on N by permuting the l_s in d [29].

Definition 1.1.3 *Let $S = \{d \in N \mid \sum_{s=1}^{n} l_s \equiv 0 \ (mod\ m)\}$. Then S is an S_n-invariant subgroup of N of order m^{n-1}. Therefore we can define a split-extension $S : S_n$ of S by S_n, where S_n acts on S as it acts on N. This group extension is denoted by $B_S(m, n)$ [35].*

It is shown in [35] that the symmetric group S_n acts on the set $Irr(S)$ of irreducible characters of S producing

(i) $\frac{1}{m}\left\{\binom{n+m-1}{m-1} - 1\right\}$ orbits of lengths $\binom{n}{k_1 \; k_2 \; \ldots \; k_m}$, not all k_js equal, and one orbit of length $\frac{1}{m}\binom{n}{k \; k \; \ldots k}$ if $m \mid n$ and $k = \frac{n}{m}$,

(ii) $\frac{1}{m}\binom{n+m-1}{m-1}$ orbits of lengths $\binom{n}{k_1 \; k_2 \; \ldots \; k_m}$, if $m \nmid n$.

Remark 1.1.4 *Following from the above, it can be shown easily that the inertia factor group of the irreducible character θ of S in $B_S(m, n)$ is a subgroup (called a young subgroup) $S_{k_1} \times S_{k_2} \times \cdots S_{k_m}$ of S_n, for some $(k_1, k_2, \ldots, k_m) \in A(n, m)$. However, if $m \mid n$ and $(k, k, \ldots, k)$, $k = \frac{n}{m}$, is one of the m-compositions of n, the corresponding inertia factor group of the irreducible character θ of S in $B_S(m, n)$ is given by the split extension $(S_{\frac{n}{m}})^m : C_m$ [35]. We note from above that $(S_{\frac{n}{m}})^m : C_m$ is a subgroup of S_n.*

As an example, we know from [28] that the simplectic group $SP(6,2)$, which is a maximal subgroup of the Fischer group $F_{i_{22}}$, has a subgroup of form $2^5 : S_6$. Further the group $2^5 : S_6$ has a subgroup isomorphic to the split extension $(S_3)^2 : C_2$ or the wreath product of the symmetric group S_3 of degree 3 with the cyclic group C_2 of order 2. In this work, we will also construct the character tables of some examples of the groups $(S_n)^m : C_m$, where m is prime. These groups will include $(S_3)^2 : C_2$, $(S_3)^3 : C_3$, $(S_4)^2 : C_2$ and $(S_4)^3 : C_3$ of orders 72, 648, 1152 and 41472 respectively.

The group $(S_n)^m : C_m$ is a subgroup of Smn of degree $m \times n$.

1.2 Aims and Methodology

Several methods for constructing the character tables of group extensions exist. However, Fischer [12] has given an effective method for constructing the character tables of some group extensions including the groups cited above. This method known as the **technique of the Fischer-Clifford matrices** makes use of Clifford Theory [7, 20]. Given a group extension $\overline{G} = H.G$ such that every irreducible character of H can be extended to its inertia group, for each conjugacy class of G we construct a matrix called a **Fischer-Clifford matrix of** $\overline{G}$. By using the Fischer-Clifford matrices of $\overline{G}$ together with the fusion maps and the character tables of inertia factor groups of $\overline{G}$, we are able to construct the character table of $\overline{G}$. We note that Fischer-Clifford matrices satisfy certain properties which may be used to construct them.

The method of the Fischer-Clifford matrices has been used in many works both on split and non-split extensions [1, 10, 12, 25, 28, 30, 34, 35]. Here we will use the method of Fischer-Clifford matrices to construct the character tables of the groups of form $2^4 : S_5$ and $2^4 : A_7$. We will also use the method of Fischer-Clifford matrices to construct the character tables of the groups $(S_3)^2 : C_2$, $(S_3)^3 : C_3$, $(S_4)^2 : C_2$ and $(S_4)^3 : C_3$.

The following is the arrangement of the chapters of this Dissertation. In chapter1 we describe the groups whose character tables we are going to construct in this project. Chapter 2 is devoted to the general theory on the work being discussed here. We discuss the theory of group extensions in Section 2.1 followed by Representations and Characters of finite groups in Section 2.2. We discuss Clifford Theory and Fischer-Clifford matrices in Sections 2.3 and 2.4 respectively. We give examples for the construction of the Fischer-Clifford matrices and character tables of the groups studied here in Chapters 3 and 4. The results given here are numbered c.s.n meaning a result number n in Section s of Chapter c. All the character-tables constructed in this work have been tested in GAP [14].

Chapter 2

Preliminaries

In this chapter we discuss basic representation theory of finite groups, especially group extensions, and Clifford theory. For further information on the following material, readers may consult the following references [3], [4], [8], [9], [11], [12], [15], [16], [17], [18], [20], [21], [22], [23], [24], [30], [31], [32], [33] and any other relevant sources.

2.1 Group Extensions

In this section we present some definitions and results on finite groups especially group extensions.

Definition 2.1.1 *Let N and G be groups. Then an **extension** of N by G is a group $\overline{G}$ such that*

(i) $N \lhd \overline{G}$.

(ii) $\overline{G}/N \cong G$.

Let $\overline{G} = N.G$ denote an extension of N by G.

Definition 2.1.2 *A group extension $\overline{G}$ of N by G is called a **split extension** if there is a homomorphism φ of G into $\overline{G}$ such that $\phi\varphi$ =identity, where ϕ is the homomorphism from $\overline{G}$ to G with kernel N.*

Let $\overline{G} = N : G$ denote a split extension of N by G.

Definition 2.1.3 *A group $\overline{G}$ is a* ***semi-direct product*** *of N by K if*

(i) N *and* K *are subgroups of* $\overline{G}$

(ii) $\overline{G} = NK$.

(ii) $N \lhd \overline{G}$.

(iii) $N \cap K = \{1\}$.

Proposition 2.1.4 *If $\overline{G}$ is a semi-direct product of N by G then there is a homomorphism $\phi : G \longrightarrow Aut(N)$ defined by $\phi(g) = \phi_g$ for each $g \in G$, where ϕ_g is an automorphism of N given by $\phi_g(n) = n^g = gng^{-1}$.*

Proof. See [19] □

Theorem 2.1.5 *Every split extension of N by G is equivalent to a semi-direct product of N by G.*

Proof. See [30] or [33] □

Let $\overline{G}$ be a split extension of N by G, then every $\overline{g} \in \overline{G}$ can be written uniquely as $\overline{g} = ng$, where $n \in N$ and $g \in G$ with the composition of elements of $\overline{G}$ given by $(n_1g_1)(n_2g_2) = n_1n_2^{g_1}g_1g_2$.

Definition 2.1.6 *Let $G = N \cdot G$ and $\{1\} \longrightarrow N \longrightarrow \overline{G} \xrightarrow{\pi} G \longrightarrow \{1\}$ be the corresponding short exact sequence. Let $g \in G$ and $\overline{g} \in \overline{G}$ such that $\pi(\overline{g}) = g$. Then $\overline{g}$ is called a lifting of g in $\overline{G}$.*

Theorem 2.1.7 *Let $\overline{G}$ be an extension of N by G where N is abelian. Then there is a homomorphism $\theta : G \longrightarrow Aut(N)$ such that $\theta_g(n) = \overline{g}n(\overline{g})^{-1}$,$n \in N$ and θ is independent of the choice of liftings of $\{\overline{g} \mid g \in G\}$.*

Proof. See [30] or [34]

2.1.1 Conjugacy Classes of Group Extensions

The conjugacy classes of a finite group give some important information about the structure of the group. Since the number of irreducible ordinary characters of a finite group is equal to the number of conjugacy classes of the group, having information about conjugacy classes of a finite group is the best point to start constructing the character table of the group. Several works have been done on the properties of conjugacy classes as well as comparisons between results on conjugacy classes and characters of finite groups. For example we have the following result.

Proposition 2.1.8 *Let $\overline{G} = N.G$, $\overline{g}$ a lifting of $g \in G$, C be the centralizer of $N\overline{g}$ in G and $\overline{C}$ be the complete preimage in $\overline{G}$ of C. Then*

(i) the union of the cosets $N\overline{x}$ which are conjugate in G to $N\overline{g}$, is the union of the conjugacy classes $L_1, L_2, \ldots, L_r$ of $\overline{G}$,

(ii) $\overline{C}$ acts on the coset $N\overline{g}$ by conjugation,

(iii) $\overline{C}$ has r orbits in its action on $N\overline{g}$ and the orbit representatives $\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_r$ are representatives of the conjugacy classes $L_1, L_2, \ldots, L_r$ of $\overline{G}$,

(iv) the centralizer $C_{\overline{G}}(\overline{g}_i)$ for $1 \leq i \leq r$ is the stabilizer of $\overline{g}_i$ in $\overline{C}$ in its action on $N\overline{g}$.

Proof. See [5]. □

In recent times the determination of the conjugacy classes of a group using computational methods has become very common. For example in [5] and [6], Butler gives various algorithms which can be used for computing conjugacy classes in finite groups and in permutation groups respectively. The technique of **coset analysis**, used for the determination of the conjugacy classes of elements of both split and non-split extensions $\overline{G} = N.G$ where N is an abelian normal subgroup of $\overline{G}$, was developed by Moori in [27]. For each conjugacy class $[g]$ in G with representative g, we analyse the coset $N\overline{g}$, where $\overline{g}$ is a lifting of g in $\overline{G}$. For each class $[g]$ of G, we define

$$C_{\overline{g}} = \{x \in \overline{G} \mid x(N\overline{g}) = (N\overline{g})x\},$$

the stabilizer of $N\overline{g}$ in $\overline{G}$ under the action on $N\overline{g}$ by conjugation of $\overline{G}$. Since N is normal in $\overline{G}$, it is clear that N is a normal subgroup of $C_{\overline{g}}$.

Lemma 2.1.9 *[34]* $C_{\overline{g}}/N = C_{\overline{G}/N}(N\overline{g})$.

Proof. Let $k \in \overline{G}$. Then

$$\begin{aligned}
Nk \in C_{\overline{G}/N}(N\overline{g}) &\Leftrightarrow Nk(N\overline{g})(Nk)^{-1} = N\overline{g} \\
&\Leftrightarrow NkN\overline{g}Nk^{-1} = N\overline{g} \\
&\Leftrightarrow NkN\overline{g}k^{-1} = N\overline{g} \\
&\Leftrightarrow NkNn\overline{g}k^{-1} = N\overline{g} \;\; \forall n \in N \\
&\Leftrightarrow Nkn\overline{g}k^{-1} = N\overline{g} \;,\; \forall n \in N \\
&\Leftrightarrow kn\overline{g}k^{-1} \in N\overline{g} \;,\; \forall n \in N \\
&\Leftrightarrow k \in C_{\overline{g}} \\
&\Leftrightarrow Nk \in C_{\overline{g}}/N \;. \qquad \square
\end{aligned}$$

Since $N \trianglelefteq C_{\overline{g}}$ and by Lemma 2.1.9, it follows that $C_{\overline{g}} = N.C_{\overline{G}/N}(N\overline{g})$. For each conjugacy class $[g]$ of G, the conjugacy classes of $\overline{G}$ where N is abelian is determined by the action by conjugation of $C_{\overline{g}}$ on the elements of $N\overline{g}$. To act $C_{\overline{g}}$ on the elements of $N\overline{g}$, we act N and then act $\{\overline{h} \mid h \in C_G(g)\}$, where $\overline{h}$ is a lifting of h in $\overline{G}$.

STEP 1: The action of N on $N\overline{g}$: Let $C_N(\overline{g})$ be the stabilizer of $\overline{g}$ in N. Then for any $n \in N$ we have

$$\begin{aligned}
x \in C_N(n\overline{g}) &\Leftrightarrow x(n\overline{g})x^{-1} = n\overline{g} \\
&\Leftrightarrow xnx^{-1}x\overline{g}x^{-1} = n\overline{g} \\
&\Leftrightarrow n(x\overline{g}x^{-1}) = n\overline{g} \;, \quad \text{since } N \text{ is abelian} \\
&\Leftrightarrow x\overline{g}x^{-1} = \overline{g} \\
&\Leftrightarrow x \in C_N(\overline{g}).
\end{aligned}$$

Thus $C_N(\overline{g})$ fixes every element of $N\overline{g}$. Now let $|C_N(\overline{g})| = k$. Then under the action of N, $N\overline{g}$ splits into k orbits $Q_1, Q_2, \ldots, Q_k$, where

$$|Q_i| = [N : C_N(\overline{g})] = \frac{|N|}{k} \;\;,$$

for $i \in \{1, 2, \ldots, k\}$.

STEP 2: The action of $\{\overline{h} \mid h \in C_G(g)\}$ on $N\overline{g}$: Since the elements of $N\overline{g}$ are now in the orbits $Q_1, Q_2, \ldots, Q_k$ from Step 1 above, we need only act $\{\overline{h} \mid h \in C_G(g)\}$

on these k orbits. Suppose that under this action f_j of these orbits $Q_1, Q_2, \ldots, Q_k$ fuse together to form one orbit Δ_j, then we have

$$|\Delta_j| = f_j \times \frac{|N|}{k} \quad .$$

Thus for $x = d_j\overline{g} \in \Delta_j$, we obtain that

$$\begin{aligned} |[x]_{\overline{G}}| &= |\Delta_j| \times |[g]_G| \\ &= f_j \times \frac{|N|}{k} \times \frac{|G|}{|C_G(g)|} \\ &= f_j \times \frac{|\overline{G}|}{k|C_G(g)|} \end{aligned}$$

and thus we obtain that

$$|C_{\overline{G}}(x)| = \frac{|\overline{G}|}{|[x]_{\overline{G}}|} = |\overline{G}| \times \frac{k|C_G(g)|}{f_j|\overline{G}|} = \frac{k|C_G(g)|}{f_j} \quad .$$

Thus to calculate the conjugacy classes of $\overline{G} = N.G$, we find the values of k and the f_j's for each class representative $g \in G$.

However for the special case of a split extension $\overline{G} = N{:}G$, we identify $C_{\overline{g}}$ with $C_g = \{x \in \overline{G} \mid x(Ng) = (Ng)x\}$, where the lifting of g in $\overline{G}$ is g itself since $G \leq \overline{G}$.

Corollary 2.1.10 *[30] If $\overline{G} = N{:}G$, then $C_g = N{:}C_G(g)$.*

Proof. We have that N is a normal subgroup of C_g. Now we show that $C_G(g) \leq C_g$ and that $N \cap C_G(g) = \{1\}$. Let $x \in C_G(g)$. Then we obtain $(Ng)^x = x(Ng)x^{-1} = xNgx^{-1} = Nxgx^{-1} = Ng$. Thus $x \in C_g$ and hence $C_G(g) \leq C_g$. Since $N \cap C_G(g) \leq N \cap G = \{1_G\}$, then we have that $N \cap C_G(g) = \{1_G\}$. Hence the result. □

Thus in the case of a split extension $\overline{G} = N{:}G$, we analyse the coset Ng instead of $N\overline{g}$. Under the action of N on Ng, we always assume that $g \in Q_1$. Also instead of acting $\{\overline{h} \mid h \in C_G(g)\}$ on the k orbits $Q_1, Q_2, \ldots, Q_k$ we just act $C_G(g)$ on these orbits.

The technique of coset analysis for computing conjugacy classes of group extensions $\overline{G} = N.G$ has since been used in several works (see [1], [2], [27], [28], [30], [34]). The orders of the elements of conjugacy classes of $\overline{G} = N.G$ may be determined in different ways. However the following results are useful in the determination of the orders of the elements of a group $\overline{G} = N{:}G$.

Theorem 2.1.11 *[30] Let $\overline{G} = N{:}G$ and $dg \in \overline{G}$ where $d \in N$ and $g \in G$ such that $o(g) = m$ and $o(dg) = k$. Then m divides k.*

Proof. We have that

$$1_{\overline{G}} = (dg)^k = dd^g d^{g^2} d^{g^3} \ldots d^{g^{k-1}} g^k \quad .$$

Since G acts on N and $d \in N$, we have $d, d^g, d^{g^2}, \ldots, d^{g^{k-1}} \in N$. Hence $dd^g d^{g^2} \ldots d^{g^{k-1}} \in N$. Thus we must have that $dd^g d^{g^2} \ldots d^{g^{k-1}} = 1_N$ and $g^k = 1_G$. Hence m divides k. □

Theorem 2.1.12 *[30] Let $\overline{G} = N{:}G$ such that N is an elementary abelian p-group, where p is prime. Let $dg \in \overline{G}$ where $d \in N$ and $g \in G$ such that $o(g) = m$ and $o(dg) = k$. Then either $k = m$ or $k = pm$.*

Proof. See Theorem 2.3.10 in [30]. □

Remark 2.1.13 *[30] Let $\overline{G} = N{:}G$ where N is an elementary abelian p-group. Let $dg \in \overline{G}$ where $d \in N$ and $g \in G$ such that $o(g) = m$ and $o(dg) = k$. Then we have that*

$$(dg)^m = dd^g d^{g^2} d^{g^3} \ldots d^{g^{m-1}} g^m \quad .$$

Since $g^m = 1_G$, we obtain that $(dg)^m = w$, where $w \in N$. Now applying Theorem2.1.12 we have if $w = 1_N$ then $k = m$ and if $w \neq 1_N$ then $k = pm$.

Remark 2.1.14 *However if both N and G are permutation groups, the order of $\overline{g}$ can easily be determined as the least common multiple of n and g with $\overline{g} = ng$ written as a product of disjoint cycles.*

In [30] Mpono has developed computer programmes in CAYLEY which are used for computing the conjugacy classes and the orders of the conjugacy class representatives of the group extension $\overline{G} = N{:}G$ where N is an elementary abelian p-group for prime p on which a linear group G acts. These programmes can similarly be applied to the group extension $\overline{G} = N{:}G$ where N is an elementary abelian p-group for prime p on which a permutation group G acts, by considering a subgroup of a linear group which is isomorphic to G.

2.2 Representations and Characters of Finite Groups

Here, we present some results on group representations and characters which are useful for the technique of the Fischer-Clifford matrices as will be discussed in Section 2.4. The work discussed here does not involve projective characters and therefore we devote our discussion to ordinary complex characters. For example we discuss the relationship between the ordinary characters of a group and its subgroups. For further reading on representations and characters, readers are encouraged to consult [9], [13], [17], [20], [21], [22], [23], [30] and other relevant sources. In the following, most of the proofs have been omitted but reference is made to [13] for an extensive treatment of character theory.

Definition 2.2.1 *Let G be a group, F a field and $GL(n,F)$ the general linear group or the multiplicative group of all nonsingular $n\times n$ matrices over F for some integer n. Then a* ***representation of** G **over** F* *is a homomorphism $\rho : G \longrightarrow GL(n,F)$. The* ***degree of the representation*** *ρ is the integer n. Define the function $\chi : G \longrightarrow F$ by $\chi(g) = trace(\rho(g))$. Then χ is called the* ***character of** G **afforded by the representation*** *ρ. The character χ has the same degree as ρ.*

Definition 2.2.2 *Let ρ_1 and ρ_2 be representations of G over F. We say ρ_1 and ρ_2 are* ***equivalent*** *if there exists $P \in GL(n,F)$ such that $\rho_1(g) = P\rho_2(g)P^{-1}$ for all $g \in G$. We say a representation ρ of G is* ***reducible*** *if it is equivalent to a representation α given by*

$$\alpha(g) = \begin{pmatrix} \beta(g) & \gamma(g) \\ 0 & \delta(g) \end{pmatrix}$$

for all $g \in G$, where β, γ, δ are representations of G. A representation ρ which is not reducible is said to be ***irreducible****.*

It is clear that equivalent representations afford the same character, since similar matrices have the same trace. The character afforded by an irreducible representation is called an *irreducible* character. Further it is elementary to prove the following properties hold.

(i) Representations of G have the same character if and only if they are equivalent.

(ii) The number of irreducible ordinary characters of G is equal to the number of conjugacy classes of elements of G.

(iii) Any character of G can be written as a sum of irreducible characters.

(iv) Sums and products of characters of G are also characters.

Theorem 2.2.3 (**Schur's Lemma**) *Let $\rho_1 : G \longrightarrow GL(n, F)$ and $\rho_2 : G \longrightarrow GL(m, F)$ be two irreducible representations of a group G over a field F. Assume that there exists a matrix P such that $P\rho_1(g) = \rho_2(g)P$ for all $g \in G$. Then either P is the zero matrix or P is nonsingular so that $\rho_1(g) = P^{-1}\rho_2(g)P$.*

Proof. See Theorem 1.8 of [26]. □

Definition 2.2.4 *Let G be a group, F a field and $\phi : G \longrightarrow F$ be a function which is constant on conjugacy classes. Then ϕ is called a **class function** of G.*

It is clear from the above definition that characters are class functions. Let $Irr(G)$ denote the set of irreducible characters of the group G.

Definition 2.2.5 *Let G be a group, χ be a character of G and $Irr(G) = \{\chi_1, \chi_2, \ldots, \chi_r\}$ such that $\chi = \sum_{i=1}^{r} n_i\chi_i$, where $n_i \in \mathbb{N} \cup \{0\}$. Then those χ_i for which $n_i > 0$ are called the **irreducible constituents** of χ.*

Lemma 2.2.6 *Let G be a group, ρ be a representation of G which affords the character χ. Let $g \in G$ such that $o(g) = n$. Then the following conditions hold*

(i) $\rho(g)$ is similar to a diagonal matrix $diag(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r)$

(ii) $\varepsilon_i^n = 1$

(iii) $\chi(g) = \sum_i \varepsilon_i$

(iv) $|\chi(g)| \leq \chi(1_G)$ = degree of χ

(v) $\chi(g^{-1}) = \overline{\chi(g)}$, where $\overline{\chi(g)}$ is the complex conjugation of $\chi(g)$.

Proof. See lemma 2.15 in [20]. □

Definition 2.2.7 *Let χ and ψ be class functions of a group G. Then the* ***inner product*** *of χ and ψ is defined by*

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)} \quad .$$

The following theorems are called the first and second orthogonality relations respectively.

Theorem 2.2.8 *[20](***First Orthogonality Relation***) Let G be a group and $Irr(G) = \{\chi_1, \chi_2, \ldots, \chi_r\}$. Then*

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \delta_{ij} = \langle \chi_i, \chi_j \rangle \quad .$$

Proof. See [20] result 2.14. □

Theorem 2.2.9 *[20](***Second Orthogonality Relation***) Let G be a group and $Irr(G) = \{\chi_1, \chi_2, \ldots, \chi_r\}$ and $\{g_1, g_2, \ldots, g_r\}$ be a set of representatives of the conjugacy classes of elements of G. Then*

$$\sum_{\chi \in Irr(G)} \chi(g_i)\overline{\chi(g_j)} = \delta_{ij}|C_G(g_i)| \quad .$$

Proof. See [20] result 2.18. □

2.2.1 Lifted Characters

Let G be a group and χ be a character of G afforded by a representation ρ. Then

$$ker(\chi) = \{g \in G \mid \chi(g) = \chi(1_G)\} \quad ,$$

can be shown easily to be a normal subgroup of G (see [34]). Further every normal subgroup of G is an intersection of some of the $ker(\chi_i)$, where $\chi_i \in Irr(G)$. If N is a normal subgroup of G and ρ is a representation of G such that $N \subseteq ker(\rho)$, then there exists a unique representation $\hat{\rho}$ of G/N defined by $\hat{\rho}(Ng) = \rho(g)$. Further ρ is irreducible if and only if $\hat{\rho}$ is irreducible. If ρ affords a character χ of G, then $\hat{\rho}$ affords a character $\hat{\chi}$ of G/N.

Theorem 2.2.10 *There is a one to one correspondence between the set of characters of G/N and the set of characters of G which contain N in their Kernel. Thus*

$$Irr(G/N) = \{\chi \in Irr(G) \mid N \subseteq ker(\chi)\}.$$

Proof. See Theorem 17.3. of [22]. □

Following from above we have the following definition.

Definition 2.2.11 *Let G be a group, N a normal subgroup of G and $\hat{\chi}$ be a character of G/N. Then the character χ of G defined by*

$$\chi(g) = \hat{\chi}(Ng)$$

is called a ***lifting*** *of $\hat{\chi}$ to G.*

Thus we can use the characters of G/N to obtain some of the characters of G by the lifting process. It is clear from the definition that the degree of the lifted character χ is equal to the degree of the character $\hat{\chi}$.

2.2.2 Restriction of Characters

Definition 2.2.12 *Let H be a subgroup of a finite group G. If ρ is a representation of G, then the* ***restriction of*** *ρ to H is a representation of H, which is denoted by ρ_H. If χ is a character of G afforded by ρ, then the* ***restriction of*** *χ to H is a character of H afforded by the representation ρ_H and is denoted by χ_H.*

The characters χ_H and χ take on the same values on the elements of H. The character χ_H is generally not irreducible. But under certain conditions χ_H is irreducible, for example in [23], Karpilovsky proves a theorem (Theorem 23.1.4) due to Gallagher(1966) that if $H \leq G$, $\chi \in Irr(G)$ such that $\chi(g) \neq 0 \; \forall \; g \in G\backslash H$, then χ_H is irreducible. We also have

Theorem 2.2.13 *Suppose that $H \leq G$ of index 2 in G and $\chi \in Irr(G)$. Let λ be a linear character of G taking value 1 on H and -1 otherwise. Then the following conditions are equivalent*

(i) χ_H *is irreducible*

(ii) $\chi(g) \neq 0$ *for some* $g \in G \backslash H$*,*

(iii) $\chi \neq \chi\lambda$

Proof. See Proposition 20.9. of [22]. □

However we can easily show that if χ_H is irreducible, then χ is irreducible in G.

2.2.3 Induced Characters

Let H be a subgroup of a group G with transversal set $\{x_1, x_2, \ldots, x_r\}$ in G. Let ϕ be a representation of H of degree n. Then we define ϕ^* on G as follows:

$$\phi^*(g) = \begin{pmatrix} \phi(x_1 g x_1^{-1}), \phi(x_1 g x_2^{-1}), \ldots, \phi(x_1 g x_r^{-1}) \\ \phi(x_2 g x_1^{-1}), \phi(x_2 g x_2^{-1}), \ldots, \phi(x_2 g x_r^{-1}) \\ \vdots \\ \phi(x_n g x_1^{-1}), \phi(x_n g x_2^{-1}), \ldots, \phi(x_n g x_r^{-1}) \end{pmatrix}$$

where $\phi(x_i g x_j^{-1})$ are $n \times n$ sub-matrices of $\phi^*(g)$ satisfying the property that

$$\phi(x_i g x_j^{-1}) = 0_{n\times n} \ \ \forall\, x_i g x_j^{-1} \notin H \quad .$$

Then ϕ^* is a representation of G of degree n.

Definition 2.2.14 *Let G, H, ϕ and ϕ^* be as above. Then the representation ϕ^* is called the **representation of G induced** from the representation ϕ of H. The induced representation of ϕ is denoted by ϕ^G.*

Definition 2.2.15 *Let G be a group and $H \leq G$. Let θ be a class function of H. Then we define θ^G as follows:*

$$\theta^G(g) = \frac{1}{|H|}\sum_{x\in G}\theta^\circ(xgx^{-1}),$$

where

$$\theta^\circ(h) = \begin{cases} \theta(h) & \text{if } h \in H \\ 0 & \text{otherwise.} \end{cases}$$

Then θ^G is a class function of G, called the ***induced class function of*** *G or* ***the class function of*** *G* ***induced from*** *θ. Further we have $deg(\theta^G) = [G : H]deg(\theta)$.*

Let ϕ be a representation of H that affords a character θ. Then θ^G is a character of G afforded by the induced representation ϕ^G of G. The character θ^G is called the **induced character** of G. The induced character is not irreducible in general. There is a relationship between restricted and induced characters of G given by the following result.

Theorem 2.2.16 *[20](***Frobenius Reciprocity Theorem***) Let G be a group, $H \leq G$ and suppose that θ is a class function of H and χ is a class function of G. Then*

$$\langle\theta, \chi_H\rangle = \langle\theta^G, \chi\rangle$$

Proof. We obtain that

$$\langle\theta^G, \chi\rangle = \frac{1}{|G|}\sum_{g\in G}\theta^G(g)\overline{\chi(g)} = \frac{1}{|G|}\frac{1}{|H|}\sum_{g\in G}\sum_{x\in G}\theta^\circ(xgx^{-1})\overline{\chi(g)} \quad .$$

Putting $y = xgx^{-1}$ and since χ is a class function, then we obtain that $\chi(y) = \chi(g)$. Hence we have

$$\begin{aligned}\langle\theta^G, \chi\rangle &= \frac{1}{|G|}\frac{1}{|H|}\sum_{g\in G}\sum_{x\in G}\theta^\circ(xgx^{-1})\overline{\chi(g)} = \frac{1}{|G|}\frac{1}{|H|}\sum_{y\in G}\sum_{x\in G}\theta^\circ(y)\overline{\chi(y)} \\ &= \frac{1}{|H|}\sum_{y\in H}\theta(y)\overline{\chi(y)} = \langle\theta, \chi_H\rangle.\end{aligned}$$

□

Theorem 2.2.17 *Let G be a group and $H \leq G$. Let θ be a character of H, $g \in G$ and $\{x_1, x_2, \ldots, x_m\}$ be a set of representatives of the conjugacy classes of elements of H which fuse into $[g]$ in G. Then we obtain that*

$$\theta^G(g) = |C_G(g)| \sum_{i=1}^{m} \frac{\theta(x_i)}{|C_H(x_i)|} \quad ,$$

where we have that $\theta^G(g) = 0$ whenever $H \cap [g] = \emptyset$.

Proof. We have that

$$\theta^G(g) = \frac{1}{|H|} \sum_{x \in G} \theta^\circ(xgx^{-1}) \quad .$$

If $H \cap [g] = \emptyset$, then $xgx^{-1} \notin H$ and thus $\theta^\circ(xgx^{-1}) = 0 \;\; \forall \, x \in G$ and hence $\theta^G(g) = 0$. Now if $H \cap [g] \neq \emptyset$, then let $h \in H \cap [g]$. Then as x runs over G, then $xgx^{-1} = h$ for exactly $|C_G(g)|$ values of x. Hence we obtain that

$$\theta^G(g) = \frac{1}{|H|} \sum_{x \in G} \theta(xgx^{-1}) = \frac{|C_G(g)|}{|H|} \sum_{h \in H \cap [g]} \theta(h) = |C_G(g)| \sum_{i=1}^{m} \frac{\theta(x_i)}{|C_H(x_i)|} \quad .$$

□

2.2.4 Permutation Characters

Let Ω be a set. A group G is said to act on a set Ω if there is a homomorphism $\phi : G \longrightarrow S_\Omega$, where S_Ω is the symmetric group on Ω. Therefore G can be identified with a subgroup of S_Ω or G is isomorphic to a **permutation group** on Ω.

Definition 2.2.18 *A group G is said to be* ***transitive*** *if G has only one orbit on Ω.*

If G acts on $\Omega = \{x_1, x_2, \ldots, x_n\}$, we define a representation $\pi : G \longrightarrow GL(n, \mathbb{C})$, where $n = |\Omega|$. For each $g \in G$ we define $\pi(g) = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1 & \text{if } x_i^g = x_j \\ 0 & \text{otherwise.} \end{cases}$$

Then $\pi(g)$ is a permutation matrix of the action of g. The representation π defined above is called the **permutation representation** of G obtained from the action of G on Ω. The character afforded by the permutation representation π denoted by

$\chi(G|\Omega)$, is called the **permutation character** of G associated with the action of G on Ω. Further for $g \in G$ we have

$$\chi(G|\Omega)(g) = |\{x \in \Omega \ : \ x^g = x\}| = \quad \text{the number of points of } \Omega \text{ fixed by } g.$$

Suppose that G acts transitively on Ω and G_x is the stabilizer of $x \in \Omega$. Then it is known that the action of G on Ω is the same as the action of G on the cosets of $H = G_x$. Hence $\forall\ g \in G$, $\chi(G|\Omega)(g)$ also gives the number of cosets of $H = G_x$ which are fixed by $g \in G$ and in this case we denote this number by $\chi(G|H)(g)$. Thus $\chi(G|H) = \chi(G|\Omega)$.

Theorem 2.2.19 *Let G be a group acting transitively on a set Ω. Let $\alpha \in \Omega$, $H = G_\alpha$ and $\chi(G|H)$ be the permutation character of this action. Then*

$$\chi(G|H) = (I_H)^G \quad .$$

Proof. We have that

$$(I_H)^G(g) = \frac{1}{|H|} \sum_{x \in G, xgx^{-1} \in H} I_H(xgx^{-1}) = \frac{1}{|H|} \sum_{x \in G, xgx^{-1} \in H} 1 \quad .$$

Now if $xgx^{-1} \in H$, then $xg \in Hx$. Thus $Hxg = Hx$ and hence Hx is fixed by $g \in G$. However the summation is taken over all $x \in G$ such that $xgx^{-1} \in H$. Hence the summation is taken over all $x \in G$ for which the coset Hx is fixed by $g \in G$. But $\forall\ y \in Hx$, $Hx = Hy$ and thus we obtain that

$$\sum_{x \in G, xgx^{-1} \in H} 1 = |H||\{Hx \mid Hxg = Hx\}|$$

and hence we obtain that

$$(I_H)^G(g) = \frac{1}{|H|}|H||\{Hx \mid Hxg = Hx\}| = |\{Hx \mid Hxg = Hx\}| = \chi(G|H)(g) \quad .$$

□

Let G be a group, $H \leq G$ and $\chi = \chi(G|H)$. The following are some properties of permutation characters.

(i) $deg(\chi)$ divides $|G|$.

(ii) $\langle \chi, \psi \rangle \leq deg(\psi)$ for all $\psi \in Irr(G)$.

(iii) $\langle \chi, I_G \rangle = 1$.

(iv) $\chi(g) \in \mathbb{N} \cup \{0\}$ for all $g \in G$.

(v) $\chi(g) \leq \chi(g^m)$ for all $g \in G$ and $m \in \mathbb{N} \cup \{0\}$.

(vi) $\chi(g) = 0$ if $o(g)$ does not divide $|G|/deg(\chi)$.

(vii) $\chi(g)\frac{|[g]|}{deg(\chi)}$ is an integer for all $g \in G$.

Proof. See Theorem 2.5.6 in [34]. □

In the following section, we discuss the general theory of the technique of Fischer-Clifford matrices. Later, we will use this technique to construct the character-tables of the groups studied in this project. For the theory on Fischer-Clifford matrices, we follow the works of Faryad [1], Mpono [30] and Whitely [34].

2.3 Clifford Theory

Definition 2.3.1 *Let G be a group, $H \leq G$ and θ be a character of H. Then for $g \in G$, we define $\theta^g : gHg^{-1} \longrightarrow \mathbb{C}$ by $\theta^g(t) = \theta(g^{-1}tg)$ for all $t \in gHg^{-1}$. Then θ^g is said to be a G-conjugate of θ. If H is a normal subgroup of G and $\theta^g = \theta$ for all $g \in G$, then θ is said to be G-**invariant**.*

It is clear that θ^g is a character of gHg^{-1}.

Theorem 2.3.2 *[20](**Clifford's Theorem**)*

Let G be a group, H a normal subgroup of G and $\chi \in Irr(G)$. Let θ be an irreducible constituent of χ_H and $\theta_1, \theta_2, \ldots, \theta_n$ be distinct conjugates of θ in G such that $\theta_1 = \theta$. Then

$$\chi_H = e\sum_{i=1}^{n} \theta_i, \quad where \quad e = \langle \chi_H, \theta \rangle \quad .$$

Proof. Now for $h \in H$

$$\theta^G(h) = \frac{1}{|H|}\sum_{x\in G}\theta^\circ(xhx^{-1}) = \frac{1}{|H|}\sum_{x\in G}\theta^x(h) \quad .$$

Thus we obtain that

$$(\theta^G)_H = \frac{1}{|H|}\sum_{x\in G}\theta^x \quad .$$

Let $\phi \in Irr(H)$ such that $\phi \notin \{\theta_i \mid 1 \leq i \leq n\}$. Then we obtain that

$$\langle\sum_{x\in G}\theta^x, \phi\rangle = 0$$

and hence $\langle(\theta^G)_H, \phi\rangle = 0$. However by the Frobenius reciprocity theorem, we obtain that $\langle\chi_H, \theta\rangle = \langle\chi, \theta^G\rangle$. Hence χ is an irreducible constituent of θ^G. Since $\langle(\theta^G)_H, \phi\rangle = 0$, then $\langle\chi_H, \phi\rangle = 0$. Thus ϕ is not an irreducible constituent of χ_H. Hence all the irreducible constituents of χ_H are among the θ_i and thus we obtain that

$$\chi_H = \sum_{i=1}^{n}\langle\chi_H, \theta_i\rangle\theta_i = \sum_{i=1}^{n}\langle\chi_H, \theta\rangle\theta_i = \langle\chi_H, \theta\rangle\sum_{i=1}^{n}\theta_i = e\sum_{i=1}^{n}\theta_i \quad ,$$

where $e = \langle\chi_H, \theta\rangle$. □

Definition 2.3.3 *Let ϕ be a representation of G and α an automorphism of G. Then ϕ^α is a representation of G given by*

$$\phi^\alpha(x) = \phi(x^\alpha) \quad \textit{and} \quad \phi^\alpha(xy) = \phi^\alpha(x)\phi^\alpha(y)$$

*for $x, y \in G$. If the representation ϕ affords a character χ of G, then the representation ϕ^α affords a character χ^α of G which is given by $\chi^\alpha(x) = \chi(x^\alpha)$ for $x \in G$. Then the representation ϕ^α and the character χ^α are called the **algebraic conjugates** of ϕ and χ respectively induced by the automorphism α.*

Let $X = (\chi_i(x_j))$ be the character table of G, where $\chi_i \in Irr(G)$, $1 \leq i \leq n$ and x_j, $1 \leq j \leq n$ are representatives of the conjugacy classes of elements of G. Then the automorphism α of G induces a permutation on the conjugacy classes of G and therefore also on the columns of X. For each $\chi_i \in Irr(G)$, we deduce that $\chi_i^\alpha \in Irr(G)$. Hence α induces a permutation on the irreducible characters χ_i of G and therefore also on the rows of X. Moreover since $\chi_i^\alpha(x_j) = \chi_i(x_j^\alpha)$, then the matrices obtained from X by these two operations are identical. We have the following result known as Brauer's Theorem.

Theorem 2.3.4 *[15](***Brauer's Theorem***) Let G be a group and K be a group of automorphisms of G. Then the number of orbits of K as a group of permutations on the irreducible characters of G is the same as the number of orbits of K as a group of permutations on the conjugacy classes of G.*

Proof. Let X be the character table of G. Then as a matrix, X is square and nonsingular. Let α be an automorphism of G such that $\alpha \in K$. Then α induces a permutation on the conjugacy classes of G and thus induces a permutation on the columns of X. Hence K acts on the conjugacy classes of G. Since $\alpha \in K$, then to each character χ of G, we obtain a character χ^α of G such that $\chi^\alpha \in Irr(G)$ whenever $\chi \in Irr(G)$. For $y \in G$, we obtain that $\chi^\alpha(y) = \chi(y^\alpha)$. Thus α induces a permutation on the rows of X. Hence K acts on the irreducible characters of G. Let X^α denote the image of X under α. Then we obtain that

$$P(\alpha)X = X^\alpha = XQ(\alpha) \quad ,$$

where $P(\alpha), Q(\alpha)$ are appropriate permutation matrices which are uniquely determined by $\alpha \in K$. Suppose that $\alpha, \beta \in K$. Then we obtain that $X^{\alpha\beta} = (X^\alpha)^\beta$. Also we have that

$$P(\alpha\beta)X = X^{\alpha\beta} = (X^\alpha)^\beta = (P(\alpha)X)^\beta = P(\beta)P(\alpha)X$$

and hence $P(\alpha\beta) = P(\beta)P(\alpha)$. We also have that $X^{\alpha\beta} = XQ(\alpha\beta)$ and $(X^\alpha)^\beta = (XQ(\alpha))^\beta = XQ(\alpha)Q(\beta)$. Since $X^{\alpha\beta} = (X^\alpha)^\beta$, we obtain that $XQ(\alpha\beta) = XQ(\alpha)Q(\beta)$. The non-singularity of X implies that $Q(\alpha\beta) = Q(\alpha)Q(\beta)$. Define mappings π_1 and π_2 on K by $\pi_1(\alpha) = (P(\alpha))^t$ and $\pi_2(\alpha) = Q(\alpha)$, where t denotes the transpose operation on matrices. Then π_1 and π_2 are permutation representations of K. Let θ_1 and θ_2 be the permutation characters afforded by π_1 and π_2 respectively. Since $X^{-1}P(\alpha)X = Q(\alpha)$, $P(\alpha)$ and $Q(\alpha)$ are similar and thus have the same trace. Since $trace(P(\alpha))^t = trace(P(\alpha))$, we have that $trace(P(\alpha))^t = trace(Q(\alpha))$. Hence $\theta_1 = \theta_2$ and π_1 and π_2 are equivalent. Let d_1, d_2 be the number of orbits of K on the irreducible characters and on the conjugacy classes of G respectively. Thus we observe that d_1 is the number of orbits of $\pi_1(K)$ in its action as a group of permutations. Also d_2 is the number of orbits of $\pi_2(K)$ in its action as a group of permutations. Since θ_1 is the permutation character of K acting on the irreducible characters of G, we obtain that $\langle \theta_1, I_K \rangle = d_1$. Also for θ_2, we obtain that $\langle \theta_2, I_K \rangle = d_2$. However $\theta_1 = \theta_2$ and thus $\langle \theta_1, I_K \rangle = \langle \theta_2, I_K \rangle$ and hence $d_1 = d_2$. □

Definition 2.3.5 *Let θ be a character of a subgroup H of a group G. Let*

$$I_G(\theta) = \{g \in N_G(H) \mid \theta^g = \theta\}.$$

*Then we call $I_G(\theta)$ the **inertia group** of θ in G. If H is normal in G, then*

$$I_G(\theta) = \{g \in G \mid \theta^g = \theta\} \quad .$$

We observe that $N_G(H)$ acts on the characters of H by $g : \theta \longmapsto \theta^g$ for all $g \in N_G(H)$. Then the inertia group of θ is the stabilizer of θ in $N_G(H)$. Hence $I_G(\theta) \leq N_G(H) \leq G$ and it is clear that H is a normal subgroup of $I_G(\theta)$.

Theorem 2.3.6 *[20] Let G be a group, H a normal subgroup of G, $\theta \in Irr(H)$ and $T = I_G(\theta)$. Let*

$$A = \{\psi \in Irr(T) \mid \langle \psi_H, \theta \rangle \neq 0\}$$

$$B = \{\chi \in Irr(G) \mid \langle \chi_H, \theta \rangle \neq 0\}$$

Then

(a) If $\psi \in A$, then $\psi^G \in Irr(G)$.

(b) If $\psi^G = \chi$ and $\psi \in A$, then $\langle \psi_H, \theta \rangle = \langle \chi_H, \theta \rangle$.

(c) If $\psi^G = \chi$ and $\psi \in A$, then ψ is the unique irreducible constituent of χ_T which sits in A.

(d) The map $\psi \longmapsto \psi^G$ is a bijection of A to B.

Proof.

(a) Let $\psi \in A$ and χ be an irreducible constituent of ψ^G. Then ψ is an irreducible constituent of χ_T. Since θ is an irreducible constituent of ψ_H, θ is an irreducible constituent of χ_H and thus $\chi \in B$. Now suppose that $\theta_1, \theta_2, \ldots, \theta_n$ are the distinct conjugates of θ in G, where $\theta_1 = \theta$. Then we obtain that $[G : T] = n$ and by Clifford's theorem, we obtain that $\chi_H = e\sum_{i=1}^{n} \theta_i$ for some $e \in \mathbb{N}$, where $e = \langle \chi_H, \theta \rangle$. Since θ is invariant in T, θ is self-conjugate in T. Hence by Clifford's theorem (applied to T, H and ψ) we get that $\psi_H = k\theta$ for some $k \in \mathbb{N}$ where $k = \langle \psi_H, \theta \rangle$. Since ψ is an irreducible constituent of χ_T, then we obtain that $k \leq e$. Hence we have

$$en\theta(1_H) = \chi(1_G) \leq \psi^G(1_G) = n\psi(1_T) = kn\theta(1_H) \leq en\theta(1_H)$$

and thus equality holds throughout. In particular, from this equality we obtain that $\psi^G(1_G) = \chi(1_G)$ and hence we obtain that $\psi^G = \chi$. Therefore $\psi^G \in Irr(G)$.

(b) We have that $\langle \chi_H, \theta \rangle = e$ and $\langle \psi_H, \theta \rangle = k$ and from the equality in part(a), we obtain that $k = e$ and thus $\langle \chi_H, \theta \rangle = \langle \psi_H, \theta \rangle$.

(c) Let $\phi \in A, \phi \neq \psi$ and ϕ is an irreducible constituent of χ_T. Then we obtain that

$$\langle \chi_H, \theta \rangle \geq \langle (\phi + \psi)_H, \theta \rangle = \langle \phi_H, \theta \rangle + \langle \psi_H, \theta \rangle > \langle \psi_H, \theta \rangle$$

which is a contradiction by part(b). Hence the result.

(d) The map $\psi \longmapsto \psi^G$ is well-defined by part(a). Also we obtain that for any $\psi \in A$, $\psi^G \in B$ by part(b). By the uniqueness assertion given by part(c), the map $\psi \longmapsto \psi^G$ is one-to-one. Then it suffices to show that the map is onto B. Let $\chi \in B$. Then θ is an irreducible constituent of χ_H and hence there exists an irreducible constituent ψ of χ_T such that $\langle \psi_H, \theta \rangle \neq 0$. Thus $\psi \in A$ and we have that χ is an irreducible constituent of ψ^G. Hence we obtain that $\chi = \psi^G$ since $\psi^G \in Irr(G)$ by part(a).

□

Remark 2.3.7 By Theorem 2.3.6, we deduce that induction to G maps the irreducible characters of T that contain θ in their restriction to H faithfully onto the irreducible characters of G that contain θ in their restriction to H.

Definition 2.3.8 *Let G be a group, H a normal subgroup of G, $\theta \in Irr(H)$ and $T = I_G(\theta)$. Since H is normal in T, we obtain the factor group T/H called the* ***inertia factor*** *of T.*

Let $\overline{G} = N{:}G$. Then for all $\theta \in Irr(N)$, define

$$\overline{H} = \{x \in \overline{G} \mid \theta^x = \theta\} = I_{\overline{G}}(\theta)$$

$$H = \{y \in G \mid \theta^y = \theta\} = I_G(\theta) \quad .$$

Then it can be shown that $\overline{H} = N{:}H$.

Definition 2.3.9 *Let G be a group, H a subgroup of G, $\theta \in Irr(H)$ and $\chi \in Irr(G)$ such that $\chi_H = \theta$. Then θ is said to be* ***extendible*** *to an irreducible character of G or θ can be* ***extended to*** *G.*

If θ is extendible to an irreducible character of G, we will say that θ is extendible to G.

Definition 2.3.10 *Let G be a group and F be a field. Then the map $\rho : G \longrightarrow GL(n, F)$ such that*

(i) $\rho(1_G) = I$, where I is the identity $n \times n$ matrix.

(ii) for all $x, y \in G$, there exists a map $\alpha : G \times G \longrightarrow F^*$ such that

$$\rho(x)\rho(y) = \alpha(x, y)\rho(xy) \quad \textit{where} \quad \alpha(x, y) \in F^* \quad .$$

Then ρ is called a ***projective representation*** *of G over F of degree n. The map α is called the* ***factor set*** *associated with ρ.*

Now if $\alpha(x, y) = 1_F$ for all $x, y \in G$, we obtain that

$$\rho(xy) = \rho(x)\rho(y),$$

that is ρ is an ordinary representation of G. Thus projective representations are generalizations of ordinary representations.

Theorem 2.3.11 *([9])(***Mackey's Theorem***) Let N be a normal subgroup of $\overline{G}$ and θ be a $\overline{G}$-invariant irreducible character of N. If N is abelian and $\overline{G}$ splits over N, then θ can be extended to $\overline{G}$.*

Proof. Let $\overline{G} = N{:}G$. Since $\overline{G}$ is a semidirect product of N by G, then any $x \in \overline{G}$ can be expressed uniquely as $x = ng$, where $n \in N, g \in G$. Define χ on $\overline{G}$ by $\chi(ng) = \theta(n)$. Since N is abelian, θ has degree 1 and thus is linear. The invariance of θ in $\overline{G}$ implies that $\theta(n) = \theta(xnx^{-1})$ for all $x \in \overline{G}$.

Now let $x_1 = n_1g_1, x_2 = n_2g_2$ be elements of $\overline{G}$. Then we obtain that

$$\begin{aligned}\chi(x_1x_2) &= \chi(n_1g_1n_2g_2) = \chi(n_1n_2^{g_1}g_1g_2) = \theta(n_1n_2^{g_1}) \\ &= \theta(n_1)\theta(n_2^{g_1}) = \theta(n_1)\theta(n_2) = \chi(x_1)\chi(x_2).\end{aligned}$$

Therefore χ is a linear character of $\overline{G}$ such that $\chi_N = \theta$. $\square$

The above result is only applicable in the case where N is abelian. However we have the following results.

Lemma 2.3.12 *Let $N \subseteq G$. Let χ be a character of G. Then*

$$< \chi_N, \chi_N > \leq [G : N] < \chi, \chi >,$$

with equality if and only if $\chi(g) = 0$ for all $g \in G \setminus N$.

Proof. See Lemma 2.29 in [20]. $\square$

Theorem 2.3.13 *Let K/N be an abelian chief factor of G. (That is, $K, N \trianglelefteq G$ and no $M \trianglelefteq G$ exists with $N \leq M \leq K$.) Suppose $\theta \in Irr(K)$ is invariant in G. Then one of the following holds:*

(a) $\theta_N \in Irr(N)$;

(b) $\theta_N = e\varphi$ for some $\varphi \in Irr(N)$ and $e^2 = [K : N]$;

(c) $\theta_N = \sum_{i=1}^{t} \varphi_i$ where the $\varphi \in Irr(N)$ are distinct and $t = [K : N]$.

Proof. Let φ be an irreducible constituent of θ_N and let $T = I_G(\varphi)$. Since θ is invariant in G, every G-conjugate of φ is a constituent of θ_N and hence is K-conjugate to φ. It follows that $[G : T] = [K : K \cap T]$ and hence $KT = G$. Since K/N is abelian, $K \cap T \lhd KT = G$ and thus either $K \cap T = K$ or $K \cap T = N$

If $K \cap T = N$, then $\theta_N = e \sum_{i=1}^{t} \varphi_i$, where $t = [K : N], \varphi_1 = \varphi$, and the φ_i are distinct. Thus $\theta(1) = e[K : N]\varphi(1)$. Since θ is a constituent of φ^K, we have $\theta(1) \leq [K : N]\varphi(1)$ and therefore $e = 1$. This is situation (c).

Now assume $K \cap T = K$ so that φ is invariant in K and $\theta_N = e\varphi$ for some e. Let $\lambda \in Irr(K/N)$. Since λ is linear, $\lambda\theta \in Irr(K)$. Also $(\lambda\theta\ )_N = \theta_N = e\varphi$. Suppose that all of the characters $\lambda\theta$ are distinct as λ runs over $Irr(K/N)$. Each of these $[K : N]$ characters is an irreducible constituent of φ^K with multiplicity e and we have

$$e[K:N]\theta(1) \le \varphi^K(1) = [K:N]\varphi(1).$$

Therefore

$$e^2\varphi(1) = e\theta(1) \le \varphi(1)$$

and $e = 1$. This is situation (a).

In the remaining case, $\lambda\theta = \mu\theta$ for some $\lambda, \mu \in Irr(K/N)$ with $\lambda \neq \mu$. Let $U = ker(\lambda\overline{\mu})$. Then $N \subseteq U \le K$ and θ vanishes on $K \setminus U$. Since θ is invariant in G, it follows that θ vanishes on $K \setminus U^g$ for all $g \in G$. Since $\bigcap_{g\in G} U^g = N$, we conclude that θ vanishes on $K \setminus N$. By Lemma 2.3.12, we have

$$[K:N] = [K:N] < \theta, \theta > = < \theta_N, \theta_N > = e^2$$

and the proof is complete. □

Corollary 2.3.14 *Let $N \lhd G$ with $[G:N] = p$, a prime. Suppose $\chi \in Irr(G)$. Then either*

(a) $\chi|_N$ is irreducible or

(b) $\chi|_N = \sum_{i=1}^{p} \theta_i$, where the θ_i are distinct and irreducible.

Proof. Follows from Theorem 2.3.13

Theorem 2.3.15 *Let $N \lhd G$ and suppose $[G:N] = p$, a prime. Let $\theta \in Irr(N)$ be invariant in G. Then θ is extendible to G.*

Proof. Follows from Corollary 2.3.14 □

Theorem 2.3.16 *([20],[34])(***Gallagher's Theorem***) Let N be a normal subgroup of $\overline{G}$, $\theta \in Irr(N)$ and $\overline{H} = I_{\overline{G}}(\theta)$. If θ can be extended to $\psi \in Irr(\overline{H})$ then as β ranges over all the irreducible characters of $\overline{H}$ which contain N in their kernels, $\beta\psi$ ranges over all the irreducible characters of $\overline{H}$ which contain θ in their restriction to N.*

Proof. Since $\overline{H} = I_{\overline{G}}(\theta)$, then θ is self-conjugate in $\overline{H}$ and thus by Clifford's theorem we obtain that $(\theta^{\overline{H}})_N = f\theta$ for some positive integer f. Comparing degrees we have $(\theta^{\overline{H}})_N = [\overline{H} : N]\theta$ and so $\langle\theta^{\overline{H}}, \theta^{\overline{H}}\rangle = \langle\theta, (\theta^{\overline{H}})_N\rangle = [\overline{H} : N]$. Now we claim that $\theta^{\overline{H}} = \sum_\beta \beta(1_{\overline{G}})\beta\psi$, where β ranges over all the irreducible characters of $\overline{H}$ that contain N in their kernels. Both $\theta^{\overline{H}}$ and $\sum_\beta \beta(1_{\overline{G}})\beta\psi$ are zero off N since for $g \notin N, xgx^{-1} \notin N$ for all $x \in \overline{G}$ and thus $\theta^{\overline{H}}(g) = 0$. Also for $g \notin N$, by the orthogonality of the columns of the character table of $\overline{H}/N$ we have that $\sum_\beta \beta(1_{\overline{G}})(\beta\psi)(g) = [\sum_\beta \beta(1_{\overline{G}})\beta(g)]\psi(g) = 0$. We also have that $(\theta^{\overline{H}})_N = [\overline{H} : N]\theta = (\sum_\beta \beta(1_{\overline{G}})\beta\psi)_N$ since for $g \in N$, $\sum_\beta \beta(1_{\overline{G}})\beta(g)\psi(g) = \sum_\beta(\beta(1_{\overline{G}}))^2\psi(g) = [\overline{H} : N]\psi(g) = [\overline{H} : N]\theta(g)$. Hence we obtain that $\theta^{\overline{H}} = \sum_\beta \beta(1_{\overline{G}})\beta\psi$. So we have

$$[\overline{H} : N] = \langle\theta^{\overline{H}}, \theta^{\overline{H}}\rangle = \langle\sum_\beta \beta(1_{\overline{G}})\beta\psi, \sum_\tau \tau(1_{\overline{G}})\tau\psi\rangle = \sum_{\beta,\tau} \beta(1_{\overline{G}})\tau(1_{\overline{G}})\langle\beta\psi, \tau\psi\rangle \quad .$$

The diagonal terms contribute at least $\sum(\beta(1_{\overline{G}}))^2 = [\overline{H} : N]$, so the $\beta\psi$ are irreducible and distinct, and are all the irreducible constituents of $\theta^{\overline{H}}$ and so are all the irreducible characters of $\overline{H}$ that contain θ in their restriction to N, since for $\phi \in Irr(\overline{H})$ such that $\langle\phi_N, \theta\rangle \neq 0$, we obtain that $\langle\phi_N, \theta\rangle = \langle\phi, \theta^{\overline{H}}\rangle$ which implies that ϕ is an irreducible constituent of $\theta^{\overline{H}}$ and hence is of the form $\beta\psi$. □

2.4 Fischer-Clifford Matrices

Let $\overline{G} = N{\cdot}G$ such that every irreducible character of N is extendible to its inertia group. We have that $\overline{G}$ permutes $Irr(N)$ by $x : \chi \longmapsto \chi^x$, where $x \in \overline{G}$ and $\chi \in Irr(N)$. Now let $\chi_1, \chi_2, \ldots, \chi_t$ be representatives of the orbits of $\overline{G}$ on $Irr(N)$, $\overline{H}_i = I_{\overline{G}}(\chi_i), 1 \leq i \leq t, \psi_i \in Irr(\overline{H}_i)$ be an extension of χ_i to $\overline{H}_i$ and $\psi \in Irr(\overline{H}_i)$ such that $N \subseteq ker(\psi)$. Then by Theorem 2.3.6 (Gallagher's theorem) and Remark 2.3.7 all irreducible characters of $\overline{G}$ will be of the form $(\psi_i\psi)^{\overline{G}}, 1 \leq i \leq t$. So

$$Irr(\overline{G}) = \bigcup_{i=1}^{t} \{(\psi_i\psi)^{\overline{G}} \mid \psi \in Irr(\overline{H}_i),\ N \subseteq ker(\psi)\} \quad .$$

Therefore the irreducible characters of $\overline{G}$ fall into blocks, each block corresponding to an inertia group $\overline{H}_i$.

2.4.1 Theory of Fischer-Clifford matrices

Let $\overline{G} = N{\cdot}G$ with the property that every irreducible character of N can be extended to its inertia group. Let $\overline{g} \in \overline{G}$ be a lifting of $g \in G$ under the natural homomorphism $\overline{G} \longrightarrow G$ and $[g]$ be a conjugacy class of elements of G with representative g. Let $X(g) = \{x_1, x_2, \ldots, x_{c(g)}\}$ be a set of representatives of the conjugacy classes of $\overline{G}$ from the coset $N\overline{g}$ whose images under the natural homomorphism $\overline{G} \longrightarrow G$ are in $[g]$ and we take $x_1 = \overline{g}$. Let $\{\chi_1, \chi_2, \ldots, \chi_t\}$ be a set of representatives of the orbits of $\overline{G}$ on $Irr(N)$ such that for $1 \leq i \leq t$, we have $\overline{H}_i = I_{\overline{G}}(\chi_i)$ with $H_i = \overline{H}_i/N \leq G$ and that $\psi_i \in Irr(\overline{H}_i)$ is an extension of χ_i to $\overline{H}_i$. Then without loss of generality suppose that $\chi_1 = I_N$ is the identity character of N. Then $\overline{H}_1 = \overline{G}$ and $H_1 = G$. Now choose $y_1, y_2, \ldots, y_r$ to be the representatives of the conjugacy classes of elements of H_i which fuse into $[g]$ in G. Since $y_k \in H_i$ for $1 \leq k \leq r$, then we define $y_{\ell_k} \in \overline{H}_i$ such that y_{ℓ_k} ranges over all the representatives of the conjugacy classes of elements of $\overline{H}_i$ which map to y_k under the homomorphism $\overline{H}_i \longrightarrow H_i$ whose kernel is N. Let $\psi \in Irr(\overline{H}_i)$ such that $N \subseteq ker(\psi)$. Then ψ is a lifting of $\hat{\psi} \in Irr(H_i)$ such that $\psi(y_{\ell_k}) = \hat{\psi}(y_k)$ for any lifting $y_{\ell_k} \in \overline{H}_i$ of $y_k \in H_i$. Then we obtain that

$$\begin{aligned}
(\psi_i\psi)^{\overline{G}}(x_j) &= \sum_{1\leq k\leq r} \sum_{\ell}{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{\ell_k})|}\psi_i\psi(y_{\ell_k}) \\
&= \sum_{1\leq k\leq r} \sum_{\ell}{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{\ell_k})|}\psi_i(y_{\ell_k})\psi(y_{\ell_k}) \\
&= \sum_{1\leq k\leq r} (\sum_{\ell}{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{\ell_k})|}\psi_i(y_{\ell_k}))\hat{\psi}(y_k)
\end{aligned}$$

where $\sum_{\ell}{}'$ is the summation over all ℓ for which $y_{\ell_k} \sim x_j$ in $\overline{G}$. Now we define a matrix $M_i(g)$ by $M_i(g) = (a_{uv})$, where $1 \leq u \leq r$ and $1 \leq v \leq c(g)$, and

$$a_{uv} = \sum_{\ell}{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{\ell_k})|}\psi_i(y_{\ell_k}) \quad .$$

Then we obtain that

$$(\psi_i\psi)^{\overline{G}}(x_j) = \sum_{1\leq k\leq r} a_{uv}\hat{\psi}(y_k) \quad .$$

By doing this for all $1 \leq i \leq t$ such that H_i contains an element in $[g]$ we obtain the matrix $M(g)$ given by

$$M(g) = \begin{bmatrix} M_1(g) \\ M_2(g) \\ \vdots \\ M_t(g) \end{bmatrix} \quad ,$$

where $M_i(g)$ is the submatrix corresponding to the inertia group $\overline{H}_i$ and its inertia factor H_i. If $H_i \cap [g] = \emptyset$, then $M_i(g)$ will not exist and $M(g)$ does not contain $M_i(g)$. The size of the matrix $M(g)$ is $p \times c(g)$ where p is the number of conjugacy classes of elements of the inertia factors H_i's for $1 \leq i \leq t$ which fuse into $[g]$ in G and $c(g)$ is the number of conjugacy classes of elements of $\overline{G}$ which correspond to the coset $N\overline{g}$. Then $M(g)$ is the **Fischer-Clifford matrix** of $\overline{G}$ corresponding to the coset $N\overline{g}$. We will see later that $M(g)$ is a $c(g) \times c(g)$ nonsingular matrix. Let

$$R(g) = \{(i, y_k) \mid 1 \leq i \leq t \ , \ H_i \cap [g] \neq \emptyset \ , \ 1 \leq k \leq r\}$$

and we note that y_k runs over representatives of the conjugacy classes of elements of H_i which fuse into $[g]$ in G. Following the notation used in [12] and [34] we denote $M(g)$ by writing $M(g) = (a_j^{(i,y_k)})$, where

$$a_j^{(i,y_k)} = \sum_{\ell}{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{\ell_k})|} \psi_i(y_{\ell_k}) \quad ,$$

with columns indexed by $X(g)$ and rows indexed by $R(g)$. Then the partial character table of $\overline{G}$ on the classes $\{x_1, x_2, \ldots, x_{c(g)}\}$ is given by

$$\begin{bmatrix} C_1(g)M_1(g) \\ C_2(g)M_2(g) \\ \vdots \\ C_t(g)M_t(g) \end{bmatrix}$$

where the Fischer-Clifford matrix $M(g)$ is divided into blocks with each block corresponding to an inertia group $\overline{H}_i$ and $C_i(g)$ is the partial character table of H_i consisting of the columns corresponding to the classes that fuse into $[g]$ in G. We can also observe that the number of irreducible characters of $\overline{G}$ is the sum of the numbers of irreducible characters of the inertia factors H_i's.

2.4.2 Properties of Fischer-Clifford Matrices

We shall discuss the properties which may be used in the computation of the Fischer-Clifford matrices. These properties have been discussed in [25], [30], [34]. Let K be a group and $A \leq Aut(K)$. Then by Brauer's theorem A acts on the conjugacy classes of elements of K and on the irreducible characters of K resulting in the same number of orbits.

Lemma 2.4.1 *Suppose we have the following matrix describing the above actions:*

$$\begin{array}{c} \\ s_1 \\ s_2 \\ \vdots \\ s_i \\ \vdots \\ s_t \end{array} \begin{array}{c} \begin{array}{cccccc} 1 = l_1 & l_2 & \cdots & l_j & \cdots & l_t \end{array} \\ \begin{pmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2t} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{it} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{tj} & \cdots & a_{tt} \end{pmatrix} \end{array}$$

where $a_{1j} = 1$ for $j \in \{1, 2, \ldots, t\}$, l_j's are lengths of orbits of A on the conjugacy classes of K, s_i's are lengths of orbits of A on $Irr(K)$ and a_{ij} is the sum of s_i irreducible characters of K on the element x_j, where x_j is an element of the orbit of length l_j. Then the following relation holds for $i, i' \in \{1, 2, \ldots, t\}$:
$\sum_{j=1}^{t} a_{ij}\overline{a_{i'j}}l_j = |K|s_i\delta_{ii'}$.

Proof. See Lemma 4.2.2 in [34].

□

Let $x_j \in X(g)$ and define $m_j = [C_{\overline{g}} : C_{\overline{G}}(x_j)]$. The Fischer-Clifford matrix $M(g)$ is partitioned row-wise into blocks, where each block corresponds to an inertia group. The columns of $M(g)$ are indexed by $X(g)$ and to each $x_j \in X(g)$, corresponds $|C_{\overline{G}}(x_j)|$ of a conjugacy class of $\overline{G}$. The rows of $M(g)$ are indexed by $R(g)$ and to each row corresponds $|C_{H_i}(y_k)|$, where y_k fuses into $[g]$ in G. The following result gives the orthogonality relation for $M(g)$.

Proposition 2.4.2 *[34](*Column orthogonality*) Let $\overline{G} = N{\cdot}G$, then*

$$\sum_{(i,y_k)\in R(g)} |C_{H_i}(y_k)| a_j^{(i,y_k)} \overline{a_{j'}^{(i,y_k)}} = \delta_{jj'}|C_{\overline{G}}(x_j)| \quad .$$

Proof. See [34], Proposition 4.2.3. □

Theorem 2.4.3 $a_j^{(1,g)} = 1$ *for all* $j \in \{1, 2, \ldots, c(g)\}$

Proof. For $y_{\ell_k} \sim x_j$ in $\overline{G}$, we have $|C_{\overline{G}}(x_j)| = |C_{\overline{H}_1}(y_{\ell_k})|$. Thus we obtain that

$$a_j^{(1,g)} = \sum_{\ell}{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_1}(y_{\ell_k})|}\psi_1(y_{\ell_k}) = \sum_{\ell}{}' 1 = 1 \quad .$$

□

Proposition 2.4.4 *([34]) The matrix* $M(1_G)$ *is the matrix with rows equal to the orbit sums of the action of* $\overline{G}$ *on* $Irr(N)$ *with duplicate columns discarded. For this matrix we have* $a_j^{(i,1_G)} = [G : H_i]$*, and an orthogonality relation for rows:*

$$\sum_{j=1}^{t} \frac{1}{|C_{\overline{G}}(x_j)|} a_j^{(i,1_G)} a_j^{(i',1_G)} = \frac{1}{|C_{H_i}(1_G)|}\delta_{ii'} = \frac{1}{|H_i|}\delta_{ii'}.$$

Proof. The $(i, 1_G), j^{\text{th}}$ entry of $M(1_G)$ is given by

$$a_j^{(i,1_G)} = \sum_{\ell}{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{\ell_k})|}\psi_i(y_{\ell_k}),$$

where we sum over representatives of conjugacy classes of $\overline{H}_i$ which fuse into $[x_j]$ in $\overline{G}$. Therefore $a_j^{(i,1_G)} = \psi_i^{\overline{G}}(x_j)$. By Theorem 2.3.6 we have $\psi_i^{\overline{G}} \in Irr(\overline{G})$ and we obtain that $\langle (\psi_i^{\overline{G}})_N, \theta_i \rangle = \langle (\psi_i)_N, \theta_i \rangle = 1$. Therefore by Clifford's theorem $(\psi_i^{\overline{G}})_N = \sum_{\alpha} \theta_{\alpha}$, where the summation is taken over all $\theta_{\alpha} \in Irr(N)$ such that θ_{α} is conjugate to θ_i. So for $x_j \in N$ we obtain that $a_j^{(i,1_G)} = \sum_{\alpha} \theta_{\alpha}(x_j)$. The orthogonality relation follows by Lemma 2.4.1. □

Following from Lemma 2.4.1, Proposition 2.4.2 and the results proved by Fischer in [12], the Fischer-Clifford matrix $M(g)$ satisfies the following properties:

(a) $|X(g)| = |R(g)|$

(b) $\sum_{j=1}^{c(g)} m_j a_j^{(i,y_k)} \overline{a_j^{(i',y'_k)}} = \delta_{(i,y_k),(i',y'_k)} \frac{|C_G(g)|}{|C_{H_i}(y_k)|}|N|$

(c) $\sum_{(i,y_k) \in R(g)} a_j^{(i,y_k)} \overline{a_{j'}^{(i,y_k)}} |C_{H_i}(y_k)| = \delta_{jj'} |C_{\overline{G}}(x_j)|$

(d) $M(g)$ is square and nonsingular.

For N is elementary abelian, $M(g)$ also satisfies the following

(e) $a_1^{(i,y_k)} = \frac{|C_G(g)|}{|C_{H_i}(y_k)|}$

(f) $|a_1^{(i,y_k)}| \geq |a_j^{(i,y_k)}|$

Further in [25], Mahmoud and List have shown that the Fischer-Clifford matrices for wreath products G_wS_n can be computed from the character table of G. Consider a wreath product G_wS_n and let σ be an element of cycle type $1^{a1}...n^{an}$ of S_n. Here the Fischer-Clifford matrix at the class containing σ is denoted by $F(G,\sigma)$ or $F(G,1^{a1}...n^{an})$. The Fischer-Clifford matrix $F(G,\sigma)$ is constructed from $F(G,1)$ as follows [25].

Theorem 2.4.5 *For any wreath product G_wS_n the following statements are true.*

(1) $F(G,1)$ is a character table of G.

(2) $F(G,k^a) = F(G,1^a)$, $k = 1,2,...$, $a = 1,2,...$

(3) $F(G,1^n)$ is obtained from the matrix of orbit sums of S_n, acting on the rows of the Kronecker product matrix $F(G,1)^{\otimes n} = F(G,1) \otimes ... \otimes F(G,1)$(n-times) in the natural way, by deleting repeated columns.

(4) $F(G,1^{a_1}2^{a_2}...k^{a_k}) = F(G,1^{a_1}) \otimes F(G,1^{a_2}) \otimes ... \otimes F(G,1^{a_k})$, where $F(G,1^0)$ is defined to be the matrix with single entry 1.

Proof. See (3.1,3.2,3.3) in [25]. □

Chapter 3

Some Groups of forms $2^4 : S_5$ and $2^4 : A_7$

The Mathieu groups M_{22} and M_{23} are examples of sporadic simple groups [32]; they were discovered by the French mathematician Emile Leonard Mathieu [4]. The group M_{22} contains a maximal subgroup of form $2^4 : S_5$ while M_{23} contains a maximal subgroup of form $2^4 : A_7$. In this chapter we will construct the Fischer-Clifford matrices of the groups of forms $2^4 : S_5$ and $2^4 : A_7$. We use the Fischer-Clifford matrices of these groups to construct their character tables. This gives some information about the structures of these groups.

3.1 A group of form $2^4 : S_5$

Let $N = 2^4$ be an elementary abelian group of order 16 generated by e_1, e_2, e_3, e_4 with $e_i^2 = 1$ for $1 \leq i \leq 4$ and $G = S_5$, the symmetric group of degree 5. From the ATLAS [8] we note that S_5 is a maximal subgroup of A_7 the alternating group of degree 7. Further from the ATLAS [8], we have that S_5 is isomorphic to a matrix group generated by the matrices

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let S_5 act on N naturally. In this section we construct the character table of the group extension $2^4 : S_5$. In subsection 3.1.1 we compute the Conjugacy Classes of $2^4 : S_5$ using coset analysis, then in subsection 3.1.2 we construct a Fischer-Clifford matrix for each class of the group S_5. In section 3.1.4 we use these Fischer-Clifford matrices to put together the character table of $2^4 : S_5$.

3.1.1 Conjugacy classes of $2^4 : S_5$

In this subsection we calculate the conjugacy classes of $\overline{G} = 2^4 : S_5$ using the coset analysis method as described in section 2.1.1. To determine the conjugacy classes of $\overline{G} = N : G$, we need to find the values of the $k's$ and the $f_j's$ for each class representative $g \in G$. Now since $\overline{G} = N : G$ is a split extension, we analyze the cosets Ng. Under the action of N on Ng, we have that $g \in Q_1$. Since $C_G(g)$ fixes g, Q_1 does not fuse with any other Q_i for $i \in \{2, 3, ..., k\}$. Hence we have $f_1 = 1$. We now find the values of $|C_{\overline{G}}(x)|$ by using the formula $|C_{\overline{G}}(x)| = \frac{k|C_G(g)|}{f_j}$, where f_j of the k blocks of the coset Ng have fused to give a class of $\overline{G}$ containing x.

(i) $g \in 1A$: Then

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The identity element fixes all the elements of N, so that $k = 16$. Thus we obtain 16 orbits containing one element each. Now we act $C_G(g)$ on these orbits. For $ng \in Ng$, $h \in C_G(g)$,

$$(ng)^h = n^h g^h = n^h g.$$

We get

$\{g\}^{C_G(g)} = \{g\}$, $\{e_1 g\}^{C_G(g)} = \{e_1g, e_2g, e_3g, e_4g, e_1e_2g, e_1e_3g, e_1e_4g, e_2e_3g, e_2e_4g$ $e_3e_4g, e_1e_2e_3g, e_1e_3e_4g, e_1e_2e_4g, e_2e_3e_4g, e_1e_2e_3e_4g\}$.

Thus we obtain two orbits with $f = 1$ and $f = 15$. So the coset Ng gives two classes of $\overline{G}$ with representatives g and e_1g. The k^{th} power of each element ng is given by the following formula

$$(ng)^k = nn^g n^{g^2} n^{g^3} ... n^{g^{k-1}} g^k,$$

from which we can determine the order of $\overline{g} = ng$. So we have that the order of g is 1 and the order of e_1g is 2. Therefore this coset gives two classes of $\overline{G}$.

class$(1a)$, $x = g$, $|C_{\overline{G}}(x)| = \frac{16\times120}{1} = 1920$

class $(2a)$, $x = e_1g$, $|C_{\overline{G}}(x)| = \frac{16\times120}{15} = 128$

(ii) $g \in 2A$: Then

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

with $|C_G(g)| = 12$ and

$$C_G(g) = \left\langle \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\rangle.$$

The action of g on N is represented by the cycle structure

$(1)(e_1\ \ e_3)(e_2)(e_4\ \ e_2e_4)(e_1e_2\ \ e_2e_3)\ (e_1e_3)(e_1e_4\ \ e_2e_3e_4)\ (e_1e_2e_3)(e_3e_4\ \ e_1e_2e_4)$ $(e_1e_3e_4\ e_1e_2e_3e_4)$.

Since there are four fixed points i.e $1,\ e_2,\ e_1e_3\ and\ e_1e_2e_3$ we have $k = 4$. Now under conjugation by N each element of Ng is conjugate to $\frac{|N|}{k}$ elements of Ng. So that Ng splits into four orbits with four elements each. For example considering $g \in Ng$ and conjugating it by N, we get

$(g)^1 = g$, $(g)^{e_1} = e_1e_3g$, $(g)^{e_2} = g$,$(g)^{e_3} = e_1e_3g$, $(g)^{e_4} = e_2g$, $(g)^{e_1e_2} = e_1e_3g$, $(g)^{e_1e_3} = g$, $(g)^{e_1e_4} = e_1e_2e_3g$, $(g)^{e_2e_3} = e_1e_3g$, $(g)^{e_2e_4} = e_2g$, $(g)^{e_3e_4} =$

$e_1e_2e_3g$, $(g)^{e_1e_2e_3} = g$,$(g)^{e_1e_3e_4} = e_2g$, $(g)^{e_1e_2e_4} = e_1e_2e_3g$,$(g)^{e_2e_3e_4} = e_1e_2e_3g$, $(g)^{e_1e_2e_3e_4} = e_2g$.

Thus the orbit of N on Ng containing g is $\{g, e_1e_3g, e_2g, e_1e_2e_3g\}$. Similarly the remaining three orbits of N on Ng are

$\{e_1g, e_3g, e_1e_2g, e_2e_3g\}$, $\{e_4g, e_2e_4g, e_1e_3e_4g, e_1e_2e_3e_4g\}$, $\{e_1e_4g, e_3e_4g, e_1e_2e_4g, e_2e_3e_4g\}$.

Now acting $C_G(g)$ on these orbits, we obtain that

$\{g, e_2g, e_1e_3g, e_1e_2e_3g\}^{C_G(g)} = \{g, e_2g, e_1e_3g, e_1e_2e_3g\}$,
$\{e_1g, e_3g, e_1e_2g, e_2e_3g\}^{C_G(g)} = \{e_1g, e_3g, e_1e_2g, e_2e_3g, e_4g, e_2e_4g, e_1e_3e_4g, e_1e_2e_3e_4g, e_1e_4g, e_3e_4g, e_1e_2e_4g, e_2e_3e_4g\}$.

Thus three of the Q_i fuse together to give one orbit. Hence we get two classes of $\overline{G}$ with $f = 1$ and $f = 3$ and representatives g and e_1g where $o(g) = 2$ and $o(e_1g) = 4$.

class $(2b)$, $x = g$, $|C_{\overline{G}}(x)| = \frac{4\times 12}{1} = 48$.

class $(4a)$, $x = e_1g$, $|C_{\overline{G}}(x)| = \frac{4\times 12}{3} = 16$.

(iii) $g \in 2B$: Then

$$g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

with $|C_G(g)| = 8$ and

$$C_G(g) = \left\langle \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \right\rangle.$$

The action of g on N is represented by the cycle structure

$(1)(e_1\ e_4)(e_2\ e_1e_3)(e_3\ e_2e_4)(e_1e_2\ e_1e_3e_4)(e_2e_3\ e_1e_2e_3e_4)(e_1e_4)(e_2e_3e_4)(e_3e_4\ e_1e_2e_4)$ $(e_1e_2e_3)$.

Now g fixes 4 elements of N so $k = 4$ and the four orbits of N on Ng are

$\{g, e_1e_4g, e_1e_2e_3g, e_2e_3e_4g\}$,$\{e_1g, e_4g, e_2e_3g, e_1e_2e_3e_4g\}$,$\{e_2g, e_1e_3g, e_3e_4g, e_1e_2e_4g\}$, $\{e_3g, e_1e_2g, e_2e_4g, e_1e_3e_4g\}$.

Now we act $C_G(g)$ on the above orbits and obtain

$\{g, e_1e_4g, e_1e_2e_3g, e_2e_3e_4g\}^{C_G(g)} = \{g, e_1e_4g, e_1e_2e_3g, e_2e_3e_4g\}$,
$\{e_1g, e_4g, e_2e_3g, e_1e_2e_3e_4g\}^{C_G(g)} = \{e_1g, e_4g, e_2e_3g, e_1e_2e_3e_4g, e_3g, e_1e_2g, e_2e_4g, e_1e_3e_4g\}$.
$\{e_2g, e_1e_3g, e_3e_4g, e_1e_2e_4g\}^{C_G(g)} = \{e_2g, e_1e_3g, e_3e_4g, e_1e_2e_4g\}$,

Hence we get three classes of $\overline{G}$ from this coset with $f = 1$, $f = 2$ and $f = 1$ and representatives g, e_1g, e_2g with orders $o(g) = 2$, $o(e_1g) = 4$ and $o(e_2g) = 4$, respectively.

class $(2c)$, $x = g$, $|C_{\overline{G}}(x)| = \frac{4\times8}{1} = 32$,

class $(4b)$, $x = e_1g$, $|C_{\overline{G}}(x)| = \frac{4\times8}{2} = 16$,

class $(4c)$, $x = e_2g$, $|C_{\overline{G}}(x)| = \frac{4\times8}{1} = 32$.

Similarly, acting $C_G(g)$ on Ng for all the remaining classes $[g]$ of G, we get that the conjugacy classes of $\overline{G}$ are as follows:

$[g]$	$1A$		$2A$		$2B$			$3A$	$4A$		$5A$	$6A$
$[\overline{g}]$	$1a$	$2a$	$2b$	$4a$	$2c$	$4b$	$4c$	$3a$	$4d$	$8a$	$5a$	$6a$
$\lvert C_{\overline{G}}(\overline{g})\rvert$	1920	128	48	16	32	16	32	6	8	8	5	6

3.1.2 Fischer-Clifford matrices of $2^4:S_5$

In this subsection we calculate the Fisher-Clifford matrices of $\overline{G}$ using the properties of Fisher-Clifford matrices as discussed in Chapter 2. Corresponding to the identity of G, we have

$$M(1A) = \begin{array}{c} \\ 120 \\ 8 \end{array} \begin{array}{c} \begin{array}{cc} 1920 & 128 \end{array} \\ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \end{array}.$$

Now by relation $a_j^{(1,g)} = 1$, $j \in \{1, 2, ..., c(g)\}$ of Theorem 2.4.3, we have $a = b = 1$. And by relation

$$a_1^{(i,y_k)} = \frac{|C_G(g)|}{|C_{H_i}(y_k)|}$$

we have $c = \frac{120}{8} = 15$. Now by relation

$$\sum_{(i,y_k)} \in R(g) a_{j'}^{(i,y_k)\overline{(i,y_k)}} |C_{H_i}(y_k)| = \delta_{jj'} |C_{\overline{G}}(x_j)|$$

we have $(120 \times 1 \times 1) + (8 \times 15 \times d) = 0$, from which it follows that $d = -1$. Therefore

$$M(1A) = \begin{array}{c} \\ 120 \\ 8 \end{array} \begin{array}{c} \begin{array}{cc} 1920 & 128 \end{array} \\ \left(\begin{array}{cc} 1 & 1 \\ 15 & -1 \end{array} \right) \end{array}.$$

Corresponding to $g \in 2B$, $M(2B)$ is a 3×3 matrix since Ng has three $\overline{G}$-conjugacy classes. Just as was done for $M(1A)$ we have that

$$M(2B) = \begin{array}{c} \\ 8 \\ 4 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccc} 32 & 16 & 32 \end{array} \\ \left(\begin{array}{ccc} 1 & 1 & 1 \\ 2 & a & c \\ 1 & b & d \end{array} \right) \end{array},$$

Table 3.1: Fischer-Clifford matrices of $2^4 : S_5$

[g]	M(g)	[g]	M(g)
1A	$\begin{pmatrix} 1 & 1 \\ 15 & -1 \end{pmatrix}$	4A	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
2A	$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	5A	$\begin{pmatrix} 1 \end{pmatrix}$
2B	$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$	6A	$\begin{pmatrix} 1 \end{pmatrix}$
3A	$\begin{pmatrix} 1 \end{pmatrix}$		

where the entries of the first row and first column follow by relations $a_j^{(1,g)} = 1$ for all $j \in \{1, 2, ..., c(g)\}$ and

$$a_1^{(i,y_k)} = \frac{|C_G(g)|}{|C_{H_i}(y_k)|},$$

respectively. To calculate a, b, c and d we use the relation

$$\sum_{(i,y_k)} \in R(g) a_{j'}^{(i,y_k)\overline{(i,y_k)}} |C_{H_i}(y_k)| = \delta_{jj'} |C_{\overline{G}}(x_j)|.$$

For the second column, $8 + 4|a|^2 + 8|b|^2 = 16 \Rightarrow |a|^2 + 2|b|^2 = 2 \Rightarrow a = 0$ and $|b| = 1$. But by columns 1 and 2, we have $8 + 8a + 8b = 0 \Rightarrow a + b = -1$. Therefore $a = 0$ and $b = -1$. Similarly, $c = -2$ and $d = 1$.

The other matrices are determined in the same way, and all the Fischer-Clifford matrices of $\overline{G}$ are given in Table 3.1.

Table 3.2: Character Table of S_5

$[g]$	1A	2A	2B	3A	4A	5A	6A
$\|C_G(g)\|$	120	12	8	6	4	5	6
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	1	-1
χ_3	4	2	0	1	0	-1	-1
χ_4	4	-2	0	1	0	-1	1
χ_5	6	0	-2	0	0	1	0
χ_6	5	1	1	-1	-1	0	1
χ_7	5	-1	1	-1	1	0	-1

Table 3.3: Character Table of $H_2 = D_8$

$[g]$	1A	2A	2B	2C	4A
$\|C_G(g)\|$	8	4	4	8	4
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	-1	1	-1	1
χ_5	2	0	-2	0	0

3.1.3 Inertia Factor Groups of $2^4 : S_5$

The action by conjugation of $G = S_5$ on $N = 2^4$ gives two orbits and so does the action of G on $Irr(N)$. These orbits are both of length 1 and 15. The inertia groups are $\overline{H}_1 = \overline{G}$ and $\overline{H}_2$, where $[\overline{G} : \overline{H}_2] = 15$. Let $H_2 = \overline{H}_2/N$, then H_2 is a subgroup of G with $[G : H_2] = 15$. And from the maximal subgroups of S_5 of order 8, we have that $H_2 \cong D_8$. The character Table of H_2 can be constructed easily [21] while the character table of $H_1 = G$ can be constructed in GAP [14]. The character Tables of H_2 and S_5 appear in Tables 3.3 and 3.2 respectively. The class fusions of H_2 into S_5 appear in Table 3.4.

Table 3.4: Fusion of H_2 in S_5

Class of H_2	Class of S_5
1A	1A
2A	2A
2B	2B
2C	2B
4A	4A

3.1.4 Character Table of $2^4 : S_5$

Now we calculate the characters of $\overline{G}$, which fall into two blocks with inertia groups $\overline{G}$ and $\overline{H}_2$, by multiplying rows of the Fischer-Clifford matrices $M(g)$ of $2^4 : S_5$ with sections of the character tables of S_5 corresponding to g according to the fusions of H_2 into S_5. At the identity of S_5 we have

$$M(1A) = \begin{pmatrix} 1 & 1 \\ 15 & -1 \end{pmatrix}.$$

Now we multiply each row of $M(1A)$ by columns of Tables 3.2 and 3.3 according to the fusions in Table 3.4 respectively to get the values of the characters of $\overline{G}$ on $\overline{G}$-classes of $(1a)$ and $(2a)$ as follows;

$$\begin{pmatrix} 1 \\ 1 \\ 4 \\ 4 \\ 6 \\ 5 \\ 5 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 4 & 4 \\ 4 & 4 \\ 6 & 6 \\ 5 & 5 \\ 5 & 5 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 15 & -1 \end{pmatrix} = \begin{pmatrix} 15 & -1 \\ 15 & -1 \\ 15 & -1 \\ 15 & -1 \\ 30 & -2 \end{pmatrix}.$$

Similarly, we can obtain the character values corresponding to class (2B) of S_5. These give the values of the characters of $\overline{G}$ on $\overline{G}$-classes $(2c)$, $(4b)$ and $(4c)$. This is done for all the other classes of S_5 to give the character table of $\overline{G}$ given in Table 3.5. It is divided into blocks each corresponding to the two inertia groups.

Table 3.5: Character Table of $\overline{G} = 2^4 : S_5$

$[\overline{g}]$	1	$2a$	$2b$	$4a$	$2c$	$4b$	$4c$	$3a$	$4d$	$8a$	$5a$	$6a$
$\lvert C_{\overline{G}}(\overline{g})\rvert$	1920	128	48	16	32	16	32	6	8	8	5	6
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	-1	1	1	1	1	-1	-1	1	-1
χ_3	4	4	2	2	0	0	0	1	0	0	-1	-1
χ_4	4	4	-2	-2	0	0	0	1	0	0	-1	1
χ_5	6	6	0	0	-2	-2	-2	0	0	0	1	0
χ_6	5	5	1	1	1	1	1	-1	-1	-1	0	1
χ_7	5	5	-1	-1	1	1	1	-1	1	1	0	-1
χ_8	15	-1	3	-1	3	-1	-1	0	1	-1	0	0
χ_9	15	-1	-3	1	3	-1	-1	0	-1	1	0	0
χ_{10}	15	-1	3	-1	-1	-1	3	0	-1	1	0	0
χ_{11}	15	-1	-3	1	-1	-1	3	0	1	-1	0	0
χ_{12}	30	-2	0	0	-2	2	-2	0	0	0	0	0

The above character Table of $2^4 : S_5$ has been tested using GAP and found to be correct.

3.2 A group of form $2^4 : A_7$

Let N be an elementary abelian group of order 16, so $N \cong V_4(2)$, the vector space of dimension four over a field of two elements. Let $G \cong A_7$, the alternating group of degree 7. In this section we construct the character table of the group extension $\overline{G} = N : G$ where $N = 2^4$ and $G = A_7$ and A_7 acts naturally on 2^4. The group A_7 is a subgroup of the symmetric group S_7 and from the ATLAS [8], we have that A_7 is isomorphic to a group generated by the matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Let $N = 2^4$ be generated by e_1, e_2, e_3, e_4 with $e_i^2 = 1$ for $1 \leq i \leq 4$. Let A_7 act on N naturally, that is $e_i^g = e_{g(i)}$ for all $g \in A_7$ and $e_i \in 2^4$. In subsection 3.2.1 we compute the Conjugacy Classes of $2^4 : A_7$ using the method of coset analysis, then in subsection 3.2.2 we construct the Fischer-Clifford matrix for each class of the group A_7. We use these matrices in section 3.2.4 to put together the character table of $2^4 : A_7$.

3.2.1 Conjugacy Classes of $2^4 : A_7$

In this subsection we determine the conjugacy classes of $2^4 : A_7$ using coset analysis as described in section 2.1.1. To determine the conjugacy classes of $\overline{G}$ we analyze the cosets Ng where g is a representative of a class of $G = A_7$. We then find the values of $|C_{\overline{G}}(x)|$ by using $|C_{\overline{G}}(x)| = \frac{k|C_G(g)|}{f}$, where f of the k blocks of the action of N on the coset Ng have fused to give a class of $\overline{G}$ containing x.

(i) $g \in 1A$: Then

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The identity of G fixes all elements of N, so $k = 16$, where k is the number of elements of N fixed by g. Thus we have 16 orbits with one element in each. Now acting $C_G(g) = G$ by conjugation on these orbits, i.e for $ng \in Ng$ and $h \in C_G(g) = G$, $(ng)^h = n^h g^h = n^h g$, we get

$\{g\}^{C_G(g)} = \{g\}$, $\{e_1g\}^{C_G(g)} = \{e_1g, e_2g, e_3g, e_4g, e_1e_2g, e_1e_3g, e_1e_4g, e_2e_3g, e_2e_4g$ $e_3e_4g, e_1e_2e_3g, e_1e_3e_4g, e_1e_2e_4g, e_2e_3e_4g, e_1e_2e_3e_4g\}$.

Therefore under the action of $C_G(g) = G$ we have two orbits with $f_1 = 1$ and $f_2 = 15$. So the coset Ng gives two classes of $\overline{G}$ with representatives g and e_1g.

We have that the order of g is 1 and the order of e_1g is 2. Therefore this coset gives two classes of $\overline{G}$.

class$(1a)$, $x = g$, $|C_{\overline{G}}(x)| = \frac{16\times 2520}{1} = 40320$,

class$(2a)$, $x = e_1g$, $|C_{\overline{G}}(x)| = \frac{16\times 2520}{15} = 2688$.

(ii) $g \in 2A$: Then

$$g = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

with

$$C_G(g) = \left\langle \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\rangle$$

and $|C_G(g)| = 24$. The action of g on N is represented by the cycle structure

$(1)(e_1\ e_2e_3)(e_2\ e_2e_3e_4)(e_3\ e_1e_2e_3e_4)(e_4\ e_1e_2)(e_1e_3\ e_1e_4)(e_2e_4\ e_1e_3e_4)(e_1e_2e_3)$
$(e_1e_2e_4)(e_3e_4)$.

Since there are four fixed points i.e $1,\ e_1e_2e_3,\ e_1e_2e_4\ and\ e_3e_4$ we have $k = 4$. Now under conjugation by N each element of Ng is conjugate to $\frac{|N|}{k}$ elements of Ng. So that Ng splits into four orbits with four elements each. For example considering $g \in Ng$ and conjugating it by N, we get that the orbit of N on Ng containing g is $\{g, e_1e_2e_3g, e_1e_2e_4g, e_3e_4g\}$. Similarly the remaining three orbits of N on Ng are

$\{e_1g, e_2e_3g, e_1e_3e_4g, e_2e_4g\}$ $\{e_2g, e_1e_3g, e_2e_3e_4g, e_1e_4g\}$, $\{e_3g, e_1e_2g, e_4g, e_1e_2e_3e_4g\}$.

Now we act $C_G(g)$ on these orbits. For $ng \in Ng$, $h \in C_G(g)$, $(ng)^h = n^h g^h = n^h g$. We obtain the following orbits;

$\{g, e_1e_2e_3g, e_1e_2e_4g, e_3e_4g\}^{C_G(g)} = \{g, e_1e_2e_3g, e_1e_2e_4g, e_3e_4g\}$,
$\{e_1g, e_2e_3g, e_1e_3e_4g, e_2e_4g\}^{C_G(g)} = \{e_1g, e_2e_3g, e_1e_3e_4g, e_2e_4g, e_2g, e_1e_3g, e_2e_3e_4g, e_1e_4g, e_3g, e_1e_2g, e_4g, e_1e_2e_3e_4g\}$.

Therefore we get two classes of $\overline{G}$ with $f_1 = 1$ and $f_2 = 3$ and representatives g and e_1g where $o(g) = 2$ and $o(e_1g) = 4$.

class $(2b)$, $x = g$, $|C_{\overline{G}}(x)| = \frac{4\times 24}{1} = 96$,

class $(4a)$, $x = e_1g$, $|C_{\overline{G}}(x)| = \frac{4\times 24}{3} = 32$.

(iii) $g \in 3A$: Then

$$g = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

with

$$C_G(g) = \left\langle \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\rangle$$

and $|C_{\overline{G}}(g)| = 36$. The action of g on N is represented by the cycle structure

$(1)(e_1\ e_2e_3e_4\ e_1e_2e_3e_4)(e_2\ e_1e_4\ e_1e_2e_4)(e_3\ e_1e_2\ e_1e_2e_3)(e_4\ e_1e_3\ e_1e_3e_4)(e_2e_3\ e_2e_4\ e_3e_4)$.

Thus $k = 1$ and we obtain one orbit with 16 elements. Hence we get one class with representative g of order 3.

class $(3a)$, $x = g$, $|C_G(x)| = 36$.

(iv) $g \in 3B$: Then

$$g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

with

$$C_G(g) = \left\langle \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \right\rangle$$

and $|C_G(g)| = 9$. The action of g on N is represented by the cycle structure

$(1)(e_1\ e_2e_4\ e_4)(e_2\ e_1e_4\ e_1e_2e_4)(e_1e_2e_3e_4\ e_2e_3\ e_1e_3e_4)(e_3)(e_1e_2)(e_1e_3\ e_2e_3e_4\ e_3e_4)(e_1e_2e_3)$.

Thus $k = 4$ and the four orbits of N on Ng are

$\{g, e_2g, e_1e_4g, e_1e_2e_4g\}$,$\{e_1g, e_4g, e_1e_2g, e_2e_4g\}$, $\{e_3g, e_2e_3g, e_1e_3e_4g, e_1e_2e_3e_4g\}$, $\{e_1e_3g, e_2e_3e_4g, e_3e_4g, e_1e_2e_3g\}$.

Now we act $C_G(g)$ on these orbits and obtain that

$\{g, e_2g, e_1e_4g, e_1e_2e_4g\}^{C_G(g)} = \{g, e_2g, e_1e_4g, e_1e_2e_4g\}$,
$\{e_1g, e_4g, e_1e_2g, e_2e_4g\}^{C_G(g)} = \{e_1g, e_4g, e_1e_2g, e_2e_4g, e_3g, e_2e_3g, e_1e_3e_4g, e_1e_2e_3e_4g, e_1e_3g, e_2e_3e_4g, e_3e_4g, e_1e_2e_3g\}$.

We get two classes of $\overline{G}$ with $f_1 = 1$ and $f_2 = 3$ and representatives g of order 3 and e_1g of order 6.

class $(3b)$, $x = g$,$|C_{\overline{G}}(x)| = \frac{4\times 9}{1} = 36$,

class $(6a)$, $x = e_1g$,$|C_{\overline{G}}(x)| = \frac{4\times 9}{3} = 12$.

(v) $g \in 4A$: Then

$$g = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

with

$$C_G(g) = \left\langle \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

and $|C_G(g)| = 4$. The action of g on N is represented by the cycle structure

$(1)(e_1\ e_2e_4\ e_2e_3\ e_1e_3e_4)(e_2\ e_1e_2e_3e_4\ e_2e_3e_4\ e_3)\ (e_4\ e_1e_4\ e_1e_2\ e_1e_3)(e_1e_2e_4\ e_3e_4)(e_1e_2e_3)$.

Thus $k = 2$ and the two orbits of N on Ng are

$\{g, e_1e_2e_4g, e_2e_3g, e_1g, e_1e_3e_4g, e_2e_4g, e_3e_4g, e_1e_2e_4g\}$,
$\{e_2g, e_1e_4g, e_1e_2e_3e_4g, e_3g, e_1e_2g, e_4g, e_2e_3e_4g, e_1e_3g\}$.

These cannot fuse together under $C_G(g)$, since Q_1 is fixed. Therefore we have two classes of $\overline{G}$ with $f_1 = 1$ and $f_2 = 1$ and representatives g of order 4 and e_2g of order 8.

class $(4b)$, $x = g$, $|C_{\overline{G}}(x)| = 2 \times 4 = 8$,

class $(8a)$,$x = e_2g$, $|C_{\overline{G}}(x)| = 2 \times 4 = 8$.

(vi) $g \in 7A$: We get two classes of $\overline{G}$:

class $(7a)$, $x = g$, $|C_{\overline{G}}(x)|$=$2 \times 7 = 14$,

class $(14a)$, $x = e_2g$, $|C_{\overline{G}}(x)| = 2 \times 7 = 14$.

(vii) $g \in 7B$: We get two classes of $\overline{G}$:

class $(7b)$, $x = g$, $|C_{\overline{G}}(x)| = 2 \times 7 = 14$,

class $(14b)$, $x = e_1 g$, $|C_{\overline{G}}(x)| = 2 \times 7 = 14$.

For classes (5A) and (6A) of A_7 we have $k = 1$, so each coset gives one class of $\overline{G}$. These are classes $(5a)$ and $(6b)$ of $\overline{G}$, with centralizers of order 5 and 12 respectively. This completes the conjugacy classes of $\overline{G}$ which are as follows:

$[g]$	1A		2A		3A	3B		4A		5A	6A
$[\overline{g}]$	$1a$	$2a$	$2b$	$4a$	$3a$	$3b$	$6a$	$4b$	$8a$	$5a$	$6b$
$\lvert C_{\overline{G}}(\overline{g})\rvert$	40320	2688	96	32	36	36	12	8	8	5	12

$[g]$	7A		7B	
$[\overline{g}]$	$7a$	$14a$	$7b$	$14b$
$\lvert C_{\overline{G}}(\overline{g})\rvert$	14	14	14	14

3.2.2 Fischer-Clifford Matrices of $2^4 : A_7$

In this subsection we construct the Fischer-Clifford matrices of $2^4 : A_7$. We will use the properties of Fischer-Clifford matrices as discussed in Chapter 2. Corresponding to the identity of G, we have the matrix

$$M(1A) = \begin{matrix} \\ 2520 \\ 168 \end{matrix} \begin{matrix} 40320 & 2688 \\ \left(\begin{matrix} a \\ c \end{matrix} \right. & \left. \begin{matrix} b \\ d \end{matrix} \right) \end{matrix}.$$

Now by the relation $a_j^{(1,g)} = 1$ for all $j \in \{1, 2, ..., c(g)\}$ we have that a=b=1. And by relation

$$a_1^{(i,y_k)} = \frac{|C_G(g)|}{|C_{H_i}(y_k)|}$$

we have c=15. Now by relation

$$\sum_{(i,y_k)} \in R(g) a_{j'}^{(i,y_k)\overline{(i,y_k)}} |C_{H_i}(y_k)| = \delta_{jj'} |C_{\overline{G}}(x_j)|$$

we have $(2520 \times 1 \times 1) + (168 \times 15 \times d) = 0$, from which we obtain $d = -1$. Therefore the matrix $M(1A)$ is as follows.

$$M(1A) = \begin{pmatrix} 1 & 1 \\ 15 & -1 \end{pmatrix}.$$

Now suppose $g \in (2A)$. Then $M(2A)$ is a 2×2 matrix. Let

$$M(2A) = \begin{matrix} & 96 & 32 \\ 24 & a & b \\ 8 & c & d \end{matrix}.$$

Now we have that $a = b = 1$ and $c = 3$ by relations $a_j^{(1,g)} = 1$ for all $j \in \{1, 2, ..., c(g)\}$ and

$$a_1^{(i,y_k)} = \frac{|C_G(g)|}{|C_{H_i}(y_k)|}$$

respectively. By relation

$$\sum_{(i,y_k)} \in R(g) a_{j'}^{(i,y_k)\overline{(i,y_k)}} |C_{H_i}(y_k)| = \delta_{jj'} |C_{\overline{G}}(x_j)|,$$

$(24 \times 1 \times 1) + (8 \times 3 \times d) = 0$ from which we obtain $d = -1$. Therefore the matrix $M(2A)$ is as follows.

$$M(2A) = \begin{matrix} & 96 & 32 \\ 24 & 1 & 1 \\ 8 & 3 & -1 \end{matrix}.$$

Table 3.6: Fischer-Clifford matrices of $2^4 : A_7$

[g]	M(g)	[g]	M(g)
1A	$\begin{pmatrix} 1 & 1 \\ 15 & -1 \end{pmatrix}$	5A	$\begin{pmatrix} 1 \end{pmatrix}$
2A	$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	6A	$\begin{pmatrix} 1 \end{pmatrix}$
3A	$\begin{pmatrix} 1 \end{pmatrix}$	7A	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
3B	$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	7B	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
4A	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$		

The other matrices are determined similarly, and all the Fischer-Clifford matrices of $\overline{G}$ are given in Table 3.6.

3.2.3 Inertia Factor Groups of $2^4 : A_7$

The action by conjugation of $G = A_7$ on $N = 2^4$ gives two orbits and so does the action of G on $Irr(N)$. These orbits are both of length 1 and 15. The inertia groups are $\overline{H_1}$=$\overline{G}$ and $\overline{H_2}$, where $[\overline{G} : \overline{H_2}] = 15$. Let $H_2 = \overline{H_2}/N$, then H_2 is a subgroup of G with $[G : H_2] = 15$. Therefore $H_2 = PSL(2, 7)$, by considering the maximal subgroups of A_7 given in the ATLAS [8]. The character table of $H_1 = G$ and $H_2 = PSL(2, 7)$ can easily be constructed in GAP [14]. The character tables of $H_1 = A_7$ and $H_2 = PSL(2, 7)$ are given in Tables 3.7 and 3.8 respectively. The class fusions of H_2 into A_7 are given in Table 3.9.

Table 3.7: Character Table of A_7

$[g]$	1A	2A	3A	3B	4A	5A	6A	7A	7B
$\|C_G(g)\|$	2520	24	36	9	4	5	12	7	7
χ_1	1	1	1	1	1	1	1	1	1
χ_2	6	2	3	0	0	1	-1	-1	-1
χ_3	10	-2	1	1	0	0	1	a	$\overline{a}$
χ_4	10	-2	1	1	0	0	1	$\overline{a}$	a
χ_5	14	2	-1	2	0	-1	-1	0	0
χ_6	14	2	2	-1	0	-1	2	0	0
χ_7	15	-1	3	0	-1	0	-1	1	1
χ_8	21	1	-3	0	-1	1	1	0	0
χ_9	35	-1	-1	-1	1	0	-1	0	0

$$a = \frac{1}{2}(-1 - \sqrt{7}i)$$

Table 3.8: Character Table of $H_2 = PSL(2,7)$

$[g]$	1A	2A	3A	4A	7A	7B
$\|C_G(g)\|$	168	8	3	4	7	7
χ_1	1	1	1	1	1	1
χ_2	3	-1	0	1	a	$\overline{a}$
χ_3	3	-1	0	1	$\overline{a}$	a
χ_4	6	2	0	0	-1	-1
χ_5	7	-1	1	-1	0	0
χ_6	8	0	-1	0	1	1

$$a = \frac{1}{2}(-1 - \sqrt{7}i)$$

Table 3.9: Fusion of H_2 in A_7

Class of H_2	Class of G
1A	1A
2A	2A
3A	3B
4A	4A
7A	7A
7B	7B

3.2.4 Character Table of $2^4 : A_7$

In this subsection we use the Fischer-Clifford matrices and the character tables of the inertia factor groups $H_1 = G$ and H_2, together with the fusions of $H_2 = -PSL(2,7)$ into G to construct the character table of $2^4 : A_7$. Thus we can calculate the characters of $\overline{G}$, which fall into two blocks according to inertia groups $\overline{G}$ and $\overline{H_2}$, by multiplying rows of the Fischer-Clifford matrices $M(g)$ of $2^4 : A_7$ with sections of the character tables $H_1 = G$ and H_2 according to the fusions. At the identity of G we have

$$M(1A) = \begin{pmatrix} 1 & 1 \\ 15 & -1 \end{pmatrix}.$$

Since the first columns of the Tables 3.7 and 3.8 correspond to the identity of G we multiply each row of $M(1A)$ by the first columns of Tables 3.7 and 3.8 respectively to get the values of the characters of $\overline{G}$ on $\overline{G}$-classes $1a$ and $2a$ as follows;

$$\begin{pmatrix} 1 \\ 6 \\ 10 \\ 10 \\ 14 \\ 14 \\ 15 \\ 21 \\ 35 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 6 & 6 \\ 10 & 10 \\ 10 & 10 \\ 14 & 14 \\ 14 & 14 \\ 15 & 15 \\ 21 & 21 \\ 35 & 35 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 3 \\ 3 \\ 6 \\ 7 \\ 8 \end{pmatrix} \times \begin{pmatrix} 15 & -1 \end{pmatrix} = \begin{pmatrix} 15 & -1 \\ 45 & -3 \\ 45 & -3 \\ 90 & -6 \\ 105 & -7 \\ 120 & -8 \end{pmatrix}.$$

Similarly we can obtain the character values corresponding to class (2A) of G. These give the values of the characters of $\overline{G}$ on $\overline{G}$-classes $(2b)$ and $(4a)$. We proceed in the same way for all classes of G, to get all the characters of $\overline{G}$. This gives the character table of $\overline{G}$ given in Table 3.10. The character table is divided into two blocks each corresponding to an inertia factor group A_7 and H_2.

Table 3.10: Character Table of $\overline{G} = 2^4 : A7$

$[\overline{g}]$	1	$2a$	$2b$	$4a$	$3a$	$3b$	$6a$	$4b$	$8a$	$5a$	$6b$	$7a$	$14a$	$7b$	$14b$
$\lvert C_{\overline{G}}(\overline{g})\rvert$	40320	2688	96	32	36	36	12	8	8	5	12	14	14	14	14
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	6	6	2	2	3	0	0	0	0	1	-1	-1	-1	-1	-1
χ_3	10	10	-2	-2	1	1	1	0	0	0	1	a	a	$\overline{a}$	$\overline{a}$
χ_4	10	10	-2	-2	1	1	1	0	0	0	1	$\overline{a}$	$\overline{a}$	a	a
χ_5	14	14	2	2	-1	2	2	0	0	-1	-1	0	0	0	0
χ_6	14	14	2	2	2	-1	-1	0	0	-1	2	0	0	0	0
χ_7	15	15	-1	-1	3	0	0	-1	-1	0	-1	1	1	1	1
χ_8	21	21	1	1	-3	0	0	-1	-1	1	1	0	0	0	0
χ_9	35	35	-1	-1	-1	-1	-1	1	1	0	-1	0	0	0	0
χ_{10}	15	-1	3	-1	0	3	-1	1	-1	0	0	1	-1	1	-1
χ_{11}	45	-3	-3	1	0	0	0	1	-1	0	0	a	$-a$	$\overline{a}$	-$\overline{a}$
χ_{12}	45	-3	-3	1	0	0	0	1	-1	0	0	$\overline{a}$	-$\overline{a}$	a	$-a$
χ_{13}	90	-6	6	-2	0	0	0	0	0	0	0	-1	1	-1	1
χ_{14}	105	-7	-3	1	0	3	-1	-1	1	0	0	0	0	0	0
χ_{15}	120	-8	0	0	0	-3	1	0	0	0	0	1	-1	1	-1

$$a = \frac{1}{2}(-1 - \sqrt{7}i),\ \overline{a} = \frac{1}{2}(-1 + \sqrt{7}i)$$

The character Table of $2^4 : A_7$ has been tested using GAP and found to be correct.

Chapter 4

A subgroup $(S_n)^m : C_m$ of S_{mn}, prime m

4.1 Introduction

Let $m, n \in \mathbb{N}$, the set of positive integers and C_m be a cyclic group of order m. In this chapter we construct the character tables of examples of groups of the form $S_n^m : C_m$, prime m, where the group $S_n^m : C_m$, is a subgroup of the symmetric group S_{mn} of degree $m \times n$. In the first two sections of this chapter, we define the group $S_n^m : C_m$ and discuss in detail the method of calculating the conjugacy classes of this group. In the third section we briefly outline the steps of constructing the character tables of the groups, including giving a method of finding the Fischer-Clifford matrices and the inertia factor groups. Lastly we determine the Fischer-Clifford matrices and use these to construct the character tables of the groups $S_n^m : C_m$, prime m.

4.2 The Group $S_n^m : C_m$, prime m

Let $N = S_n^m$ be the direct product of m copies of S_n, where the j^{th} copy of S_n given by

$${}^jS_n := Group(((j-1)n+1 \quad (j-1)n+2 \quad \dots jn), ((j-1)n+1 \quad (j-1)n+2)),$$

$j = 1, 2, \dots, m$.
Let $C_m =< g >$, and we may take

$$g = (1\ \ n+1\ \ 2n+1\ \ \dots(m-1)n+1)(2\ \ n+2\ \ 2n+2\ \ \dots(m-1)n+2)$$
$$(3\ \ n+3\ \ 2n+3\ \ \dots(m-1)n+3)\dots(n\ \ 2n\ \ 3n\ \ \dots mn).$$

Then g is an element of S_{mn} of type $[m^n]$, it permutes the elements of the copies S_n of N. We note that g^s permutes cycles from distinct copies of S_n, that is g^s permutes copies of cycles of N. It follows that N is invariant under conjugation by elements of C_m and $N \bigcap C_m = 1$. Thus we can define the split extension $\overline{G} = N : C_m$ of N by C_m with action conjugation. Since $\overline{G}$ is a split extension of N by C_m, every element $\overline{g}$ of $\overline{G}$ is of form $\overline{g} = ug^r$, where $u \in N$ and $g^r \in C_m$, $r = 1, 2, \dots, m$. We have

$$\overline{g} = \begin{cases} u \in N & \text{if } g^r = 1_{C_m} \\ g^r \in C_m & \text{if } u = 1_N \\ ug^r \in \overline{G} \setminus \{N, C_m\} & \text{if } u \neq 1_N, g^r \neq 1_{C_m}. \end{cases}$$

As indicated earlier, the group $\overline{G} = S_n^m : C_m$ is a subgroup of the symmetric group S_{mn} of degree $m \times n$. In the next section, we give a method for constructing the conjugacy classes of the group $\overline{G} = S_n^m : C_m$, where m is prime.

4.3 Conjugacy Classes of $\overline{G} = S_n^m : C_m$, prime m

It is well known [24] that the conjugacy classes of the symmetric group S_n are in one-to-one correspondence with partitions of n, called **types** of the conjugacy classes of S_n. The conjugacy classes of the symmetric group S_n are easily constructed (even in GAP [14]). Since conjugation in S_n is equivalent to applying the conjugating element to the symbols in the elements of S_n [24], any conjugate elements of a subgroup of S_n have the same type as elements of S_n. However two elements may have the same type in S_n but may not be conjugate in the subgroup. Therefore here, our reference to the type of elements of a subgroup of S_n will not mean the elements are conjugate in the subgroup. Consequently the fusion of the conjugacy classes of a subgroup of S_n to the conjugacy classes of S_n is useful in the determination of the conjugacy classes of the subgroup. It is clear that the conjugacy classes of $N = S_n^m$ consist of direct products of the elements in the conjugacy classes of the copies of S_n. Since $N = S_n^m$ is not abelian, we cannot apply coset analysis as in [27, 28] to construct the conjugacy classes of $\overline{G}$. Here we give a method similar to coset analysis for constructing the conjugacy classes of a subgroup $S_n^m : C_m$ of the symmetric group S_{mn} of degree $m \times n$, where m is prime. Thus, in the following we consider prime m as this case provides interesting examples [35].

Let $\overline{h}, \overline{g} \in \overline{G}$. In the following we will write conjugation of $\overline{h}$ by $\overline{g}$ as $(\overline{h})^{\overline{g}} = (\overline{g})^{-1}\overline{h}\overline{g}$. It is clear that for $\overline{h} = xg^s$ and $\overline{g} = ug^r$, we have $(\overline{h})^{\overline{g}} = x^{\overline{g}}(g^s)^{\overline{g}}$ and $(\overline{h})^{\overline{g}} = (\overline{h})^{ug^r} = ((\overline{h})^u)^{g^r}$.

Lemma 4.3.1 *Let $x(\neq 1) \in N$ and $g^s(\neq 1) \in C_m$, $s = 1, 2, \ldots, m-1$. Then*

- *(i)* $x^{g^s} \in N$.
- *(ii)* $(g^s)^x \in \overline{G} \backslash N$. *Further if* $x^{g^s} \neq x$ *we have* $(g^s)^x \in \overline{G} \backslash (N \cup C_m)$.

Proof.

- (i) Clear since by definition of $\overline{G}$, the group N is C_m-invariant.
- (ii) We have $(g^s)^x = x^{-1}g^s x = x^{-1}x^{g^{-s}}g^s \in \overline{G}\backslash N$ since $g^s \neq 1$. Further if $x^{g^s} \neq x$ then $x^{g^{-s}} \neq x$ so that $x^{-1}x^{g^{-s}} \neq 1$. Now from above we have $(g^s)^x = x^{-1}x^{g^{-s}}g^s(\neq g^s) \notin C_m$. Hence $(g^s)^x \in \overline{G}\backslash(N \cup C_m)$.

□

Definition 4.3.2 *For each $g^s \in C_m$, we call the set $\{x^{-1}x^{g^{-s}} \mid x \in N\}$ the g^s-**basic set** in $\overline{G}$.*

Since $x^{g^s} \neq x$ for some $x \in N$, it is clear that each g^s-basic set in $\overline{G}$ is distinct, for $s = 1, 2, \ldots, m-1$.

Lemma 4.3.3 *Let $x(\neq 1) \in N$ and $g^s \in C_m$. Let $\overline{g} = ug^r \in \overline{G}$. Then*

- *(i)* $x^{\overline{g}} = y^{g^r}$, *where* $y = x^u \in x^N$.
- *(ii)* $(g^s)^{\overline{g}} = v^{-1}v^{g^{-s}}g^s$, *where* $v = u^{g^r}$.

Proof. With $\overline{g} = ug^r$ where $u \in N$ and $g^r \in C_m$, we have

(i) $x^{\overline{g}} = x^{ug^r} = (x^u)^{g^r} = y^{g^r}$, where $y = x^u \in x^N$.

(ii)

$$\begin{aligned}(g^s)^{\overline{g}} &= ((g^s)^u)^{g^r} \\ &= (u^{-1}g^s u)^{g^r} \\ &= (u^{-1}u^{g^{-s}}g^s)^{g^r} \\ &= (u^{-1}u^{g^{-s}})^{g^r}g^s \\ &= (u^{g^r})^{-1}(u^{g^r})^{g^{-s}}g^s \\ &= v^{-1}v^{g^{-s}}g^s.\end{aligned}$$

□

Remark 4.3.4 *Let $x, u \in N$ and $g^s, g^r \in C_m$. We see from Lemma 4.3.3 that*

(i) since $x^u \in N$, conjugating $x \in N$ by $\overline{g} = ug^r$ is equivalent to conjugating a conjugacy class representative of x^N by g^r.

(ii) since $(g^s)^{g^r} = g^s$, conjugating g^s by $\overline{g} = ug^r$ is equivalent to computing the element $(g^s)^u$.

(iii) $(g^s)^u = u^{-1}u^{g^{-s}}g^s$.

We note that to construct the conjugacy classes of $\overline{G}$, we need to consider the actions by conjugation of each group N and C_m on the conjugacy classes of the other. Therefore from now onwards we shall assume that the conjugacy classes of N and C_m are known and write x^{g^r} in place of $x^{\overline{g}} = (x^u)^{g^r} = y^{g^r}$ where $y \in x^N$.

Let $C_N(g^s)$ be the centralizer of g^s in N and $x \in C_N(g^s)$. Since $C_N(g^s)$ is a group, it follows that $x^{-1} \in C_N(g^s)$. The following result describes $C_N(g)$, the elements in N that fix g.

Proposition 4.3.5 *Let $g(\neq 1) \in C_m$. Then $C_N(g) = \{x \in N \mid g^x = g\}$ consists of elements of N of type $[\lambda]^m = [1^{m\lambda_1}2^{m\lambda_2}\cdots n^{m\lambda_n}]$ in S_{mn}, where $[\lambda]$ ranges over the types of conjugacy classes of S_n.*

Proof. Let $g^x = g$. Then $x^{-1}gx = g$ which implies that $x^g = x$, that is g fixes x. Thus g permutes the cycles of x amongst themselves. Now since g permutes cycles from distinct copies of S_n it follows that x is a product of elements of the same lengths, one element from each distinct copy of S_n. Without loss of generality let

$$x = \cdots uv \cdots$$

where u and v are any two elements, of the same length, from distinct copies of S_n such that $u^g = v$. Then for each cycle u_i of u there is a cycle v_j of v of the same length as u_i such that $u_i^g = v_j$. Thus u and v have the same type $[\lambda]$. Since x must contain one element of type $[\lambda]$ from each copy of S_n and N contains m copies of S_n, it follows that x has type $[\lambda]^m = [1^{m\lambda_1}2^{m\lambda_2}\cdots n^{m\lambda_n}]$. □

Lemma 4.3.6 *Let $g^s(\neq 1) \in C_m$ where m is prime. Then $C_N(g^s) = C_N(g)$ for all s.*

Proof. Let $x \in C_N(g)$. Then $xg = gx$ which implies that

$$xg^s = xgg^{s-1} = gxg^{s-1} = g^2xg^{s-2} = g^3xg^{s-3} = \ldots = g^{s-1}xg = g^sx.$$

Thus $x \in C_N(g^s)$. Hence $C_N(g) \leq C_N(g^s)$. Now m is prime implies that $o(g) = o(g^s) = m$. Since $o(g^s)$ is the least common multiple of the orders of the cycles of g^s, we have that g^s and g only have cycles of the same length m. It follows that g^s and g move the same symbols, and by Proposition 4.3.5 they fix the same elements of N. Hence $C_N(g^s) = C_N(g)$. □

Corollary 4.3.7 *Let $g^s(\neq 1) \in C_m$, where m is prime. Then $|C_N(g^s)| = n!$ for all s.*

Proof. By Proposition 4.3.5, $|C_N(g)|$ is the number of elements of N of type $[\lambda]^m$ for all types $[\lambda]$ of conjugacy classes of S_n. However the number of elements of type $[\lambda]^m$ in $C_N(g)$ is equal to the number of elements of type $[\lambda]$ in S_n. Since each conjugacy class of type $[\lambda]$ in S_n contains $\frac{n!}{1^{\lambda_1}(\lambda_1)!2^{\lambda_2}(\lambda_2)!\cdots n^{\lambda_n}(\lambda_n)!}$ elements [24], we have

$$\begin{aligned} |C_N(g)| &= \sum_{[\lambda]} \frac{n!}{1^{\lambda_1}(\lambda_1)!2^{\lambda_2}(\lambda_2)!\cdots n^{\lambda_n}(\lambda_n)!} \\ &= |S_n| \\ &= n!. \end{aligned}$$

Now since m is prime, by Lemma 4.3.6 we have $|C_N(g^s)| = |C_N(g)| = n!$. □

Proposition 4.3.8 *Let $g^s(\neq 1) \in C_m$, where m is prime. Define $[g^s]_N = \{(g^s)^x \mid x \in N\}$. Then*

$$|[g^s]_N| = (n!)^{m-1}.$$

Proof. By the orbit-centralizer relation and Corollary 4.3.7 we have

$$\begin{aligned} |[g^s]_N| &= [N : C_N(g^s)] \\ &= \frac{(n!)^m}{n!} \\ &= (n!)^{m-1}. \end{aligned}$$

□

Let $x \in N$ and $g^s \in C_m$. Since $x^{g^s} \in N$, conjugating the elements of N by elements of C_m does not produce any further conjugacy classes of $\overline{G}$ besides those of N. However some conjugacy classes of N fuse on conjugation by C_m as we show in the following result.

Theorem 4.3.9 *Let $x, z \in N$ such that $z \notin x^N$. Let $g^s \in C_m$. If*

(i) x and z have the same type as elements of S_{mn} and

(ii) for each disjoint cycle z_j of z there is a disjoint cycle x_i in x of the same length such that $z_j = x_i^{g^s}$,

then z is conjugate to x in $\overline{G}$.

Proof. Let $x^u \neq z$ for any $u \in N$. By condition (i) let x and z have the same type in S_{mn}. By condition (ii) let each disjoint cycle z_j of z be such that there is a disjoint cycle x_i of x of the same length as z_j and $z_j = x_i^{g^s}$. Since the cycles are disjoint we can, if necessary, re-arrange the disjoint cycles x_i of x in the order of the cycles z_j in z such that $z_j = x_{i'}^{g^s}$. Then

$$z = \prod_j z_j = \prod_{i'} x_{i'}^{g^s} = x^{g^s}.$$

Thus z is conjugate to x in $\overline{G}$. □

Remark 4.3.10 *Since g^s permutes cycles from distinct copies of S_n, Theorem 4.3.9 implies that the cycles x_i and $z_j = x_{i'}^{g^s}$ belong to distinct copies of S_n.*

Corollary 4.3.11 *Let $x, z \in N$ and $g^s(\neq 1) \in C_m$.*

(i) If $x \in C_N(g)$, then the class x^N of N is also a conjugacy class equal to $x^{\overline{G}}$ of $\overline{G}$ of size $|x^N|$.

(ii) If $z \notin x^N$ and $x^{g^s} = z$, then the conjugacy classes x^N and z^N of N fuse in $\overline{G}$.

Proof. Follows from Theorem 4.3.9. □

Lemma 4.3.12 ***(Action of** C_m **on** N)* *Let m be prime. Then the number of orbits of the action of C_m on N is given by*

$$\frac{n!\left[(n!)^{m-1} + m - 1\right]}{m},$$

that is $\frac{n!}{m}\left[(n!)^{m-1} - 1\right])$ orbits of length m and $n!$ orbits of length 1.

Proof. There are $(n!)^m$ elements in N and by Corollary 4.3.7 only $n! = |C_N(g^s)|$ of these are fixed by C_m, and thus each of these forms its own orbit under conjugation by C_m. This leaves $(n!)^m - n!$ elements of N, some of which must fuse under conjugation by C_m. However since elements of C_m permute cycles from distinct copies of S_n and there are m copies of S_n, it follows that each orbit of the action of C_m on $N \setminus C_N(g)$ contains m elements. It follows that the $(n!)^m - n!$ elements of N form $\frac{(n!)^m - n!}{m}$ orbits on conjugation by C_m. Thus the total number of orbits of the action of C_m on N is

$$\frac{(n!)^m - n!}{m} + n!$$

which gives the result. The orbit sizes are clear from the number $|C_N(g)| = n!$ of elements in N fixed by C_m and the m elements in $N \setminus C_N(g)$ that are permuted by C_m. □

Proposition 4.3.13 *Let m be prime. Let $P(n)$ be the number of partitions of n. Let $[\lambda] = [\lambda_1][\lambda_2]\cdots[\lambda_m]$ be the type of x in N, where $[\lambda_i] = [\lambda_{i1}\lambda_{i2}\cdots\lambda_{in}]$ is a type of an element of the i^{th} copy of S_n in N. Then the number of orbits of the action of C_m on the conjugacy classes of N is given by*

$$\frac{P(n)\left[(P(n))^{m-1}+m-1\right]}{m},$$

where an orbit containing a class of N with representative x contains the following numbers of elements of N:

(i) $\prod_{i=1}^{m} \frac{n!}{1^{\lambda_{i1}}(\lambda_{i1})!2^{\lambda_{i2}}(\lambda_{i2})!\cdots n^{\lambda_{in}}(\lambda_{in})!}$, *if* $x \in C_N(g)$.

(ii) $\prod_{i=1}^{m} \frac{m(n!)}{1^{\lambda_{i1}}(\lambda_{i1})!2^{\lambda_{i2}}(\lambda_{i2})!\cdots n^{\lambda_{in}}(\lambda_{in})!}$, *if* $x \notin C_N(g)$.

Proof. Since there are $P(n)$ conjugacy classes of S_n, it follows from the direct products of the classes of S_n that N has $(P(n))^m$ conjugacy classes each of size

$$\prod_{i=1}^{m} \frac{n!}{1^{\lambda_{i1}}(\lambda_{i1})!2^{\lambda_{i2}}(\lambda_{i2})!\cdots n^{\lambda_{in}}(\lambda_{in})!},$$

where $\frac{n!}{1^{\lambda_{i1}}(\lambda_{i1})!2^{\lambda_{i2}}(\lambda_{i2})!\cdots n^{\lambda_{in}}(\lambda_{in})!}$ is the size of the class of S_n of type $[\lambda_i] = [\lambda_{i1}\lambda_{i2}\cdots\lambda_{in}]$. Since there are $P(n)$ types in $C_N(g)$, the $P(n)$ classes of N are fixed by C_m by Corollary 4.3.11 (i), and thus each of these forms its own orbit under conjugation by C_m. This leaves $(P(n))^m - P(n)$ conjugacy classes of N, some of which must fuse under conjugation by C_m. However we know from Lemma 4.3.12 that each orbit of the action of C_m on $N \backslash C_N(g)$ contains m elements each of which belongs to a distinct class of N. It follows that the $(P(n))^m - P(n)$ conjugacy classes of N form $\frac{(P(n))^m - P(n)}{m}$ orbits on conjugation by C_m. Thus the total number of orbits of the action of C_m on the conjugacy classes of N is

$$\frac{(P(n))^m - P(n)}{m} + P(n)$$

which gives the result. The orbit sizes follow from the products of the sizes

$$\frac{n!}{1^{\lambda_{i1}}(\lambda_{i1})!2^{\lambda_{i2}}(\lambda_{i2})!\cdots n^{\lambda_{in}}(\lambda_{in})!}$$

for each class of S_n of type $[\lambda_i]$. Further the classes are of sizes those of type $[\lambda]$ in N if $x \in C_N(g)$ or we multiply the above sizes by m where m conjugacy classes of N fuse that is if $x \notin C_N(g)$. □

We remark that if N is abelian, then Proposition 4.3.13 would be equivalent to Lemma 4.3.12.

Proposition 4.3.14 *Let $z \notin x^N$ and $x^{g^s} = z$. Then xg^s is conjugate to zg^s in $\overline{G}$.*

Proof. On conjugating xg^s with $\overline{g} = g^s$ we get $(xg^s)^{g^s} = x^{g^s}g^s = zg^s$. □

Examples 4.3.15 *In $\overline{G} = S_3^3 : C_3$, let $g = (1,4,7)(2,5,8)(3,6,9)$ with $g^2 = (1,7,4)(2,8,5)(3,9,6) \in C_3$. Then*

(i) take $x = (1,2)$ and $z = (4,5)$ which are not conjugate in N. See that $(1,2)^g = (4,5)$ so that $((1,2)g^2)^g = (4,5)g^2$.

(ii) take $x = (1,2)(4,5,6)$ and $z = (4,5)(7,8,9)$ which are not conjugate in N. See that $((1,2)(4,5,6))^g = (4,5)(7,8,9)$ so that $((1,2)(4,5,6)g^2)^g = (4,5)(7,8,9)g^2$. See that $((4,5)(7,8,9))^g = (7,8)(1,2,3)$ so that $((4,5)(7,8,9)g^2)^g = (7,8)(1,2,3)g^2$.

The following proposition generalizes Lemma 4.3.3.

Proposition 4.3.16 *Let $x^{ug^r} = z$ in $\overline{G}$. Then*

(i) xg^s is not conjugate to zg^t in $\overline{G}$ unless $t = s$.

(ii) we have

$$(xg^s)^{ug^r} = \begin{cases} zg^s & \text{if } u \in C_N(g^s) \\ z(v^{-1}v^{g^{-s}})g^s & \text{otherwise,} \end{cases} \quad \text{where } v = u^{g^r}.$$

Proof.

(i) Suppose xg^s is conjugate to zg^t in $\overline{G}$. Then for some $\overline{g} \in \overline{G}$ we have $(\overline{g})^{-1}(xg^s)\overline{g} = zg^t$, that is $(\overline{g})^{-1}x\overline{g}(\overline{g})^{-1}g^s\overline{g} = zg^t$. This implies $(g^s)^{\overline{g}} = g^t$, that is g^s is conjugate to g^t which is not possible by Lemmas 4.3.1 (ii) and 4.3.3 (ii).

(ii) We have $(xg^s)^{ug^r} = x^{ug^r}(g^s)^{ug^r} = (x^u)^{g^r}((g^s)^u)^{g^r} = y^{g^r}g^s$ if $u \in C_N(g^s)$. Now if $u \notin C_N(g^s)$, then $(xg^s)^{ug^r} = (x^u)^{g^r}((g^s)^u)^{g^r} = y^{g^r}(u^{-1}g^su)^{g^r} = y^{g^r}(u^{-1}u^{g^{-s}})^{g^r}g^s$ $= z(v^{-1}v^{g^{-s}})g^s$. □

Examples 4.3.17 *In $\overline{G} = S_3^3 : C_3$, let $g = (1,4,7)(2,5,8)(3,6,9)$, $g^2 = (1,7,4)(2,8,5)(3,9,6) \in C_3$. Then*

(i) take $x = (1,2)$ and $y = (2,3)$ which are conjugate in N. Then we have $u = (1,2,3)(4,5,6)(7,8,9), u^{-1} = (1,3,2)(4,6,5)(7,9,8) \in C_N(g^s)$ as stated in Proposition 4.3.5. Now see that $(1,2)^u = (2,3)$ so that $((1,2)g)^{ug^2} = (2,3)^{g^2}g = (8,9)g$ and $((1,2)g^2)^{ug} = (2,3)^g g^2 = (5,6)g^2$.

(ii) *take* $x = (1,2)(4,5,6)$ *and* $y = (2,3)(5,6,4)$ *which are conjugate in* N. *Note that* $((1,2)(4,5,6))^u = (2,3)(5,6,4)$ *so that*

$$((1,2)(4,5,6)g)^{ug^2} = ((2,3)(5,6,4))^{g^2}g = (1,2,3)(8,9)g$$

and

$$((1,2)(4,5,6))g^2)^{ug} = ((2,3)(5,6,4))^g g^2 = (5,6)(8,9,7)g^2.$$

Remark 4.3.18 *Proposition 4.3.16 (i) implies that only the elements within a coset* Ng^s *can be conjugate in* $\overline{G}$. *Thus to determine the conjugacy classes of* $\overline{G}$ *we may consider elements within a coset of* N *in* $\overline{G}$.

Theorem 4.3.19 ***(Coset Analysis Equivalent)*** *Let* $\{y\}$ *be the set of elements of* $C_N(g^s)$ *of type* $[\lambda]^m$, *where* $[\lambda]$ *is a type of a conjugacy class of* S_n *and* m *is prime. Let* $Ng^s \neq N$ *be a non-identity coset of* N *in* $\overline{G}$. *Then on conjugation by elements of* $\overline{G}$, *the coset* Ng^s *breaks into sets of form*

$$\{\sigma\}[g^s]_N = \{\sigma t \mid \sigma \in \{\sigma\}, t \in [g^s]_N\},$$

where σ *is one of the products of cycles of* y *being permuted within* y. *Further each set* $\{\sigma\}[g^s]_N$ *is a conjugacy class of* $\overline{G}$.

Proof. By Remark 4.3.18, to determine the conjugacy classes of $\overline{G}$ we consider elements within a coset Ng^s of N in $\overline{G}$. Since $|Ng^s| = |N| = |C_N(g^s)||[g^s]_N|$, we analyze the set $[g^s]_N$. Let $t \in [g^s]_N$. Then $t = (g^s)^u = (u^{-1}u^{g^{-s}})g^s \in Ng^s$. Now by Proposition 4.3.16, an element $\overline{h}$ of $\overline{G}$ is conjugate to an element of form $\sigma(v^{-1}v^{g^{-s}})g^s$ for some $\sigma = (x^u)^{g^r}$, that is σ is conjugate under g^r to some element x^u of N. But the only elements of N of this form are those parts of an element y of $C_N(g^s)$. This suggests that σ is a product of cycles from some copies S_n of N. Let $y = \sigma_1\sigma_2\cdots\sigma_m$. If $\sigma_i^{g^s} = \sigma_{i+1}$ for $i = 1,2\ldots,m-1$ and $\sigma_m^{g^s} = \sigma_1$, then $\sigma = \sigma_i$, one fixed $i = 1,2\ldots,m$ and take $(\sigma)^{g^r} = \sigma_{i'}$.

Now notice that the elements of $[g^s]_N$ may be considered as products of the elements of $[g^s]_N$ by $\sigma = 1_{S_n}$ of 1_N. Following this notation, we multiply the elements of $[g^s]_N$ by $\sigma(\neq 1)$ as $\sigma(u^{-1}u^{g^{-s}}g^s) \in \{\sigma\}[g^s]_N$. We show that the element is conjugate to $\overline{h}$ in $\overline{G}$. Consider an element of Ng^s as an element $\sigma u^{-1}u^{g^{-s}}g^s$ of $\{\sigma\}[g^s]_N$. By the remarks following Remark 4.3.4, taking conjugation by $\overline{g} = ug^r$ as conjugation effectively by g^r, we have

$$\begin{aligned}(\sigma(v^{-1}v^{g^{-s}})g^s)^{g^r} &= \sigma^{g^r}(v^{-1})^{g^r}v^{g^{-s+r}}g^s \\ &= \sigma_{i'}(v^{g^r})^{-1}(v^{g^r})^{g^{-s}}g^s \\ &= \sigma_{i'}w^{-1}w^{g^{-s}}g^s \\ &= \sigma_{i'}(g^s)^w,\end{aligned}$$

which belongs to $\{\sigma\}[g^s]_N$. This shows that $\{\sigma\}[g^s]_N$ is a conjugacy class of $\overline{G}$. □

Corollary 4.3.20 *Let $g^s \in C_m$ where m is prime. Then for each $s = 1, 2, \ldots, m-1$, the set $[g^s]_N = \{(g^s)^x \mid x \in N\}$ is a conjugacy class of $\overline{G}$ of size $(n!)^{m-1}$.*

Proof. Let $[\lambda] = [1^n]$ in Theorem 4.3.19. Then $\{1_{S_n}\}[g^s]_N = [g^s]_N$ for each $s = 1, 2, \ldots, m-1$ is a conjugacy class of $\overline{G}$. The size is clear from Proposition 4.3.8. □

Definition 4.3.21 *We call the classes $[g^s]_N$, $s = 1, 2, \ldots, m-1$, **basic conjugacy classes** of $\overline{G}$.*

Theorem 4.3.22 *(**Classes of** $\overline{G} = S_n^m : C_m$)*

(i) Let $g^s \in C_m$ where m is prime. Let x_i^N, $i = 1, 2, \ldots, k$ be the conjugacy classes of N such that $x_i^{g^s} = x_j$ for some $j = 1, 2, \ldots, k$, that is the classes of N which fuse under conjugation by elements of C_m. Then $C = \bigcup_{i=1}^k x_i^N$ is a conjugacy class of $\overline{G}$ of size $\sum_{i=1}^k |x_i^N|$.

(ii) Let $\{y\}$ be the set of elements of $C_N(g^s)$ of type $[\lambda]^m$, where m is prime and $[\lambda]$ is a type of a conjugacy class of S_n. Let $\{\sigma_i\}$ be the set of elements σ_i in $y = \sigma_1\sigma_2\cdots\sigma_m \in \{y\}$ such that $\sigma_i^{g^s} = \sigma_{i+1}$ for $i = 1, 2\ldots, m-1$ and $\sigma_m^{g^s} = \sigma_1$. Let $\sigma = \sigma_i$. Then for each $s = 1, 2, \ldots, m-1$ and each set $\{\sigma\}$, the set $\{\sigma\}[g^s]_N = \{\sigma t \mid \sigma \in \{\sigma\}, t \in [g^s]_N\}$ is a conjugacy class of $\overline{G}$ of size $|\{y\}| \times |[g^s]_N|$.

(iii) The classes in (i) and (ii) above form a complete set of conjugacy classes of $\overline{G}$.

Proof.

(i) Follows from Theorem 4.3.9 and Proposition 4.3.16.

(ii) Follows from Theorem 4.3.19. The sizes of the conjugacy classes follow from Proposition 4.3.8, Corollary 4.3.7 and Proposition 4.3.5.

(iii) By Theorem 4.3.19, we consider the cosets of N in $\overline{G}$.

(a) If $Ng^s = N$, we obtain the classes of $\overline{G}$ as given in Theorem 4.3.22 (i) above.

(b) If $Ng^s \neq N$, we obtain the classes of $\overline{G}$ as given in Theorem 4.3.22 (ii) above.

Now since $\overline{G} = \bigcup_{s=1}^{m} Ng^s$, the result follows. □

Proposition 4.3.23 *Let $P(n)$ be the number of partitions of n. Let m be prime. Then the total number of conjugacy classes of $\overline{G} = S_n^m : C_m$ is*

$$\frac{P(n)[(P(n))^{m-1} + m^2 - 1]}{m}.$$

Proof. By Proposition 4.3.13 the number of conjugacy classes of $\overline{G}$ obtained by uniting the conjugacy classes of N that fuse on conjugation by elements of C_m is $\frac{P(n)[(P(n))^{m-1}+m-1]}{m}$. Also since for each class $\{y\}$ in $C_N(g^s)$ of type $[\lambda]^m$ there is a class $\{\sigma\}[g^s]_N$ of $\overline{G}$, the number n_{g^s} of classes $\{y\}$ gives the number of classes of $\overline{G}$ obtained from the non-identity cosets. Thus from Theorem 4.3.22, we obtain that the total number of conjugacy classes of $\overline{G}$ is

$$\frac{P(n)[(P(n))^{m-1} + m - 1]}{m} + \sum_{s=1}^{m-1} n_{g^s}.$$

Since m is prime, by Proposition 4.3.5 and Corollary 4.3.7 we have $n_{g^s} = P(n)$ for each s. Also there are $m-1$ non-identity elements of C_m. The formula above now becomes

$$\frac{P(n)[(P(n))^{m-1} + m - 1]}{m} + (m-1)P(n),$$

which gives the result. □

Examples 4.3.24 *Let $m = 2$ with $n = 3$. The group $S_3^2 : C_2$ is a subgroup of S_6 of order 72. Using the theory in Section 4.3, we obtain 6 conjugacy classes of $S_3^2 : C_2$ from the coset N, these are of sizes 1,6,4,9,12,4. Since $|N| = (3!)^2 = 36$ and $m = 2$ is prime we have $C_N(g^s) = C_N(g) = \{1, (1,2,3)(4,5,6), (1,3,2)(4,6,5), (2,3)(5,6), (1,2)(4,5), (1,3)(4,6)\}$ of order 3!=6, so that $[N : C_N(g)] = 6$. Thus $|[g^s]| = 6$. We obtain 3 conjugacy classes of $S_3^2 : C_2$ from the coset $Ng \neq N$, these are of sizes $1 \times 6 = 6$, $3 \times 6 = 18$ and $2 \times 6 = 12$. Thus $S_3^2 : C_2$ has a total of 9 conjugacy classes.*

4.4 Character tables of $(S_n)^m : C_m$, prime m

In this section we outline the steps we follow in constructing the character tables of examples of groups of the form $S_n^m : C_m$, prime m. We use the method of Fischer-Clifford matrices to construct these tables. The method is applicable since by Theorem 2.3.15 the group $S_n^m : C_m$ has the property that every Irreducible character of $N = (S_n)^m$ is extendible to its inertia group.

(i) We first calculate the conjugacy classes using the method discussed in section 4.3. Thus by Theorems 4.3.19, 4.3.22 and Proposition 4.3.23 we are able to construct all the conjugacy classes of a group $S_n^m : C_m$, prime m.

(ii) To calculate the Fischer-Clifford matrix $M(1A)$ of $S_n^m : C_m$, prime m we apply Proposition 2.4.4 and for the matrices $M(g^s)$ we apply Theorem 2.4.5(1) (see [25]). In using Theorem 2.4.5 we consider S_n in the place of G and C_m in the place of S_n since in this case C_m is a subgroup of S_n. We obtain m Fischer-Clifford matrices for each group $S_n^m : C_m$ as follows.

   (a) The matrix $M(1A)$ is obtained from the orbit sums of the action of C_m on $Irr((S_n)^m)$ and deleting the repeated columns.

   (b) We consider the group $S_n^m : C_m$ as the wreath product $S_n w C_m$, where the group C_m is taken as a group generated by a cycle of order m in S_n rather than a multiple of n cycles of order m in S_{mn}. The Fischer-Clifford matrices $M(g^s)$ at g^s, where $s \neq 1$ are each equal to the character table of S_n.

(iii) Since m is prime we have only two inertia factor groups, that is C_1 and C_m. The fusion of the irreducible characters of C_1 into C_m is trivial.

Table 4.1: Character Table of $(S_3)^2$

$[g]$	1a	2a	3a	2b	2c	6a	3b	6b	3c
$\|C_G g\|$	72	24	36	24	8	12	36	12	18
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	1	1	-1	1
χ_3	2	0	-1	2	0	-1	2	0	-1
χ_4	1	1	1	-1	-1	-1	1	1	1
χ_5	1	-1	1	-1	1	-1	1	-1	1
χ_6	2	0	-1	-2	0	1	2	0	-1
χ_7	2	2	2	0	0	0	-1	-1	-1
χ_8	2	-2	2	0	0	0	-1	1	-1
χ_9	4	0	-2	0	0	0	-2	0	1

4.4.1 The group $(S_3)^2 : C_2$

As shown in Examples 4.3.24, the group $(S_3)^2 : C_2 =< (1,2),(1,2,3),(4,5),(4,5,6),(1,4)(2,5)(3,6) >$ has a total of 9 conjugacy classes of sizes 1,6,4,9,12,4, 6,18 and 12. We obtain the Fischer-Clifford matrices of $(S_3)^2 : C_2$ using the information as given in (ii) above as follows. The character table of $(S_3)^2$ is as given in Table 4.1.

Now we act $C_2 =< (1,4)(2,5)(3,6) >$ on $Irr((S_3)^2)$ and we obtain that three characters χ_1, χ_5 and χ_9 are fixed while $\{\chi_2,\chi_4\},\{\chi_3,\chi_7\}$ and $\{\chi_6,\chi_8\}$ form the rest of the orbits. Thus we add rows 2 and 4, 3 and 7, 6 and 8 of the character table of $(S_3)^2$, and deleting the repeated columns to obtain the Fischer-Clifford matrix $M(1A)$ which is as follows.

$$M(1A) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ 4 & 0 & -2 & 0 & 0 & 1 \\ 2 & 0 & 2 & -2 & 0 & 2 \\ 4 & 2 & 1 & 0 & -1 & -2 \\ 4 & -2 & 1 & 0 & 1 & -2 \end{pmatrix}.$$

The Fischer-Clifford matrix $M(2A)$ is as follows.

$$M(2A) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{pmatrix}.$$

The character table of $(S_3)^2 : C_2$ is obtained by multiplying the partial character tables of the inertia factor groups C_1 and C_2 with the corresponding rows of the above Fischer-Clifford matrices according to the fusions. The character table of $(S_3)^2 : C_2$ is given in Table 4.2.

Table 4.2: Character Table of $(S_3)^2 : C_2$

$[g]$	1A						2B		
$[\overline{g}]$	1a	2a	3a	2b	6a	3b	2c	4a	6b
$\lvert C_{\overline{G}}(\overline{g})\rvert$	72	12	18	8	6	18	12	4	6
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	-1	-1	-1
χ_3	1	-1	1	1	-1	1	1	-1	1
χ_4	1	-1	1	1	-1	1	-1	1	-1
χ_5	4	0	-2	0	0	1	2	0	-1
χ_6	4	0	-2	0	0	1	-2	0	1
χ_7	2	0	2	-2	0	2	0	0	0
χ_8	4	2	1	0	-1	-2	0	0	0
χ_9	4	-2	1	0	1	-2	0	0	0

4.4.2 The group $(S_3)^3 : C_3$

The group $(S_3)^3 : C_3 =< (1,2),(1,2,3),(4,5),(4,5,6),(7,8),(7,8,9),$ $(1,4,7)(2,5,8)(3,6,9) >$ is a subgroup of S_9 of order 648. We obtain 11 conjugacy classes of $(S_3)^3 : C_3$ from the coset N, these are of sizes 1,9,6,27,18,18,12,27,54,36,8. Since $|N| = (3!)^3 = 216$ and $m = 3$ is prime we have that for all s, $C_N(g^s) = C_N(g) = \{1,(1,2,3)(4,5,6)(7,8,9),(1,3,2)(4,6,5)(7,9,8),(2,3)(5,6)(8,9),(1,2)(4,5)(7,8),$ $(1,3)(4,6)(7,9)\}$ of order 3!=6, so that $[N : C_N(g)] = 36$. Thus $|[g^s]| = 36$. We obtain 6 conjugacy classes of $(S_3)^3 : C_3$ from the cosets $Ng^s \neq N$, these are of sizes $1\times 36 = 36$, $3\times 36 = 108$, $2\times 36 = 72$ and $1\times 36 = 36$, $3\times 36 = 108$, $2\times 36 = 72$, for Ng and Ng^2 respectively. Thus $(S_3)^3 : C_3$ has a total of 17 conjugacy classes. We obtain the Fischer-Clifford matrices of $(S_3)^3 : C_3$ as described above. Thus

$$M(1A) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 8 & 0 & -4 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 3 & 1 & 3 & -1 & 1 & 1 & 3 & -3 & -1 & 1 & 3 \\ 3 & -1 & 3 & -1 & -1 & -1 & 3 & 3 & -1 & -1 & 3 \\ 6 & -4 & 3 & 2 & -1 & -1 & 0 & 0 & -1 & 2 & -3 \\ 6 & 4 & 3 & 2 & 1 & 1 & 0 & 0 & -1 & -2 & -3 \\ 6 & 0 & 3 & -2 & -3 & 3 & 0 & 0 & 1 & 0 & -3 \\ 6 & 0 & 3 & -2 & 3 & -3 & 0 & 0 & 1 & 0 & -3 \\ 12 & -4 & 0 & 0 & 2 & 2 & -3 & 0 & 0 & -1 & 3 \\ 12 & 4 & 0 & 0 & -2 & -2 & -3 & 0 & 0 & 1 & 3 \end{pmatrix}$$

The Fischer-Clifford matrices for $g \in 3A$ and $g \in 3B$ are the same, thus:

$$M(3A) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix} = M(3B).$$

Table 4.3: Character Table of $(S_3)^3 : C_3$

$[g]$	1A											3A			3B		
$[\overline{g}]$	1a	2a	3a	2b	6a	6b	3b	2c	6c	6d	3c	3d	6e	9a	3e	6f	9b
$\lvert C_{\overline{G}}(\overline{g})\rvert$	648	72	108	24	36	36	54	24	12	18	81	18	6	9	18	6	9
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	1	1	1	α	α	α	$\overline{\alpha}$	$\overline{\alpha}$	$\overline{\alpha}$
χ_3	1	1	1	1	1	1	1	1	1	1	1	$\overline{\alpha}$	$\overline{\alpha}$	$\overline{\alpha}$	α	α	α
χ_4	8	0	-4	0	0	0	2	0	0	0	-1	2	0	-1	2	0	-1
χ_5	8	0	-4	0	0	0	2	0	0	0	-1	2α	0	$-\alpha$	$2\overline{\alpha}$	0	$-\overline{\alpha}$
χ_6	8	0	-4	0	0	0	2	0	0	0	-1	$2\overline{\alpha}$	0	$-\overline{\alpha}$	2α	0	$-\alpha$
χ_7	1	-1	1	1	-1	-1	1	-1	1	-1	1	1	-1	1	1	-1	1
χ_8	1	-1	1	1	-1	-1	1	-1	1	-1	1	α	$-\alpha$	α	$\overline{\alpha}$	$-\overline{\alpha}$	$\overline{\alpha}$
χ_9	1	-1	1	1	-1	-1	1	-1	1	-1	1	$\overline{\alpha}$	$-\overline{\alpha}$	$\overline{\alpha}$	α	$-\alpha$	α
χ_{10}	3	1	3	-1	1	1	3	-3	-1	1	3	0	0	0	0	0	0
χ_{11}	3	-1	3	-1	-1	-1	3	3	-1	-1	3	0	0	0	0	0	0
χ_{12}	6	-4	3	2	-1	-1	0	0	-1	2	-3	0	0	0	0	0	0
χ_{13}	6	4	3	2	1	1	0	0	-1	-2	-3	0	0	0	0	0	0
χ_{14}	6	0	3	-2	-3	3	0	0	1	0	-3	0	0	0	0	0	0
χ_{15}	6	0	3	-2	3	-3	0	0	1	0	-3	0	0	0	0	0	0
χ_{16}	12	-4	0	0	2	2	-3	0	0	-1	3	0	0	0	0	0	0
χ_{17}	12	4	0	0	-2	-2	-3	0	0	1	3	0	0	0	0	0	0

$$\alpha = \frac{-1+\sqrt{3i}}{2}$$

The character table of $(S_3)^3 : C_3$ is obtained in the same way by multiplying the partial character tables of the inertia factor groups C_1 and C_3 with the corresponding rows of the above Fischer-Clifford matrices according to the fusions. The character table of $(S_3)^3 : C_3$ is given in Table 4.3.

4.4.3 The group $(S_4)^2 : C_2$

The group $(S_4)^2 : C_2 =< (1,2), (1,2,3,4), (5,6), (5,6,7,8), (1,5)(2,6)(3,7)(4,8) >$ is a subgroup of S_8 of order 1152. We obtain 15 conjugacy classes of $(S_4)^2 : C_2$ from the coset N, these are of sizes 1,12,16,6,12,36,96,36,72,64,48,96,9,36,36. Since $|N| = (4!)^2 = 576$ and $m = 2$ is prime we have $C_N(g^s) = C_N(g)$ of order 4!=24, so that $[N : C_N(g)] = 24$. Thus $|[g^s]| = 24$. We obtain 5 conjugacy classes of $(S_4)^2 : C_2$ from the coset $Ng \neq N$, these are of sizes $1 \times 24 = 24$, $6 \times 24 = 144$, $8 \times 24 = 192$, $3 \times 24 = 72$ and $6 \times 24 = 144$. Thus $(S_4)^2 : C_2$ has a total of 20 conjugacy classes. We obtain the Fischer-Clifford matrices of $(S_4)^2 : C_2$ as described above. Thus

$$M(1A) = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 0 & 2 & 2 & 0 & -2 & 0 & 0 & -2 & 2 & 2 & 0 & 2 & 0 & -2 \\
4 & 2 & 1 & 4 & 2 & 0 & -1 & 2 & 0 & -2 & 1 & -1 & 4 & 2 & 0 \\
6 & 4 & 3 & 2 & 2 & 2 & 1 & 0 & 0 & 0 & -1 & -1 & -2 & -2 & -2 \\
6 & 2 & 3 & 2 & 4 & -2 & -1 & -2 & 0 & 0 & -1 & 1 & -2 & 0 & 2 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 \\
4 & -2 & 1 & 4 & -2 & 0 & 1 & -2 & 0 & -2 & 1 & 1 & 4 & -2 & 0 \\
6 & -2 & 3 & 2 & -4 & -2 & 1 & 2 & 0 & 0 & -1 & -1 & -2 & 0 & 2 \\
6 & -4 & 3 & 2 & -2 & 2 & -1 & 0 & 0 & 0 & -1 & 1 & -2 & 2 & -2 \\
4 & 0 & -2 & 4 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 4 & 0 & 0 \\
12 & 2 & -3 & 4 & -2 & 0 & -1 & 2 & 0 & 0 & 1 & 1 & -4 & -2 & 0 \\
12 & -2 & -3 & 4 & 2 & 0 & 1 & -2 & 0 & 0 & 1 & -1 & -4 & 2 & 0 \\
18 & 0 & 0 & -6 & 0 & -2 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & -2 \\
9 & 3 & 0 & -3 & -3 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\
9 & -3 & 0 & -3 & 3 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & -1 & 1
\end{pmatrix}$$

The Fischer-Clifford matrix $M(2A)$ is as follows.

$$M(2A) = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & -1 \\
2 & 0 & -1 & 2 & 0 \\
3 & 1 & 0 & -1 & -1 \\
3 & -1 & 0 & -1 & 1
\end{pmatrix}$$

The character table of $(S_4)^2 : C_2$ is obtained in the same way by multiplying the partial character tables of the inertia factor groups C_1 and C_2 with the corresponding rows of the above Fischer-Clifford matrices according to the fusions. The character table of $(S_4)^2 : C_2$ is given in Table 4.4.

Table 4.4: Character Table of $(S_4)^2 : C_2$

$[g]$	1A															2A				
$[\overline{g}]$	1a	2a	3a	2b	4a	2c	6a	2d	4b	3b	6b	12a	2e	4c	4d	2f	4e	6c	4f	8a
$\lvert R_{\overline{G}}(\overline{g})\rvert$	1152	96	72	192	96	32	12	32	16	18	24	12	128	32	32	48	8	6	16	8
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1
χ_3	1	-1	1	1	-1	1	-1	-1	1	1	1	-1	1	-1	1	1	-1	1	1	-1
χ_4	1	-1	1	1	-1	1	-1	-1	1	1	1	-1	1	-1	1	-1	1	-1	-1	1
χ_5	4	0	-2	4	0	0	0	0	0	1	-2	0	4	0	0	2	0	-1	2	0
χ_6	4	0	-2	4	0	0	0	0	0	1	-2	0	4	0	0	-2	0	1	-2	0
χ_7	9	3	0	-3	-3	1	0	-1	-1	0	0	0	1	1	1	3	1	0	-1	-1
χ_8	9	3	0	-3	-3	1	0	-1	-1	0	0	0	1	1	1	-3	-1	0	1	1
χ_9	9	-3	0	-3	3	1	0	1	-1	0	0	0	1	-1	1	3	-1	0	-1	1
χ_{10}	9	-3	0	-3	3	1	0	1	-1	0	0	0	1	-1	1	-3	1	0	1	-1
χ_{11}	2	0	2	2	0	-2	0	0	-2	2	2	0	2	0	-2	0	0	0	0	0
χ_{12}	4	2	1	4	2	0	-1	2	0	-2	1	-1	4	2	0	0	0	0	0	0
χ_{13}	6	4	3	2	2	2	1	0	0	0	-1	-1	-2	-2	-2	0	0	0	0	0
χ_{14}	6	2	3	2	4	-2	-1	-2	0	0	-1	1	-2	0	2	0	0	0	0	0
χ_{15}	4	-2	1	4	-2	0	1	-2	0	-2	1	1	4	-2	0	0	0	0	0	0
χ_{16}	6	-2	3	2	-4	-2	1	2	0	0	-1	-1	-2	0	2	0	0	0	0	0
χ_{17}	6	-4	3	2	-2	2	-1	0	0	0	-1	1	-2	2	-2	0	0	0	0	0
χ_{18}	12	2	-3	4	-2	0	-1	2	0	0	1	1	-4	-2	0	0	0	0	0	0
χ_{19}	12	-2	-3	4	2	0	1	-2	0	0	1	-1	-4	2	0	0	0	0	0	0
χ_{20}	18	0	0	-6	0	-2	0	0	2	0	0	0	2	0	-2	0	0	0	0	0

4.4.4 The group $(S_4)^3 : C_3$

The group $(S_4)^3 : C_3 =< (1,2),(1,2,3,4),(5,6),(5,6,7,8),(9,10),(9,10,11,12),$ $(1,5,9)(2,6,10)(3,7,11)(4,8,12) >$ is a subgroup of S_{12} of order 41472. We obtain 45 conjugacy classes of $(S_4)^3 : C_3$ from the coset N, these are of sizes $1, 9, 27, 27, 24, 72, 72,$ $216, 192, 576, 512, 18, 18, 54, 54, 54, 54, 162, 162, 144, 144, 432, 432, 144, 144, 432, 432, 1152,$ $1152, 108, 108, 108, 108, 324, 324, 324$, $324, 864, 864, 864, 864, 216, 648, 648, 216$. Since $|N| = (4!)^3 = 13824$ we have

$$C_N(g^s) =< 1, (1,2,3,4)(5,6,7,8)(9,10,11,12), (1,2,3)(5,6,7)(9,10,11), (1,2)(5,6)(9,10) >,$$

of order $4! = 24$, so that $[N : C_N(g^s)] = 576$. Thus $|[g^s]| = 576$. We obtain 10 conjugacy classes of $(S_4)^3 : C_3$ from the cosets $Ng^s \neq N$, these are of sizes 1×576, 6×576, 3×576, 8×576, 6×576 and 1×576, 6×576, 3×576, 8×576, 6×576 respectively for Ng and Ng^2. Thus $(S_4)^3 : C_3$ has a total of 55 conjugacy classes. We obtain the Fischer-Clifford matrices of $(S_4)^3 : C_3$ as described above. Thus

$$
M(1A) = \left(\begin{array}{rrrrrrrrrrrrrrr}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
9 & 7 & 5 & 6 & 5 & 5 & 3 & 4 & 3 & 3 & 1 & 2 & 1 & 4 & 2 \\
27 & 15 & 3 & 9 & 3 & 7 & -1 & 3 & -1 & -1 & -5 & -3 & -5 & 3 & -3 \\
27 & 9 & -9 & 0 & -9 & 3 & -3 & 0 & -3 & -3 & 3 & 0 & 3 & 0 & 0 \\
18 & 8 & 10 & 3 & 4 & 2 & 0 & -1 & -2 & 8 & 2 & 7 & 4 & -1 & -5 \\
6 & 4 & 6 & 3 & 4 & 2 & 4 & 1 & 2 & 4 & 6 & 3 & 4 & 1 & 3 \\
18 & 8 & 10 & 3 & 4 & 2 & 8 & -1 & 2 & 0 & 2 & -5 & -4 & -1 & 7 \\
54 & 12 & 6 & -9 & -12 & 2 & 4 & -3 & -2 & 4 & -10 & 3 & -4 & -3 & 3 \\
12 & 4 & 12 & 0 & 4 & 0 & 4 & -2 & 0 & 4 & 12 & 0 & 4 & -2 & 0 \\
36 & 4 & 20 & -12 & -4 & 0 & 4 & -2 & 0 & 4 & 4 & -4 & -4 & -2 & -4 \\
8 & 0 & 8 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & -4 & 0 & 0 & -4 \\
9 & 5 & 5 & 6 & 7 & 1 & 1 & 2 & 3 & 1 & 1 & 2 & 3 & 2 & 2 \\
27 & 9 & 3 & 9 & 9 & -1 & -7 & -3 & -5 & 1 & -5 & -3 & -7 & 3 & -3 \\
18 & 4 & 10 & 3 & 8 & -2 & 4 & -5 & -2 & -4 & 2 & -5 & 0 & 1 & 7 \\
27 & 9 & 3 & 9 & 9 & -1 & 1 & 3 & 7 & -7 & -5 & -3 & 1 & -3 & -3 \\
81 & 9 & -27 & 0 & -9 & -3 & -3 & 0 & 3 & -3 & 9 & 0 & 3 & 0 & 0 \\
54 & 0 & 6 & -9 & 0 & -2 & 8 & -3 & 2 & -8 & -10 & 3 & 8 & 3 & 3 \\
18 & 4 & 10 & 3 & 8 & -2 & -4 & 1 & 2 & 4 & 2 & 7 & 8 & -5 & -5 \\
54 & 0 & 6 & -9 & 0 & -2 & -8 & 3 & 2 & 8 & -10 & 3 & -8 & -3 & 3 \\
36 & -4 & 20 & -12 & 4 & 0 & -4 & 2 & 0 & -4 & 4 & -4 & 4 & 2 & -4 \\
27 & 3 & 3 & 9 & 15 & -5 & -5 & -3 & -1 & -5 & -5 & -3 & -1 & -3 & -3 \\
81 & -9 & -27 & 0 & 9 & -3 & 3 & 0 & 3 & 3 & 9 & 0 & -3 & 0 & 0 \\
54 & -12 & 6 & -9 & 12 & 2 & -4 & 3 & -2 & -4 & -10 & 3 & 4 & 3 & 3 \\
27 & -9 & -9 & 0 & 9 & 3 & 3 & 0 & -3 & 3 & 3 & 0 & -3 & 0 & 0 \\
3 & 1 & 3 & 3 & 1 & -1 & 1 & 1 & -1 & 1 & 3 & 3 & 1 & 1 & 3 \\
9 & 1 & 5 & 6 & -1 & -3 & -3 & -2 & -5 & 5 & 1 & 2 & 3 & 4 & 2 \\
6 & 0 & 6 & 3 & 0 & -2 & 0 & -3 & -2 & 0 & 6 & 3 & 0 & 3 & 3 \\
9 & -1 & 5 & 6 & 1 & -3 & -5 & -4 & -1 & 3 & 1 & 2 & 5 & 2 & 2 \\
9 & 1 & 5 & 6 & -1 & -3 & 5 & 4 & -1 & -3 & 1 & 2 & -5 & -2 & 2 \\
27 & -3 & 3 & 9 & -15 & -5 & 5 & 3 & -1 & 5 & -5 & -3 & 1 & 3 & -3 \\
18 & -4 & 10 & 3 & -8 & -2 & 4 & -1 & 2 & -4 & 2 & 7 & -8 & 5 & -5 \\
27 & -9 & 3 & 9 & -9 & -1 & -1 & -3 & 7 & 7 & -5 & -3 & -1 & 3 & -3 \\
6 & 0 & 6 & 3 & 0 & -2 & 0 & 3 & -2 & 0 & 6 & 3 & 0 & -3 & 3 \\
18 & -4 & 10 & 3 & -8 & -2 & -4 & 5 & -2 & 4 & 2 & -5 & 0 & -1 & 7 \\
12 & -4 & 12 & 0 & -4 & 0 & -4 & 2 & 0 & -4 & 12 & 0 & -4 & 2 & 0 \\
18 & -8 & 10 & 3 & -4 & 2 & -8 & 1 & 2 & 0 & 2 & -5 & 4 & 1 & 7 \\
9 & -1 & 5 & 6 & 1 & -3 & 3 & 2 & -5 & -5 & 1 & 2 & -3 & -4 & 2 \\
27 & -9 & 3 & 9 & -9 & -1 & 7 & 3 & -5 & -1 & -5 & -3 & 7 & -3 & -3 \\
18 & -8 & 10 & 3 & -4 & 2 & 0 & 1 & -2 & -8 & 2 & 7 & -4 & 1 & -5 \\
27 & -15 & 3 & 9 & -3 & 7 & 1 & -3 & -1 & 1 & -5 & -3 & 5 & -3 & -3 \\
3 & -1 & 3 & 3 & -1 & -1 & -1 & -1 & -1 & -1 & 3 & 3 & -1 & -1 & 3 \\
9 & -5 & 5 & 6 & -7 & 1 & -1 & -2 & 3 & -1 & 1 & 2 & -3 & -2 & 2 \\
6 & -4 & 6 & 3 & -4 & 2 & -4 & -1 & 2 & -4 & 6 & 3 & -4 & -1 & 3 \\
9 & -7 & 5 & 6 & -5 & 5 & -3 & -4 & 3 & -3 & 1 & 2 & -1 & -4 & 2 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1
\end{array} \right|
$$

$$
M(1A\ contd.) = \left|\begin{array}{ccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 2 & 3 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & -1 & 0 & -1 & 0 & 1 \\
0 & -3 & -1 & -5 & -3 & -5 & 3 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 0 \\
0 & 0 & -3 & 3 & 0 & 3 & 1 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\
-3 & -5 & 2 & -4 & 1 & -2 & 0 & 2 & -1 & 0 & 0 & 3 & 2 & -1 & -1 \\
0 & 1 & 2 & 4 & 1 & 2 & 0 & 2 & -1 & 0 & 4 & 1 & 2 & 1 & -2 \\
-3 & 1 & -2 & 4 & -5 & -2 & 0 & 2 & -1 & 0 & 0 & -1 & -2 & 3 & -1 \\
0 & 3 & -2 & -4 & 3 & 2 & 0 & 2 & -1 & 0 & -4 & 1 & -2 & 1 & 0 \\
-3 & -2 & 0 & 4 & -2 & 0 & 0 & 0 & 0 & 0 & 4 & -2 & 0 & -2 & 1 \\
3 & 2 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 4 & -2 & 0 & -2 & 1 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 4 & 3 & 3 & 4 & 5 & -3 & -3 & -2 & -1 & -3 & -2 & -1 & -2 & -1 \\
0 & -3 & 7 & 1 & 3 & -1 & -3 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 0 \\
-3 & -1 & 2 & 8 & -1 & 2 & 0 & -2 & 1 & 0 & -4 & 1 & 2 & -1 & 1 \\
0 & 3 & -5 & -7 & -3 & -1 & -3 & -1 & -1 & 1 & 1 & 1 & 3 & -1 & 0 \\
0 & 0 & 3 & 3 & 0 & -3 & -3 & 1 & 0 & 3 & 1 & 0 & -1 & 0 & 0 \\
0 & -3 & 2 & -8 & 3 & -2 & 0 & -2 & 1 & 0 & 0 & -1 & 2 & 1 & 0 \\
-3 & -1 & -2 & 0 & -1 & 2 & 0 & -2 & 1 & 0 & -4 & -1 & -2 & 1 & 1 \\
0 & 3 & 2 & 8 & -3 & -2 & 0 & -2 & 1 & 0 & 0 & 1 & 2 & -1 & 0 \\
3 & -2 & 0 & 4 & -2 & 0 & 0 & 0 & 0 & 0 & -4 & 2 & 0 & 2 & -1 \\
0 & 3 & -1 & -1 & 3 & 7 & 3 & 3 & 1 & -1 & 3 & 1 & -1 & 1 & 0 \\
0 & 0 & 3 & -3 & 0 & -3 & 3 & 1 & 0 & -3 & -1 & 0 & -1 & 0 & 0 \\
0 & -3 & -2 & 4 & -3 & 2 & 0 & 2 & -1 & 0 & 4 & -1 & -2 & -1 & 0 \\
0 & 0 & -3 & -3 & 0 & 3 & -1 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\
3 & 1 & -1 & 1 & 1 & -1 & -3 & -1 & -1 & -3 & 1 & 1 & -1 & 1 & 1 \\
3 & 2 & -1 & -5 & -4 & -3 & -3 & 1 & 0 & -1 & 1 & 0 & 3 & 2 & 1 \\
0 & 3 & -2 & 0 & -3 & -2 & 0 & -2 & 1 & 0 & 0 & 3 & -2 & -3 & 0 \\
3 & 4 & -5 & -3 & -2 & -3 & 3 & 1 & 0 & 1 & -1 & -2 & -1 & 0 & -1 \\
3 & -4 & -5 & 3 & 2 & -3 & -3 & 1 & 0 & -1 & 1 & 2 & -1 & 0 & 1 \\
0 & -3 & -1 & 1 & -3 & 7 & -3 & 3 & 1 & 1 & -3 & -1 & -1 & -1 & 0 \\
-3 & 1 & -2 & 0 & 1 & 2 & 0 & -2 & 1 & 0 & 4 & 1 & -2 & -1 & -1 \\
0 & -3 & -5 & 7 & 3 & -1 & 3 & -1 & -1 & -1 & -1 & -1 & 3 & 1 & 0 \\
0 & -3 & -2 & 0 & 3 & -2 & 0 & -2 & 1 & 0 & 0 & -3 & -2 & 3 & 0 \\
-3 & 1 & 2 & -8 & 1 & 2 & 0 & -2 & 1 & 0 & 4 & -1 & 2 & 1 & -1 \\
-3 & 2 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & -4 & 2 & 0 & 2 & -1 \\
-3 & -1 & -2 & -4 & 5 & -2 & 0 & 2 & -1 & 0 & 0 & 1 & -2 & -3 & 1 \\
3 & -2 & -1 & 5 & 4 & -3 & 3 & 1 & 0 & 1 & -1 & 0 & 3 & -2 & -1 \\
0 & 3 & 7 & -1 & -3 & -1 & 3 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 0 \\
-3 & 5 & 2 & 4 & -1 & -2 & 0 & 2 & -1 & 0 & 0 & -3 & 2 & 1 & 1 \\
0 & 3 & -1 & 5 & 3 & -5 & -3 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 0 \\
3 & -1 & -1 & -1 & -1 & -1 & 3 & -1 & -1 & 3 & -1 & -1 & -1 & -1 & -1 \\
3 & -4 & 3 & -3 & -4 & 5 & 3 & -3 & -2 & 1 & 3 & 2 & -1 & 2 & 1 \\
0 & -1 & 2 & -4 & -1 & 2 & 0 & 2 & -1 & 0 & -4 & -1 & 2 & -1 & 2 \\
3 & -2 & 3 & -1 & -2 & 1 & -3 & 1 & 2 & -1 & 1 & 0 & -1 & 0 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1
\end{array}\right|,
$$

$$M(1A\ contd.) = \left|\begin{array}{rrrrrrrrrrrrrrr}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & -1 & 0 & -1 & -3 & -2 & -3 & -1 & -2 & -3 & -2 & 0 & -1 & -2 & -3 \\
-1 & -1 & -1 & -1 & 3 & 1 & 3 & 0 & 1 & 3 & 1 & 0 & 0 & 1 & 3 \\
0 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
-1 & -2 & 1 & 0 & -6 & -1 & -4 & 1 & -1 & -2 & 1 & 0 & 1 & 1 & 0 \\
-1 & 2 & -1 & 0 & 6 & 3 & 4 & 0 & 1 & 2 & 1 & -3 & -2 & -1 & 0 \\
1 & 2 & -1 & 0 & -6 & -1 & -4 & 1 & 1 & -2 & -1 & 0 & 1 & 1 & 0 \\
1 & -2 & 1 & 0 & 6 & -1 & 4 & 0 & -1 & 2 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 12 & 0 & 4 & -3 & -2 & 0 & -2 & 3 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -12 & 4 & -4 & -1 & 2 & 0 & 2 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 8 & -4 & 0 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & -3 & -2 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & 2 & 3 \\
1 & 3 & 1 & 1 & 3 & 1 & 1 & 0 & 1 & -1 & -1 & 0 & 0 & -1 & -3 \\
-1 & -2 & 1 & 0 & -6 & -1 & 0 & 1 & -1 & 2 & 3 & 0 & -1 & -1 & 0 \\
1 & -1 & 1 & 1 & 3 & 1 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & -1 & -3 \\
0 & -1 & 0 & -3 & -3 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 3 \\
-1 & 2 & -1 & 0 & 6 & -1 & 0 & 0 & 1 & -2 & -1 & 0 & 0 & 1 & 0 \\
1 & 2 & -1 & 0 & -6 & -1 & 0 & 1 & 3 & 2 & -1 & 0 & -1 & -1 & 0 \\
-1 & 2 & -1 & 0 & 6 & -1 & 0 & 0 & -1 & -2 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -12 & 4 & 4 & -1 & -2 & 0 & -2 & 0 & 1 & 0 & 0 \\
-1 & -1 & -1 & -1 & 3 & 1 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & 3 \\
0 & -1 & 0 & 3 & -3 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -3 \\
1 & -2 & 1 & 0 & 6 & -1 & -4 & 0 & 1 & 2 & 1 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\
-1 & -1 & -1 & -3 & 3 & 3 & 1 & 3 & 1 & -1 & 1 & 3 & 1 & -1 & -3 \\
2 & -1 & -2 & 1 & -3 & -2 & -1 & -1 & -2 & 1 & 0 & 0 & -1 & 0 & 3 \\
1 & -2 & 1 & 0 & 6 & 3 & 0 & 0 & 3 & -2 & -3 & -3 & 0 & 1 & 0 \\
-2 & 3 & 2 & -1 & -3 & -2 & 1 & -1 & 0 & 1 & 2 & 0 & 1 & 0 & -3 \\
-2 & 3 & 2 & 1 & -3 & -2 & -1 & -1 & 0 & 1 & -2 & 0 & -1 & 0 & 3 \\
-1 & -1 & -1 & 1 & 3 & 1 & 1 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & -3 \\
1 & 2 & -1 & 0 & -6 & -1 & 0 & 1 & -3 & 2 & 1 & 0 & 1 & -1 & 0 \\
1 & -1 & 1 & -1 & 3 & 1 & -1 & 0 & 1 & -1 & -1 & 0 & 0 & -1 & 3 \\
1 & -2 & 1 & 0 & 6 & 3 & 0 & 0 & -3 & -2 & 3 & -3 & 0 & 1 & 0 \\
-1 & -2 & 1 & 0 & -6 & -1 & 0 & 1 & 1 & 2 & -3 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 12 & 0 & -4 & -3 & 2 & 0 & 2 & 3 & -1 & 0 & 0 \\
1 & 2 & -1 & 0 & -6 & -1 & 4 & 1 & -1 & -2 & 1 & 0 & -1 & 1 & 0 \\
2 & -1 & -2 & -1 & -3 & -2 & 1 & -1 & 2 & 1 & 0 & 0 & 1 & 0 & -3 \\
1 & 3 & 1 & -1 & 3 & 1 & -1 & 0 & -1 & -1 & 1 & 0 & 0 & -1 & 3 \\
-1 & -2 & 1 & 0 & -6 & -1 & 4 & 1 & 1 & -2 & -1 & 0 & -1 & 1 & 0 \\
-1 & -1 & -1 & 1 & 3 & 1 & -3 & 0 & -1 & 3 & -1 & 0 & 0 & 1 & -3 \\
-1 & -1 & -1 & 3 & 3 & 3 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & 3 \\
0 & -1 & 0 & -1 & -3 & -2 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & 2 & -3 \\
-1 & 2 & -1 & 0 & 6 & 3 & -4 & 0 & -1 & 2 & -1 & -3 & 2 & -1 & 0 \\
0 & -1 & 0 & 1 & -3 & -2 & 3 & -1 & 2 & -3 & 2 & 0 & 1 & -2 & 3 \\
1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1
\end{array}\right).$$

The Fischer-Clifford matrices $M(3A)$ and $M(3B)$ are equal as follows.

$$M(3A) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & -1 & 0 & -1 & 1 \\ 2 & 2 & -1 & 0 & 0 \\ 3 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 \end{pmatrix} = M(3B).$$

The character table of $(S_4)^3 : C_3$ is obtained in the same way by multiplying the partial character tables of the inertia factor groups C_1 and C_3 with the corresponding rows of the above Fischer-Clifford matrices according to the fusions. The character table of $(S_4)^3 : C_3$ is given in Tables 4.5, 4.6, 4.7, 4.8.

Table 4.5: Character Table of $S_4^3 : C_3$

$[g]$	1A														
$[\overline{g}]$	1a	2a	2b	3a	4a	2c	2d	6a	4b	2e	2f	6b	4c	6c	6d
$\lvert C_{\overline{G}}(\overline{g})\rvert$	41472	4608	1536	1536	1728	576	576	192	216	72	81	2304	2304	768	768
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_4	9	7	5	6	5	5	3	4	3	3	1	2	1	4	2
χ_5	27	15	3	9	3	7	-1	3	-1	-1	-5	-3	-5	3	-3
χ_6	27	9	-9	0	-9	3	-3	0	-3	-3	3	0	3	0	0
χ_7	27	9	-9	0	-9	3	-3	0	-3	-3	3	0	3	0	0
χ_8	27	9	-9	0	-9	3	-3	0	-3	-3	3	0	3	0	0
χ_9	18	8	10	3	4	2	0	-1	-2	8	2	7	4	-1	-5
χ_{10}	6	4	6	3	4	2	4	1	2	4	6	3	4	1	3
χ_{11}	18	8	10	3	4	2	8	-1	2	0	2	-5	-4	-1	7
χ_{12}	54	12	6	-9	-12	2	4	-3	-2	4	-10	3	-4	-3	3
χ_{13}	12	4	12	0	4	0	4	-2	0	4	12	0	4	-2	0
χ_{14}	36	4	20	-12	-4	0	4	-2	0	4	4	-4	-4	-2	-4
χ_{15}	8	0	8	-4	0	0	0	0	0	0	8	-4	0	0	-4
χ_{16}	8	0	8	-4	0	0	0	0	0	0	8	-4	0	0	-4
χ_{17}	8	0	8	-4	0	0	0	0	0	0	8	-4	0	0	-4
χ_{18}	9	5	5	6	7	1	1	2	3	1	1	2	3	2	2
χ_{19}	27	9	3	9	9	-1	-7	-3	-5	1	-5	-3	-7	3	-3
χ_{20}	18	4	10	3	8	-2	4	-5	-2	-4	2	-5	0	1	7
χ_{21}	27	9	3	9	9	-1	1	3	7	-7	-5	-3	1	-3	-3
χ_{22}	81	9	-27	0	-9	-3	-3	0	3	-3	9	0	3	0	0
χ_{23}	54	0	6	-9	0	-2	8	-3	2	-8	-10	3	8	3	3
χ_{24}	18	4	10	3	8	-2	-4	1	2	4	2	7	8	-5	-5
χ_{25}	54	0	6	-9	0	-2	-8	3	2	8	-10	3	-8	-3	3
χ_{26}	36	-4	20	-12	4	0	-4	2	0	-4	4	-4	4	2	-4
χ_{27}	27	3	3	9	15	-5	-5	-3	-1	-5	-5	-3	-1	-3	-3
χ_{28}	81	-9	-27	0	9	-3	3	0	3	3	9	0	-3	0	0
χ_{29}	54	-12	6	-9	12	2	-4	3	-2	-4	-10	3	4	3	3
χ_{30}	27	-9	-9	0	9	3	3	0	-3	3	3	0	-3	0	0
χ_{31}	27	-9	-9	0	9	3	3	0	-3	3	3	0	-3	0	0
χ_{32}	27	-9	-9	0	9	3	3	0	-3	3	3	0	-3	0	0
χ_{33}	3	1	3	3	1	-1	1	1	-1	1	3	3	1	1	3
χ_{34}	9	1	5	6	-1	-3	-3	-2	-5	5	1	2	3	4	2
χ_{35}	6	0	6	3	0	-2	0	-3	-2	0	6	3	0	3	3
χ_{36}	9	-1	5	6	1	-3	-5	-4	-1	3	1	2	5	2	2
χ_{37}	9	1	5	6	-1	-3	5	4	-1	-3	1	2	-5	-2	2
χ_{38}	27	-3	3	9	-15	-5	5	3	-1	5	-5	-3	1	3	-3
χ_{39}	18	-4	10	3	-8	-2	4	-1	2	-4	2	7	-8	5	-5
χ_{40}	27	-9	3	9	-9	-1	-1	-3	7	7	-5	-3	-1	3	-3
χ_{41}	6	0	6	3	0	-2	0	3	-2	0	6	3	0	-3	3
χ_{42}	18	-4	10	3	-8	-2	-4	5	-2	4	2	-5	0	-1	7
χ_{43}	12	-4	12	0	-4	0	-4	2	0	-4	12	0	-4	2	0
χ_{44}	18	-8	10	3	-4	2	-8	1	2	0	2	-5	4	1	7
χ_{45}	9	-1	5	6	1	-3	3	2	-5	-5	1	2	-3	-4	2
χ_{46}	27	-9	3	9	-9	-1	7	3	-5	-1	-5	-3	7	-3	-3
χ_{47}	18	-8	10	3	-4	2	0	1	-2	-8	2	7	-4	1	-5
χ_{48}	27	-15	3	9	-3	7	1	-3	-1	1	-5	-3	5	-3	-3
χ_{49}	3	-1	3	3	-1	-1	-1	-1	-1	-1	3	3	-1	-1	3
χ_{50}	9	-5	5	6	-7	1	-1	-2	3	-1	1	2	-3	-2	2
χ_{51}	6	-4	6	3	-4	2	-4	-1	2	-4	6	3	-4	-1	3
χ_{52}	9	-7	5	6	-5	5	-3	-4	3	-3	1	2	-1	-4	2
χ_{53}	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	-1	1
χ_{54}	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	-1	1
χ_{55}	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	-1	1

Table 4.6: Character Table of $S_4^3 : C_3$ contd.

$[g]$															
$[\overline{g}]$	3b	12a	4d	4e	12b	4f	2g	2h	6e	4g	2i	6f	4h	6g	6h
$\lvert C_{\overline{G}}(\overline{g})\rvert$	768	768	256	256	288	288	96	96	288	288	96	96	36	36	384
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_4	3	2	3	1	2	1	3	1	2	1	-1	0	-1	0	1
χ_5	0	-3	-1	-5	-3	-5	3	-1	1	-1	-1	-1	-1	-1	0
χ_6	0	0	-3	3	0	3	1	-1	0	-1	1	0	1	0	0
χ_7	0	0	-3	3	0	3	1	-1	0	-1	1	0	1	0	0
χ_8	0	0	-3	3	0	3	1	-1	0	-1	1	0	1	0	0
χ_9	-3	-5	2	-4	1	-2	0	2	-1	0	0	3	2	-1	-1
χ_{10}	0	1	2	4	1	2	0	2	-1	0	4	1	2	1	-2
χ_{11}	-3	1	-2	4	-5	-2	0	2	-1	0	0	-1	-2	3	-1
χ_{12}	0	3	-2	-4	3	2	0	2	-1	0	-4	1	-2	1	0
χ_{13}	-3	-2	0	4	-2	0	0	0	0	0	4	-2	0	-2	1
χ_{14}	3	2	0	-4	2	0	0	0	0	0	4	-2	0	-2	1
χ_{15}	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{16}	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{17}	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{18}	3	4	3	3	4	5	-3	-3	-2	-1	-3	-2	-1	-2	-1
χ_{19}	0	-3	7	1	3	-1	-3	-1	-1	1	1	-1	-1	1	0
χ_{20}	-3	-1	2	8	-1	2	0	-2	1	0	-4	1	2	-1	1
χ_{21}	0	3	-5	-7	-3	-1	-3	-1	-1	1	1	1	3	-1	0
χ_{22}	0	0	3	3	0	-3	-3	1	0	3	1	0	-1	0	0
χ_{23}	0	-3	2	-8	3	-2	0	-2	1	0	0	-1	2	1	0
χ_{24}	-3	-1	-2	0	-1	2	0	-2	1	0	-4	-1	-2	1	1
χ_{25}	0	3	2	8	-3	-2	0	-2	1	0	0	1	2	-1	0
χ_{26}	3	-2	0	4	-2	0	0	0	0	0	-4	2	0	2	-1
χ_{27}	0	3	-1	-1	3	7	3	3	1	-1	3	1	-1	1	0
χ_{28}	0	0	3	-3	0	-3	3	1	0	-3	-1	0	-1	0	0
χ_{29}	0	-3	-2	4	-3	2	0	2	-1	0	4	-1	-2	-1	0
χ_{30}	0	0	-3	-3	0	3	-1	-1	0	1	-1	0	1	0	0
χ_{31}	0	0	-3	-3	0	3	-1	-1	0	1	-1	0	1	0	0
χ_{32}	0	0	-3	-3	0	3	-1	-1	0	1	-1	0	1	0	0
χ_{33}	3	1	-1	1	1	-1	-3	-1	-1	-3	1	1	-1	1	1
χ_{34}	3	2	-1	-5	-4	-3	-3	1	0	-1	1	0	3	2	1
χ_{35}	0	3	-2	0	-3	-2	0	-2	1	0	0	3	-2	-3	0
χ_{36}	3	4	-5	-3	-2	-3	3	1	0	1	-1	-2	-1	0	-1
χ_{37}	3	-4	-5	3	2	-3	-3	1	0	-1	1	2	-1	0	1
χ_{38}	0	-3	-1	1	-3	7	-3	3	1	1	-3	-1	-1	-1	0
χ_{39}	-3	1	-2	0	1	2	0	-2	1	0	4	1	-2	-1	-1
χ_{40}	0	-3	-5	7	3	-1	3	-1	-1	-1	-1	-1	3	1	0
χ_{41}	0	-3	-2	0	3	-2	0	-2	1	0	0	-3	-2	3	0
χ_{42}	-3	1	2	-8	1	2	0	-2	1	0	4	-1	2	1	-1
χ_{43}	-3	2	0	-4	2	0	0	0	0	0	-4	2	0	2	-1
χ_{44}	-3	-1	-2	-4	5	-2	0	2	-1	0	0	1	-2	-3	1
χ_{45}	3	-2	-1	5	4	-3	3	1	0	1	-1	0	3	-2	-1
χ_{46}	0	3	7	-1	-3	-1	3	-1	-1	-1	-1	1	-1	-1	0
χ_{47}	-3	5	2	4	-1	-2	0	2	-1	0	0	-3	2	1	1
χ_{48}	0	3	-1	5	3	-5	-3	-1	1	1	1	1	-1	1	0
χ_{49}	3	-1	-1	-1	-1	-1	3	-1	-1	3	-1	-1	-1	-1	-1
χ_{50}	3	-4	3	-3	-4	5	3	-3	-2	1	3	2	-1	2	1
χ_{51}	0	-1	2	-4	-1	2	0	2	-1	0	-4	-1	2	-1	2
χ_{52}	3	-2	3	-1	-2	1	-3	1	2	-1	1	0	-1	0	-1
χ_{53}	1	-1	1	-1	-1	1	-1	1	1	-1	-1	-1	1	-1	-1
χ_{54}	1	-1	1	-1	-1	1	-1	1	1	-1	-1	-1	1	-1	-1
χ_{55}	1	-1	1	-1	-1	1	-1	1	1	-1	-1	-1	1	-1	-1

Table 4.7: Character Table of $S_4^3 : C_3$ contd.

$[g]$															
$[\overline{g}]$	12c	4i	12d	4j	2j	6i	4k	6j	12e	4l	12f	3c	12g	12h	4m
$\lvert C_{\overline{G}}(\overline{g})\rvert$	384	384	384	128	128	128	128	48	48	48	48	192	64	64	192
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_4	0	-1	0	-1	-3	-2	-3	-1	-2	-3	-2	0	-1	-2	-3
χ_5	-1	-1	-1	-1	3	1	3	0	1	3	1	0	0	1	3
χ_6	0	1	0	1	-1	0	-1	0	0	-1	0	0	0	0	-1
χ_7	0	1	0	1	-1	0	-1	0	0	-1	0	0	0	0	-1
χ_8	0	1	0	1	-1	0	-1	0	0	-1	0	0	0	0	-1
χ_9	-1	-2	1	0	-6	-1	-4	1	-1	-2	1	0	1	1	0
χ_{10}	-1	2	-1	0	6	3	4	0	1	2	1	-3	-2	-1	0
χ_{11}	1	2	-1	0	-6	-1	-4	1	1	-2	-1	0	1	1	0
χ_{12}	1	-2	1	0	6	-1	4	0	-1	2	-1	0	0	-1	0
χ_{13}	0	0	0	0	12	0	4	-3	-2	0	-2	3	1	0	0
χ_{14}	0	0	0	0	-12	4	-4	-1	2	0	2	0	-1	0	0
χ_{15}	0	0	0	0	8	-4	0	2	0	0	0	-1	0	0	0
χ_{16}	0	0	0	0	8	-4	0	2	0	0	0	-1	0	0	0
χ_{17}	0	0	0	0	8	-4	0	2	0	0	0	-1	0	0	0
χ_{18}	0	-1	0	1	-3	-2	-1	-1	0	1	0	0	1	2	3
χ_{19}	1	3	1	1	3	1	1	0	1	-1	-1	0	0	-1	-3
χ_{20}	-1	-2	1	0	-6	-1	0	1	-1	2	3	0	-1	-1	0
χ_{21}	1	-1	1	1	3	1	1	0	-1	-1	1	0	0	-1	-3
χ_{22}	0	-1	0	-3	-3	0	-1	0	0	1	0	0	0	0	3
χ_{23}	-1	2	-1	0	6	-1	0	0	1	-2	-1	0	0	1	0
χ_{24}	1	2	-1	0	-6	-1	0	1	3	2	-1	0	-1	-1	0
χ_{25}	-1	2	-1	0	6	-1	0	0	-1	-2	1	0	0	1	0
χ_{26}	0	0	0	0	-12	4	4	-1	-2	0	-2	0	1	0	0
χ_{27}	-1	-1	-1	-1	3	1	-1	0	-1	-1	-1	0	0	1	3
χ_{28}	0	-1	0	3	-3	0	1	0	0	1	0	0	0	0	-3
χ_{29}	1	-2	1	0	6	-1	-4	0	1	2	1	0	0	-1	0
χ_{30}	0	1	0	-1	-1	0	1	0	0	-1	0	0	0	0	1
χ_{31}	0	1	0	-1	-1	0	1	0	0	-1	0	0	0	0	1
χ_{32}	0	1	0	-1	-1	0	1	0	0	-1	0	0	0	0	1
χ_{33}	-1	-1	-1	-3	3	3	1	3	1	-1	1	3	1	-1	-3
χ_{34}	2	-1	-2	1	-3	-2	-1	-1	-2	1	0	0	-1	0	3
χ_{35}	1	-2	1	0	6	3	0	0	3	-2	-3	-3	0	1	0
χ_{36}	-2	3	2	-1	-3	-2	1	-1	0	1	2	0	1	0	-3
χ_{37}	-2	3	2	1	-3	-2	-1	-1	0	1	-2	0	-1	0	3
χ_{38}	-1	-1	-1	1	3	1	1	0	1	-1	1	0	0	1	-3
χ_{39}	1	2	-1	0	-6	-1	0	1	-3	2	1	0	1	-1	0
χ_{40}	1	-1	1	-1	3	1	-1	0	1	-1	-1	0	0	-1	3
χ_{41}	1	-2	1	0	6	3	0	0	-3	-2	3	-3	0	1	0
χ_{42}	-1	-2	1	0	-6	-1	0	1	1	2	-3	0	1	-1	0
χ_{43}	0	0	0	0	12	0	-4	-3	2	0	2	3	-1	0	0
χ_{44}	1	2	-1	0	-6	-1	4	1	-1	-2	1	0	-1	1	0
χ_{45}	2	-1	-2	-1	-3	-2	1	-1	2	1	0	0	1	0	-3
χ_{46}	1	3	1	-1	3	1	-1	0	-1	-1	1	0	0	-1	3
χ_{47}	-1	-2	1	0	-6	-1	4	1	1	-2	-1	0	-1	1	0
χ_{48}	-1	-1	-1	1	3	1	-3	0	-1	3	-1	0	0	1	-3
χ_{49}	-1	-1	-1	3	3	3	-1	3	-1	-1	-1	3	-1	-1	3
χ_{50}	0	-1	0	-1	-3	-2	1	-1	0	1	0	0	-1	2	-3
χ_{51}	-1	2	-1	0	6	3	-4	0	-1	2	-1	-3	2	-1	0
χ_{52}	0	-1	0	1	-3	-2	3	-1	2	-3	2	0	1	-2	3
χ_{53}	1	1	1	-1	1	1	-1	1	-1	1	-1	1	-1	1	-1
χ_{54}	1	1	1	-1	1	1	-1	1	-1	1	-1	1	-1	1	-1
χ_{55}	1	1	1	-1	1	1	-1	1	-1	1	-1	1	-1	1	-1

Table 4.8: Character Table of $S_4^3 : C_3$ contd.

$[g]$	3A					3B				
$[\overline{g}]$	3d	6k	9a	12i	6l	3e	6m	9b	12j	6n
$\lvert C_{\overline{G}}(\overline{g})\rvert$	72	12	24	9	12	72	12	24	9	12
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	ξ	ξ	ξ	ξ	ξ	ξ^2	ξ^2	ξ^2	ξ^2	ξ^2
χ_3	ξ^2	ξ^2	ξ^2	ξ^2	ξ^2	ξ	ξ	ξ	ξ	ξ
χ_4	0	0	0	0	0	0	0	0	0	0
χ_5	0	0	0	0	0	0	0	0	0	0
χ_6	3	-1	0	-1	1	3	-1	0	-1	1
χ_7	$3*\xi$	$-\xi$	0	$-\xi$	ξ	$3*\xi^2$	$-\xi^2$	0	$-\xi^2$	ξ^2
χ_8	$3*\xi^2$	$-\xi^2$	0	$-\xi^2$	ξ^2	$3*\xi$	$-\xi$	0	$-\xi$	ξ
χ_9	0	0	0	0	0	0	0	0	0	0
χ_{10}	0	0	0	0	0	0	0	0	0	0
χ_{11}	0	0	0	0	0	0	0	0	0	0
χ_{12}	0	0	0	0	0	0	0	0	0	0
χ_{13}	0	0	0	0	0	0	0	0	0	0
χ_{14}	0	0	0	0	0	0	0	0	0	0
χ_{15}	2	2	-1	0	0	2	2	-1	0	0
χ_{16}	$2*\xi$	$2*\xi$	$-\xi$	0	0	$2*\xi^2$	$2*\xi^2$	$-\xi^2$	0	0
χ_{17}	$2*\xi^2$	$2*\xi^2$	$-\xi^2$	0	0	$2*\xi$	$2*\xi$	$-\xi$	0	0
χ_{18}	0	0	0	0	0	0	0	0	0	0
χ_{19}	0	0	0	0	0	0	0	0	0	0
χ_{20}	0	0	0	0	0	0	0	0	0	0
χ_{21}	0	0	0	0	0	0	0	0	0	0
χ_{22}	0	0	0	0	0	0	0	0	0	0
χ_{23}	0	0	0	0	0	0	0	0	0	0
χ_{24}	0	0	0	0	0	0	0	0	0	0
χ_{25}	0	0	0	0	0	0	0	0	0	0
χ_{26}	0	0	0	0	0	0	0	0	0	0
χ_{27}	0	0	0	0	0	0	0	0	0	0
χ_{28}	0	0	0	0	0	0	0	0	0	0
χ_{29}	0	0	0	0	0	0	0	0	0	0
χ_{30}	3	-1	0	1	-1	3	-1	0	1	-1
χ_{31}	$3*\xi$	$-\xi$	0	ξ	$-\xi$	$3*\xi^2$	$-\xi^2$	0	ξ^2	$-\xi^2$
χ_{32}	$3*\xi^2$	$-\xi^2$	0	ξ^2	$-\xi^2$	$3*\xi$	$-\xi$	0	ξ	$-\xi$
χ_{33}	0	0	0	0	0	0	0	0	0	0
χ_{34}	0	0	0	0	0	0	0	0	0	0
χ_{35}	0	0	0	0	0	0	0	0	0	0
χ_{36}	0	0	0	0	0	0	0	0	0	0
χ_{37}	0	0	0	0	0	0	0	0	0	0
χ_{38}	0	0	0	0	0	0	0	0	0	0
χ_{39}	0	0	0	0	0	0	0	0	0	0
χ_{40}	0	0	0	0	0	0	0	0	0	0
χ_{41}	0	0	0	0	0	0	0	0	0	0
χ_{42}	0	0	0	0	0	0	0	0	0	0
χ_{43}	0	0	0	0	0	0	0	0	0	0
χ_{44}	0	0	0	0	0	0	0	0	0	0
χ_{45}	0	0	0	0	0	0	0	0	0	0
χ_{46}	0	0	0	0	0	0	0	0	0	0
χ_{47}	0	0	0	0	0	0	0	0	0	0
χ_{48}	0	0	0	0	0	0	0	0	0	0
χ_{49}	0	0	0	0	0	0	0	0	0	0
χ_{50}	0	0	0	0	0	0	0	0	0	0
χ_{51}	0	0	0	0	0	0	0	0	0	0
χ_{52}	0	0	0	0	0	0	0	0	0	0
χ_{53}	1	1	1	-1	-1	1	1	1	-1	-1
χ_{54}	ξ	ξ	ξ	$-\xi$	$-\xi$	ξ^2	ξ^2	ξ^2	$-\xi^2$	$-\xi^2$
χ_{55}	ξ^2	ξ^2	ξ^2	$-\xi^2$	$-\xi^2$	ξ	ξ	ξ	$-\xi$	$-\xi$

$$\xi^3 = 1.$$

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Index