African Perspectives of Research in Teaching and Learning Journal
Issue 8, Volume 9, 2025
ISSN Print: 2521-0262
ISSN Online: 2662-012X
ORIGINAL RESEARCH
https://doi.org/10.70875/v9i8article12

Using diagnostic assessment to determine the goal of the research lesson: A Lesson Study case

Mahlatse Radingwane, David Sekao, Mandlenkosi Richard Sibiya, Koketso Clinton Moremi

Department of Science, Mathematics and Technology Education, University of Pretoria, South Africa

***Corresponding author, email**: koketso.moremi@up.ac.za*

Abstract

Two-tier diagnostic assessment, unlike single-tier diagnostic assessment, is credited for affording teachers a deep understanding of learners' misconceptions in mathematics, thereby improving teaching and learning. The purpose of this interpretivist case-study paper was to identify grade 6 learners' misconceptions of fractions, using a two-tier diagnostic assessment, thereby determining the goal of the research lesson within the Lesson Study (LS) context. Data was collected via administration of a two-tier diagnostic test. Although some procedural and factual misconceptions were revealed, conceptual misconceptions appeared to be most dominant in the learning of fractions. However, the interplay between the three categories of misconceptions seemed to be inevitable. Therefore, we conclude and recommend that, instead of focusing on only one of the three categories of misconceptions related to fractions, the goal of the research lesson within the LS setting should encompass all three categories of misconceptions and address them in an integrated manner.

Keywords: Fractions, Lesson Study, misconceptions, two-tier diagnostic assessment, research lesson

Introduction

The concept of fractions is not only important in everyday life but also in accessing other mathematical topics, including percentages, proportions, and rates. Notwithstanding its importance, fractions are a universal phenomenon that learners often struggle with. Grasping the concept of fractions often leads to misconceptions and difficulties in problem-solving (Aliustaoğlu et al., 2018; Alkhateeb, 2019; Ingram et al., 2013).

Mohyuddin and Khalil (2016) highlighted the importance of noting the difference between an 'error' and 'misconceptions' in mathematics, since both words are regarded as equivalent due to their tendency to provide incorrect results. Errors are mistakes that may occur due to incompetence (Kshetree et al., 2021) or human fallibility (Owusu et al., 2023), but the learner can easily correct them once they are made aware of how to handle them (Ndemo & Ndemo, 2023). On the other hand, misconceptions, however, are cognitively entrenched in learners' incorrect interpretations and understanding of mathematical concepts (Aliustaoğlu *et al*., 2018). They are, in fact, a lack of understanding, or misapplications or generalisations of mathematical rules (Mohyuddin & Khalil, 2016). Misconceptions may be categorised as being conceptual, procedural, or factual (Lai, 2012). Although researchers are often not explicit about what type of misconceptions need to be prioritised (Manandhar et al., 2022), it often proves difficult to deal with one type of misconception in isolation, because often, there is an interplay between the three categories of misconceptions (Kilpatrick et al., 2001; Nahdi & Jatisunda, 2020). Notwithstanding, Moru and Mathunya (2022) asserted that misconceptions arise during the process of accommodation—a key construct of the constructivist view of learning, where knowledge is refined and reorganised in learners' respective schemas. In fact, Sehole et al. (2023, p. 83) expressed this view succinctly, thus… "the newly acquired knowledge can erroneously be connected with and fit into the discordant already-existing knowledge, thereby creating a misconception."

Consequently, misconceptions must be recognised as an essential part of the learning process. Simply put, if there is no connection between mathematical concepts, then conceptual knowledge will be compromised, thus creating fertile ground for misconceptions to develop (Hurrell, 2021).

Although numerous factors may contribute to learners' difficulties with fractions, teachers' lack of knowledge, awareness, and understanding of these difficulties is likely to perpetuate their prevalence (Mabena et al., 2021). In fact, the primary problem driving the current study is the lack of guidance on identifying learners' difficulties with fractions within the LS setting. Plugging this gap of identifying learners' struggles and hence misconceptions is achieved by using a two-tier diagnostic assessment within the LS setting. Identifying learners' misconceptions in fractions empowers teachers to make instructional decisions in response to learners' mathematical needs (Ketterlin-Geller & Yovanoff, 2009). Two-tier diagnostic assessment, unlike single-tier diagnostic assessment, has been credited with providing teachers with a deep understanding of learners' misconceptions and can therefore improve teaching and learning. Typically, a two-tier diagnostic assessment requires learners to provide an answer to a question and then justify their responses. Wu et al. (2025) stated that analysing learners' responses stemming from a two-tier diagnostic test affords teachers an opportunity to discern learners' conceptual understanding and fundamental misconceptions.

Essentially, teacher collaboration, a fundamental attribute of LS (Richit et al., 2024), provides opportunities for developing and enhancing teachers' mathematics knowledge and pedagogical skills (Alsaeed, 2022). Therefore, when teachers work collaboratively to design and administer diagnostic assessments and subsequently analyse learners' responses, they gain insights into multiple perspectives on learners' misconceptions. Multiple perspectives, when teachers work collaboratively, enhance teachers' knowledge and understanding of learners' misconceptions, unlike a single perspective, which is often achieved when a teacher works alone. Situating the current study within the LS setting – a teacher-led and practice-embedded model of teacher professional development (Sekao & Engelbrecht, 2022) – provided affordances for multiple perspectives regarding learners' misconceptions of fractions.

The purpose of this paper is therefore to determine the goal of the research lesson within the LS context by identifying grade 6 learners' misconceptions of fractions through a two-tier diagnostic test. A research lesson is a unique and detailed lesson plan within the LS context (Fujii, 2016). The research lesson is unique in two ways, namely (1) a group of teachers collaboratively plan it, based on their findings from a diagnostic assessment/analysis, and (2) the goal of the research lesson is based on evidence and aims to address learners' mathematical needs (Sekao, 2023).

Literature review

The importance of and misconceptions about fractions

Fractions are commonly defined as part of a whole (Moyo & Machaba, 2021). However, fractions can assume different meanings, depending on the context in which they are used (Van de Walle et al., 2016). For instance, the most elementary meaning of a fraction is part of a whole, while in other contexts, a fraction can be viewed as ratio to relate same quantities (Yıldızhan & Ertekin, 2022). In the South African education system, fractions are introduced in the second grade of schooling and are considered fundamental for learners' progress in other mathematical areas (Department of Basic Education, 2011). Researchers assert that understanding fractions is essential for understanding other mathematics content areas such as algebra and probability (Aliustaoğlu et al., 2018; Alkhateeb, 2019; Hansen et al., 2017; Siegler et al., 2012; Reinhold et al., 2020).

Notwithstanding this, it appears that a significant number of learners have misconceptions about fractions (Braithwaite & Siegler, 2021). Learners' misconceptions related to the concept of fractions can be attributed to, *inter alia*, whole number bias, teachers' pedagogical deficiencies, and learners' inherent misconceptions, such as where they perceive a fraction as two whole numbers rather than one number (Braithwaite & Siegler, 2021; Moyo & Machaba, 2021). In other words, learners often handle fractions the same way they handle whole numbers. For instance, when learners are required to add or subtract fractions, they are often fixated on using whole number procedures (i.e. add or subtract the numerator with the numerator and denominator with denominator) to solve that problem. Mtumtum (2020) classifies this as whole number bias, which is a result of functional fixedness. The concept of functional fixedness is traced back to Maslow's hammer (Maslow, 1966, p. 15) metaphor, which states that: "… it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail". Mohyuddin and Khalil (2016) and Vitoria et al. (2017) add that this difficulty is compounded by teachers' inability to effectively teach fractions in a way that fosters understanding and minimises misconceptions.

Misconceptions are unavoidable steps in the acquisition of knowledge (Moru & Mathunya, 2022). In education, misconceptions should not be viewed merely as errors to be corrected, but rather treated as open windows to guide the teaching and learning process. In fact, errors and misconceptions should be viewed as opportunities to teach and learn effectively (Boser, 2024). Ingram et al. (2013), Gardee and Brodie (2022), and Boser (2024) posited that misconceptions provide teachers with valuable information about learners' mathematical reasoning, help identify areas of difficulty and assist in designing targeted instructional methods.

In many instances, learners often reveal erroneous conceptions of fractions as the teaching and learning process unfolds, prompting teachers to seize the moment and capitalise on pivotal teaching moments to address these misconceptions. Pivotal teaching moments (PTMs) in this case refer to moments that happen during teaching that disrupt the flow of the lesson, but which afford the teacher an opportunity to either change or modify how learners think mathematically (Stockero & Van Zoest, 2013). There are several documented misconceptions learners present regarding fractions. Some common misconceptions about fractions are discussed in the following paragraphs.

Difficulty transitioning from whole numbers to rational numbers

Several scholars stated that a common error that learners often make is solving fractions using whole-number procedures. Moyo and Machaba (2021) argued that the root of learners' misconceptions of fractions stems from their inability to transition from whole numbers to rational numbers. Although Reinhold et al. (2020) described this inability as natural number bias (NNB), the same principle is applicable to whole number bias (WNB) as described by Mtumtum (2020). Both the NNB and WNB originate from learners' inability to master conceptual change from whole numbers to fractions. For instance, when given $\frac{4}{8}+\frac{1}{4}$ learners tend to add 4+1 to obtain 5 and 8+4 to obtain 12, resulting in $\frac{5}{12}$. Simon et al. (2018) suggested that this could be because fractions are taught after learners have been introduced to the addition and subtraction of whole numbers. In other words, learners' knowledge of whole numbers could therefore influence their interpretation and understanding of rational numbers, especially if they are not taught well. In that case, learners are seen treating fractions as they would treat whole numbers (Alkhateeb, 2019; Moyo & Machaba, 2021; Vitoria et al., 2017), leading to erroneous responses.

Lack of attention to equivalence

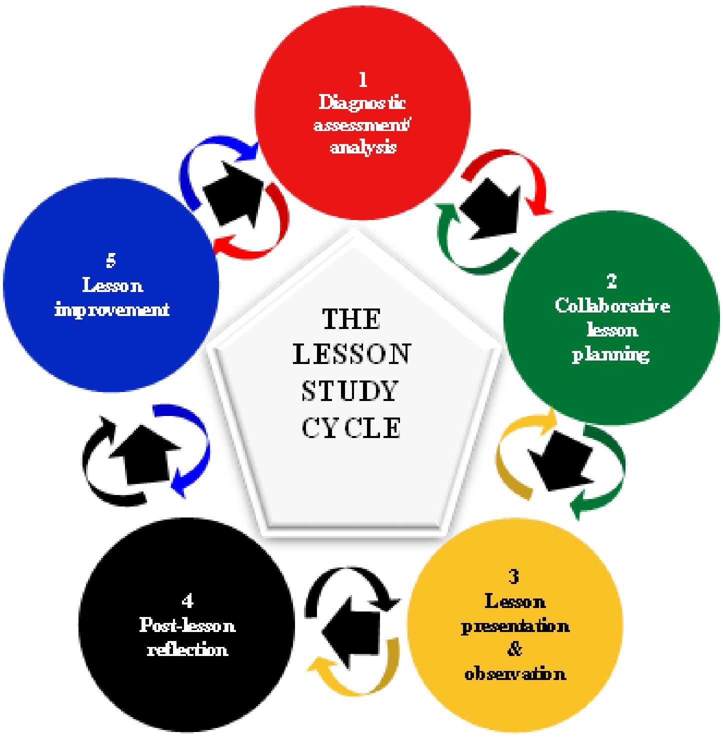
To add or subtract fractions, the first step is to make the denominators the same to solve the problem. However, many learners demonstrate minimal skill in this regard. Many learners are not able to recognise the importance of establishing equivalent fractions and understanding the relationship between fractions with different denominators (Alkhateeb, 2019; Moyo & Machaba, 2021; Ojose, 2015; Vitoria et al., 2017). This means that learners find it challenging to work with fractions with unlike denominators. For example, instead of recognising $\frac{4}{8}$ as equivalent to $\frac{1}{2}$ when adding $\frac{1}{2}+\frac{3}{8}$, most learners tend to add numerators and denominators separately, leading to an incorrect $\frac{4}{10}$.

Using Lesson Study to collaboratively identify and address learners' misconceptions

LS (*jugyokenkyu* or 授業研究 in Japanese), originally developed and practised in Japan since 1872 (Makinae, 2010), involves collaboration among teachers who jointly research, plan, and deliver a research lesson aimed at improving teaching and learning (Elliott, 2019; Lewis et al., 2009; Sekao, 2023; Takahashi & McDougal, 2018). LS gained traction in various countries due to Japan's consistently high performance in TIMSS (Fujii, 2019).

Recognising its success, South Africa, like many other countries globally (Fluminhan et al., 2024; Stigler & Hiebert, 2016), adopted and adapted the idea to suit their local educational needs. The adapted version of LS predominantly used in South African mathematics classrooms comprises five stages (Figure 1), namely: diagnostic assessment/analysis, collaborative lesson planning, lesson presentation and observation, post-lesson reflection, and lesson improvement (Sekao & Engelbrecht, 2022). Although the study primarily focuses on identifying learners' misconceptions, which is an attribute of stage 1 of the LS cycle used in South Africa, we have shed some light on the other four stages to demonstrate how the first stage informs the entire cycle.

Figure 1: Lesson Study cycle (Sekao, 2023)



The first stage – Diagnostic assessment/analysis – is a stage that gives the South African LS model its unique identity (Sekao, 2023). This unique stage of LS in South Africa was necessitated by the lack of explicit guidelines on how to identify the research lesson's goal. Several LS models across the globe emphasised firstly determining the goal of the research lesson but were silent on how this could be achieved. To this effect, this stage often involves administering a diagnostic test aimed at uncovering learners' misconceptions to inform the goal of the research lesson. In the context of this study, we developed and utilised a two-tier diagnostic test that focused on the addition and subtraction of fractions to gain insight into the misconceptions that learners display.

In the second stage – Collaborative lesson planning – teachers collaboratively plan a research lesson to address the identified misconceptions in the first stage (Sekao, 2022). A thorough study of curriculum materials, also referred to as *kyozaikenkyu* in Japanese (Fujii, 2015), is typically undertaken to gain a deeper understanding of concepts related to learners' misconceptions (Vetter, 2022). 'Curriculum materials' is a collective term encompassing different teaching and learning support materials. However, the most prominent curriculum material within the LS context is a textbook.

In the third stage – Lesson presentation and observation – one teacher from the LS team presents the collaboratively planned research lesson while other teachers observe and document their observations (Fujii, 2015). The primary purpose of this stage is for the LS team to implement their collective ideas in the classroom in real-time, focusing on and collecting data about learners' mathematical thinking (Huang & Shimizu, 2016) – hence the term "research lesson." It is at this stage that the knowledgeable other (*koshi*, in Japanese) is invited to observe the lesson to make meaningful contributions during post-lesson reflection (Foster & Trundley, 2024).

Due to the dynamic and unpredictable nature of classroom activities, unanticipated events are likely to occur; therefore, teacher *contingency* becomes inevitable. Contingency refers to the teachers' ability to think and make on-the-spot decisions in response to learners' classroom contributions (Rowland et al., 2005). Through teacher contingency, when unanticipated events occur, the teacher offering the lesson ought to be able to recognise and exploit a PTM (Stockero & Van Zoest, 2013). However, Stockero and Van Zoest (2013) indicated that an event can constitute a PTM even if the teacher offering the lesson does not recognise it as such. This is why the observing teachers are crucial in being part of the lesson presentation as they are at a vantage point to notice more goings-on than the teacher offering the lesson (Sekao, 2023). The missed PTMs can then be addressed later during the post-lesson reflection.

In the fourth stage – Post-lesson reflection – teachers collectively reflect on the presented lesson in the presence of an expert, a knowledgeable other, or a '*koshi*' (Sekao, 2023). The purpose of the *koshi* is to provide final inputs from a neutral and expert perspective. In fact, Foster and Trundley (2024) argued that the participation of a *koshi* is non-negotiable if LS is to be practised in its truest sense. Central to the post-lesson reflection is whether the lesson objectives were achieved (or not), as well as the possible reasons thereof (Shimizu & Kang, 2022). In fact, reflection tends to gravitate more towards learners' mathematical thinking.

In the fifth stage – Lesson improvement – the insights and ideas collected in the fourth stage are used to improve the lesson (Botes et al., 2022; García et al., 2024). A report is generated and can be shared with other teachers and other interested colleagues (Dudley, 2014). Scholars such as Robinson and Leikin (2011) and Kanellopoulou and Darra (2018) believe it is up to the discretion of the LS team to reteach the lesson to test the effectiveness of their refined lesson. Conversely, Fujii (2014, p.12) argued that re-teaching a lesson is "disrespectful of students' [learners'] right to the best education one can provide them." Fujii (2016) stated that re-teaching a lesson is a misconception that foreign countries developed when they implemented the Japanese LS in their native lands.

Two-tier diagnostic assessment and its relevance to this study

The two-tier diagnostic assessment gained traction in the educational setting over several years (Tasci, 2024) due to the limitations of single-tier diagnostic assessments (Verma & Choudhuri, 2025). The first tier typically consists of a multiple-choice question (MCQ) that assesses learners' factual knowledge (Yang et al., 2015). The second tier requires learners to justify their selected answer in the first tier (Nti *et al*., 2023). The justification aspect is an affordance that a single-tier diagnostic assessment does not provide, thereby making it susceptible to guesswork on the part of the learners. Essentially, implementing a two-tier diagnostic assessment is "... an efficient and effective way of investigating students' knowledge, misconceptions or alternative conceptions ..." (Yang et al., 2015, p. 367).

A typical MCQ consists of the actual question (also called a stem) and plausible options, one of which is a correct answer (called a key) and incorrect answers (called distractors) (Arhin, 2024). In a two-tier diagnostic test, often the first tier is a typical MCQ while the second tier could either be an MCQ or a constructed-response question (CRQ). A CRQ is a free-response question that allows learners to use any strategy to answer the question. The benefit of the second tier for teachers is that it enables them to gain insights into the learners' understanding of the concepts in the first tier, whether their answers are correct or not. In other words, if a learner chooses a correct option in the first tier and an incorrect justification in the second tier, then there was a possibility of guesswork (Tasci, 2024). The two-tier diagnostic test could, therefore, be an effective way of informing the goal of the research lesson within the LS context.

Theoretical frameworks

Constructivist theory

In this paper, we were guided by Piaget's constructivist theory because of the centrality of learners' knowledge construction. The constructivist theory posits that learners actively construct their own understanding of concepts through experiences and interactions within their environment (Moyo & Machaba, 2021). There are two cognitive processes that characterise knowledge construction within the constructivist realm: assimilation and accommodation (Faizah et al., 2022; Moyo & Machaba, 2021). As indicated in the Introduction of this paper, Sehole et al. (2023) remind us that assimilation requires connecting new knowledge with and fitting it to existing knowledge, thereby reinforcing conception. However, if assimilation cannot occur, accommodation is invoked to modify existing knowledge, enabling it to connect and fit with new knowledge. If accommodation is done erroneously by connecting unrelated concepts, then misconceptions are created.

However, what teachers may view as misconceptions by learners are, in fact, what learners perceive as alternative conceptions (Fujii, 2020). One could argue that misconceptions in classroom situations are inevitable. As such, they can be viewed as opportunities for both teachers and learners to engage in teaching and learning. When learners display misconceptions, teachers can capitalise on such occurrences and use them as PTMs to reinforce learners' accurate understanding of a mathematical concept. Two-tier diagnostic tests, especially the second tier, help teachers to gain a deep understanding of learners' thinking about mathematics concepts. Using the constructivist theory, we were able to unpack learners' responses to understand their mathematical thinking regarding fractions.

Categorisation of misconceptions

When one is building on constructivist theory, it is essential to recognise that misconceptions in mathematics can manifest in various forms. This, therefore, justifies the adoption of a second framework – Lai's categorisation of misconceptions (Lai, 2012) – to classify the emergent misconceptions stemming from the analysis of learners' responses in the current paper. Lai (2012) categorised misconceptions into three main categories, namely; (1) Conceptual errors – which occur due to a faulty understanding of the basic principles of mathematical concepts, (2) Procedural errors – which occur due to incorrect mathematical computation or procedure, and (3) Factual errors – which occur due to a lack of factual knowledge (Lenz et al., 2024). Lai (2012) stated that identifying learners' misconceptions is the first step in providing them with corrective instruction. Thus, identifying learners' misconceptions in fractions using this framework ties-in well with the identification of the goal of the research lesson in the LS context. After all, the research lesson is planned based on evidence from learners' apparent misconceptions (Sekao, 2023).

Methods

Research design

We adopted a single case study design – a design predominantly used in qualitative studies – to gather an in-depth understanding of the phenomenon under investigation within its natural setting (Yin, 2018), which, in this case, involves learners' misconceptions of fractions, which are then used to inform the goal of the research lesson. The case of interest was grade 6 mathematics learners from an independent primary school that offers grades R to 7, in one province in South Africa.

We chose an independent school because it is generally assumed that independent schools offer better quality education than public schools (Azzopardi Meli, 2023). In fact, Ojo and Mathabathe (2021) argued that learners from independent schools were more likely to gain university admission, compared to their counterparts from public schools, due to the abundant resources at their disposal.

Therefore, exploring learners' difficulties with fractions in an affluent school that is not as affected by as many severe challenges as a typical public school provided a better perspective on learners' misconceptions. In other words, independent schools are assumed to recruit highly qualified teachers and academically better-performing learners; therefore, the data gleaned from this environment would not be muddied by factors apparent in public schools, such as less qualified teachers, socio-economic status, and poorly academically performing learners.

Sampling

The sample for this study comprised 18 South African grade 6 learners, ranging from 11 to 12 years old, who were purposively sampled and conveniently selected based on their characteristics and relevance to the study (Nikolopou, 2023). The focus on grade 6 learners was influenced by the constant struggles they experienced with fractions (Machaba & Moloto, 2021). Grade 6 learners' difficulties in fractions could have a ripple effect on other topics related to fractions, such as rates, percentages, etc. The school where the study took place was conveniently selected due to its close proximity to the researchers during the data collection period.

Ethical considerations

This study adhered to the ethical considerations outlined in guidelines that dictate how researchers should research to ensure it is done in a moral and acceptable manner (Mirza et al., 2023). Permission was sought and granted by the Ethics Committee of the university to which the researchers are affiliated (approval number EDU147/24). In addition, permission was sought from and granted by the principal of the school from which data was collected. Subsequently, a grade 6 teacher permitted us to administer a diagnostic test in their class. Due to the learners being minors, parental consent was obtained for their participation. In all instances where consent and assent were sought, the ethical principles of privacy, including anonymity and confidentiality, the right to withdraw, and informed consent were strictly adhered to.

Data collection

Setting the scene for data collection

In this study, the LS team was not comprised of teachers at the school where the data collection took place. Instead, the first author and other postgraduate students constituted a five-member LS team who explored learners' misconceptions in various mathematics topics. In this paper, we report findings on learners' misconceptions about fractions, thereby informing the goal of the research lesson. The primary purpose of this paper was on the process of identifying the goal of the research lesson, but not necessarily on the undertaking of the entire LS cycle. The two-tier diagnostic test (Table 1) was collaboratively developed by the LS team and moderated by knowledgeable others.

The test comprised questions involving subtraction (1.1 and 4.1), addition (2.1), conceptual understanding of fraction (3.1), and application of fractions within the context of probability (5.1). For the purpose of this paper, we have circled the correct options.

Process of data collection

Data were collected through document analysis (i.e., learners' written responses stemming from the two-tier diagnostic test). The diagnostic test was photocopied and administered by Author 1 to 18 grade 6 learners for 30 minutes. Learners' written responses were immediately collected for analysis.

Data analysis

Although we used Lai's (2012) categorisation of misconceptions to frame the study, we also used the same categorisation to inform our analysis. The categorisation includes factual, conceptual and procedural misconceptions. However, in our categorisation, we also drew from Braun and Clarke's (2006) process of thematic analysis. For instance, we familiarised ourselves with the data, classified it into relevant categories, and then produced the report.

To enhance the trustworthiness of the findings, we invoked the principle of inter-rater analysis, where the individual members of the LS team conducted an analysis of learners' responses and categorised their findings according to Lai's categorisation of misconceptions. Thereafter, the LS team collaboratively discussed and corroborated their findings.

Table 1: Two-tier diagnostic test

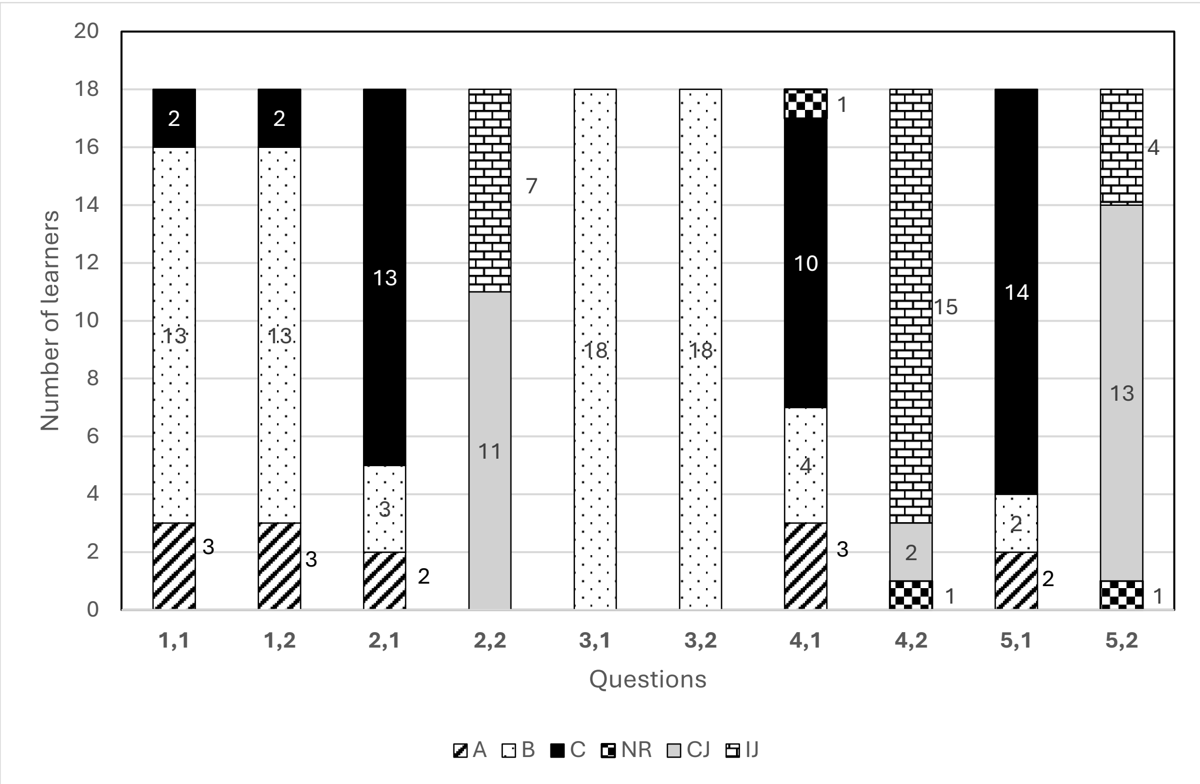
1st Tier	**2nd Tier**
1.1. Subtract $\frac{3}{8}$ from 1. Choose the correct answer: A. $\frac{2}{8}$ B. $\frac{5}{8}$ C. $-\frac{2}{8}$	1.2. Explain your choice in 1.1 by choosing the reason below: A. I chose $\frac{2}{8}$ because 1 - $\frac{3}{8}$ means $\frac{3-1}{8}$ which results in $\frac{2}{8}$ since 3 is bigger than 1. B. I chose $\frac{5}{8}$ because 1 can be represented as $\frac{8}{8}$, which means $\frac{8}{8}$ - $\frac{3}{8}$ results in $\frac{5}{8}$. C. I chose $-\frac{2}{8}$ because 1- $\frac{3}{8}$ means $\frac{1-3}{8}$ which results in $\frac{-2}{8}$.
2.1. Add the fractions $\frac{1}{4}$ and $\frac{3}{8}$. Choose the correct answer A. $\frac{4}{8}$ B. $\frac{4}{12}$ C. $\frac{5}{8}$	2.2. Calculate $\frac{1}{4}+\frac{3}{8}$. Show all your workings.
3.1. What fraction represents the black-shaded parts of the circle? A. $\frac{5}{3}$ B. $\frac{5}{8}$ C. $\frac{3}{5}$ 	3.2. Explain your choice in 3.1. by choosing your reason below: A. $\frac{5}{3}$; there are 5 black-shaded parts and 3 grey-shaded parts. B. $\frac{5}{8}$; there are 5 black-shaded parts out of a total of 8 parts. C. $\frac{3}{5}$; there are 3 grey-shaded parts against 5 black-shaded parts.
4.1. Thando had a full bar of chocolate divided into 24 equal pieces. She ate $\frac{1}{4}$ of the chocolate bar and then gave $\frac{1}{2}$ of what was left to her friend. After eating $\frac{1}{4}$ of the bar, how much of the bar was left in fractional form; and how much of the bar did she give to her friend, expressed as a fraction? A. $\frac{1}{2}$ of the bar left; $\frac{1}{4}$ of the bar given to her friend. B. $\frac{1}{6}$ of the bar left; $\frac{1}{9}$ of the bar given to her friend. C. $\frac{3}{4}$ of the bar left; $\frac{3}{8}$ of the bar given to her friend.	4.2. Show your workings to find the answer in 4.1.
5.1. Laurel has a bag containing six coloured balls: 1 black, 2 green and 3 red balls. She puts her hand in the bag and draws a ball. What is the chance that she will draw a green ball? Write your answer in simplest fractional form. A. $\frac{1}{6}$ B. $\frac{1}{2}$ C. $\frac{1}{3}$	5.2. Show your workings to find the answer in 5.1.

Results

In this section, we present the findings according to Lai's categorisation of misconceptions, i.e., conceptual, procedural, and factual misconceptions. The overall findings are summarised in Figure 2. It should be noted that for ease of reference, codes A, B, and C denote the answer options as reflected in Table 1, CJ denotes correct justification, IJ denotes incorrect justification, and NR denotes no response.

As indicated in Table 1, through question 3.1, we aimed to assess learners' conceptual understanding of fractions. As shown in Figure 2, none of the learners presented any misconception in question 3.1 (1st tier) and its associated justification (3.2, which is the 2nd tier). Notwithstanding learners' conceptual understanding, the findings revealed that their dominant misconceptions were related to the subtraction of fractions (questions 1.1 and 4.1) and the addition of fractions (question 2.1).

Figure 2: Overview of the findings by questions



Conceptual errors

The types of misconceptions stemming from this category are related to an incorrect understanding of the principles underlying mathematical problems (Delastri & Lolang, 2023; Lai, 2012). These misconceptions occur because learners have misunderstood the underlying concept, leading to the application of incorrect logic (Chamundeswari, 2014).

When asked to subtract $\frac{3}{8}$ from 1, Learner 15 (L15) subtracted 1 from 3 and left the denominator as it was, resulting in an incorrect $\frac{2}{8}$ (Figure 3). L16 seems to have followed a similar approach as L15, but for them, they subtracted 1 on both the numerator and denominator, resulting in $\frac{2}{7}$. Since $\frac{2}{7}$ was not part of the distractors, L16 opted for $\frac{2}{8}$ as their final answer, which is the 'next best thing'. With this misconception, it appears that learners are influenced by their knowledge of whole numbers, treating the fraction as two separate numbers.

Figure 3: Conceptual misconception on subtraction of common fractions by L15 and L16

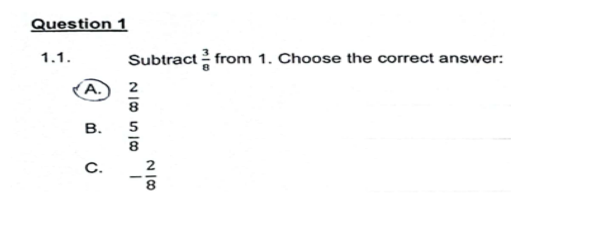
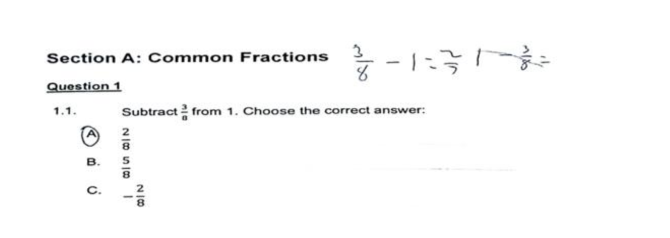
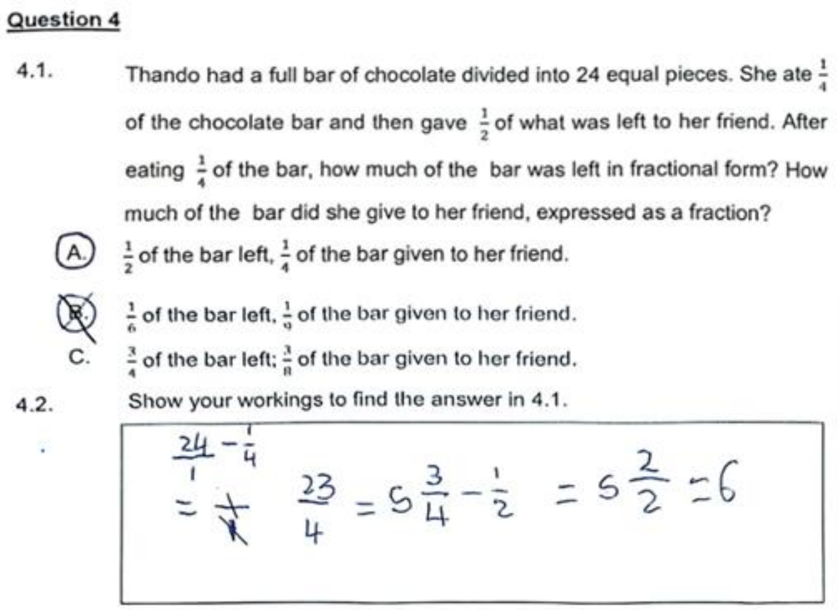
	
Learner 15	Learner 16

Figure 4: Conceptual misconceptions on word problems involving fractions (L18)



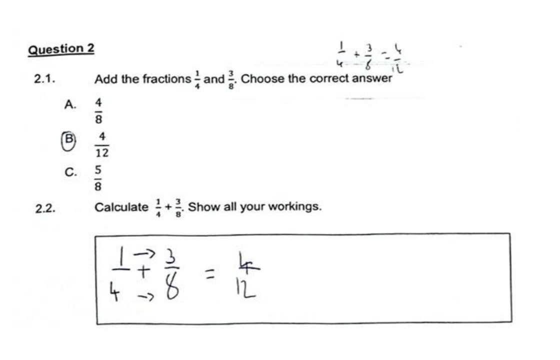
Another intriguing misconception emerged from L18 (Figure 4), where a disconnect existed between the option chosen in tier 1 and the justification/solution provided in tier 2. This disconnect suggested that L18 might have guessed the option chosen in the first tier. Notwithstanding, the intriguing part is the justification/solution in the second tier, which reveals three underlying phenomena: a conceptual understanding of fractions, whole number bias, and a lack of conceptual understanding of subtracting fractions, i.e., using equivalence to make the denominators the same.

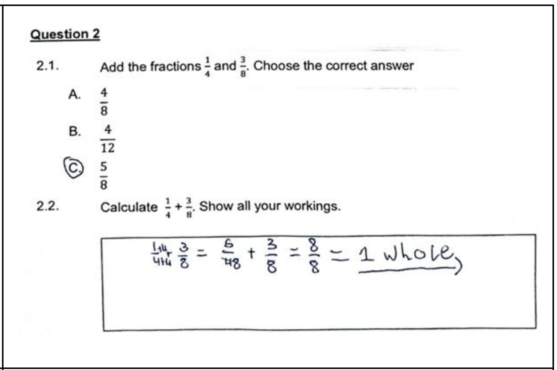
Regarding the lack of conceptual understanding of fractions, L18's conception of eating a $\frac{1}{4}$ of 24 is the same as $24 - \frac{1}{4}$. Regarding whole number bias, a common misconception across many of the learners, L18 subtracted $\frac{24}{1} - \frac{1}{4}$ to obtain $\frac{23}{4}$. Lastly, although $\frac{24}{1} - \frac{1}{4}$ is incorrect in the context of question 4.1, regarding the lack of conceptual understanding of the subtraction of fractions. L18 seems not to understand the basic principle of subtracting fractions, i.e., making the denominators the same through equivalence.

Another prevalent conceptual misconception identified was when learners were required to add $\frac{1}{4}$ and $\frac{3}{8}$ in question 2.1, where they added the numerators together and the denominators together. For instance, L15 added 1+3 and 4+8 resulting in $\frac{4}{12}$ (Figure 5). This misconception stems from the influence of working with whole numbers, where learners view a fraction as two separate whole numbers. Again, the phenomenon of whole number bias is at play.

L18's approach to answering this question was different (Figure 5). In the first tier, the learner selected the correct option, but when they had to show their workings, the learner adopted a different approach by adding 4 to both the numerator and denominator of $\frac{1}{4}$ resulting in $\frac{5}{8}$ and then proceeded to add $\frac{3}{8}$ to get $\frac{8}{8}$=1. This was a rather interesting response, and we exploited this anomaly to gain further insights into the learner's mathematical thinking. Evidently, L18 has an understanding that fractions can be added or subtracted only if the denominators are the same. Our initial view was that L18 understands that in the process of making the denominators the same, the principle of using 'one' as an identity element should be invoked; however, L18 erroneously added $\frac{4}{4}$ to $\frac{1}{4}$ showing the misunderstanding of 'one' as an identity element for multiplication. The correct approach would have been to multiply $\frac{1}{4}$ by 'one' in the form of $\frac{2}{2}$ to obtain $\frac{2}{8}$, instead of adding 'one' in the form $\frac{4}{4}$. In fact, the principle of equivalent fractions can be attained by using 'one' as an identity element for multiplication.

Figure 5: Conceptual misconception on the addition of common fractions





Learner 15	Learner 18

Procedural misconceptions

Misconceptions of a procedural nature stem from incorrect application of rules or algorithms when solving problems (Delastri & Lolang, 2023; Lai, 2012). In this section, we share findings related to learners' incorrect application of mathematical rules when solving problems involving fractions. One of the questions that elicited procedural misconceptions was question 1.1 (1^{st} tier), where learners were required to subtract $\frac{3}{8}$ from 1. However, it should be noted that conceptual misconceptions were also manifested in the same question. In fact, the justifications in question 1.2 (2^{nd} tier) were structured in a way that revealed both conceptual and procedural understanding. To this effect, 3 learners chose the incorrect option A, indicating their erroneous understanding that $1-\frac{3}{8}$ implies $\frac{3-1}{8}$, subsequently $\frac{2}{8}$. One could attribute the switching of 3 and 1 to the flawed idea that you cannot "subtract a bigger number from a smaller number", i.e., 1-3 is not possible, at least in the lower grades where integers are not yet introduced to learners. Although learners are not yet introduced to negative numbers, they are supposed to know that $1-3$ is not the same as $3-1$. Therefore, the 2 learners who chose the incorrect option C, i.e., $1-\frac{3}{8}=\frac{1-3}{8}=-\frac{2}{8}$, (see Figure 2), also displayed a lack of procedural understanding since $\frac{1-3}{8}$ in this context, it contravenes the mathematical rules of subtracting fractions.

Similarly, in question 2.1, where learners were required to add $\frac{1}{4}$ and $\frac{3}{8}$, 13 of them chose the correct option in the 1^{st} tier; however, 7 of them provided an incorrect justification in the 2^{nd} tier.

The dominant incorrect justification involves a flawed procedure where learners add numerators together and denominators together – a typical example of fixedness to the rules involving the addition of whole numbers, also known as whole number bias (Mtumtum, 2020).

Factual misconceptions

Factual misconceptions arise from learners' limited knowledge of mathematical facts, a lack of vocabulary and digit identification, and misapplication of operation signs (Mtumtum, 2020; Muthukrishnan et al., 2019). Setiawan and Surahmat (2021) argued that these types of misconceptions stem from errors in understanding the questions or errors in identifying the information contained in the problem. For instance, in question 5.1 (1st tier), learners were required to determine the probability of drawing a green ball out of a bag containing 1 black ball, 2 green and 3 red balls. L16 chose an incorrect option C, and further provided a factually flawed justification in the 2nd tier (Figure 6).

Figure 6: L16's justification for question 5.1

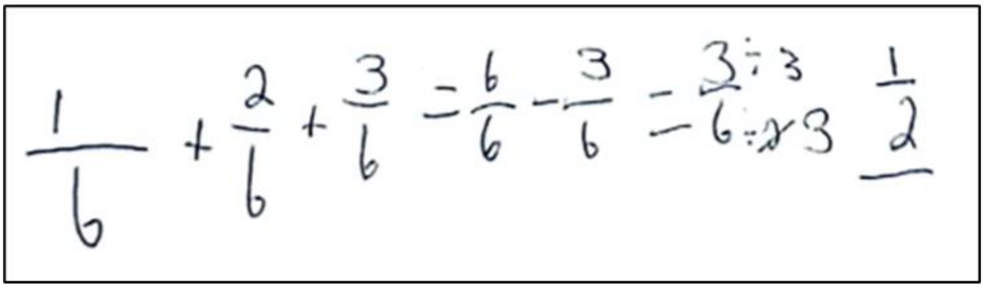
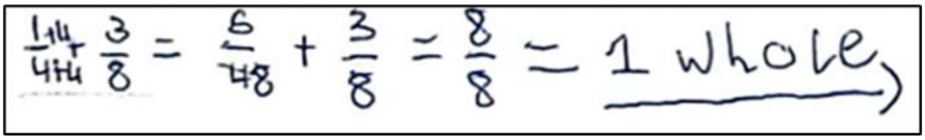


Figure 7: L18's justification for question 2.1



As seen in Figure 6, our view is that L16 added the probabilities of drawing out all the balls, i.e., $\frac{1}{6}+\frac{2}{6}+\frac{3}{6}$, and obtained $\frac{6}{6}$, with the intention of subtracting the probabilities of balls that are not green. However, subtracting $\frac{3}{6}$ instead of $\frac{4}{6}$ $\left(\text{i.e.}, \frac{1}{6}+\frac{3}{6}\right)$ is factually incorrect.

Similarly, L18 used a factually flawed operation to express $\frac{1}{4}$ as an equivalent fraction with a denominator of 8 by adding 4 to the numerator and the denominator instead of multiplying by 'one' in the form of $\frac{2}{2}$ (Figure 5). Although adding a 4 to the denominator yielded 8 as a desired denominator, the operation L18 violates a mathematical rule of converting a fraction into its equivalent form, therefore producing a factually incorrect answer.

Although we presented our findings per category, it seems inevitable that they tend to overlap. In other words, there is an interplay between conceptual, procedural and factual misconceptions. For instance, L18's justification in Figure 7 can be interpreted from the conceptual perspective (misunderstanding of one as an identity element for multiplication and equivalent fractions), procedural perspective (adding a 4 to the numerator and the denominator in $\frac{1}{4}$), and the factual perspective (factually flawed operation to express an equivalent form of $\frac{1}{4}$).

Discussion and Conclusion

Although the identification of the goal of the research lesson is a practice that cuts across all LS models globally, there remain uncertainties pertaining to how the goal is identified or determined. The uniqueness of the South African LS model lies in its diagnostic assessment/analysis stage, which intentionally informs the process of determining the goal of the research lesson (Sekao & Engelbrecht, 2022). In this study, we employed a two-tier diagnostic test to identify grade 6 learners' misconceptions of fractions, thereby determining the goal of the research lesson within the LS context. As asserted by Wu et al. (2025), two-tier diagnostic assessment enabled us to gain deeper insights into learners' misconceptions of fractions through the justifications embedded in the 2nd tier. Notwithstanding, the collaborative nature of LS (Richit et al., 2024) enabled us to view learners' misconceptions from multiple perspectives, making LS the perfect setting for the identification of learners' misconceptions, especially when using a two-tier diagnostic test (Boser, 2024; Gardee & Brodie, 2022), thereby determining the goal of the research lesson.

The findings from this study confirmed the views of scholars such as Aliustaoğlu et al. (2018), Alkhateeb (2019) and Ingram et al. (2013) who posited that misconceptions related to fractions are universal, and not dependent on whether the school is affluent or not. One major observation from the current study is that there is often an overlap of the three categories of misconceptions (conceptual, procedural, and factual) evident in learners' written responses. As such, instead of addressing these misconceptions individually, it seems rational to consider them collectively. In fact, the conundrum of which one between conceptual and procedural knowledge should be prioritised has long been a debate in research (Manandhar et al., 2022); however, there is a consensus about the interplay between these knowledge domains (Kilpatrick et al., 2001; Nahdi & Jatisunda, 2020).

Drawing on Hurrell (2021), who highlights that conceptual knowledge involves connecting mathematical concepts to enhance understanding. A lack of connections builds a breeding ground for misconceptions. This was evident in learners' lack of awareness of the links between the addition/subtraction of fractions and the concept of equivalent fractions, as well as the property of one (1) as an identity element for multiplication (Ojose, 2015; Vitoria et al., 2017). The prevalence of whole number bias (Alkhateeb, 2019; Moyo & Machaba, 2021) or functional fixedness can therefore be attributed to learners' lack of conceptual link-making. It is our view that addressing the links between concepts will enhance learners' conceptual and procedural knowledge regarding the addition/subtraction of fractions, thereby minimising misconceptions. As we have already articulated, there is often an interplay between misconceptions (Hurrell, 2021); the previous view, by implication, also resonates with factual misconceptions.

Considering the above, the goal of the research lesson in the current study can be themed as *Effective teaching and learning of addition/subtraction of fractions*. To achieve the goal, teachers would have to draw from their key learning experiences stemming from the analysis of learners' responses. The implications of the goal of the research lesson in the current study would guide teachers' study of curriculum materials (*kyouzaikenkyuu* or 教材研究 in Japanese) and pedagogical decisions within the LS setting. We note, however, that our understanding of fractions and mathematics holistically might have influenced how we conceptualise the goal of the research lesson, as reflected in the findings of this paper. Essentially, the analysis, interpretation and categorisation of the misconceptions might have been done subjectively. Hence, conducting interviews with learners could have possibly revealed different results.

Limitations and Recommendations

We assumed that the 2nd tier would provide us with sufficient justifications/reasons for learners' chosen options in the 1st tier. However, it emerged that in certain instances in the 2nd tier, learners provided a textual/narrative account of what led them to choose the option in the 1st tier, which made it difficult for us to decipher what would make mathematical sense. In other words, we needed more details to gain deeper insights into the learners' justifications. We therefore recommend that, when using a two-tier diagnostic test, follow-up interviews be conducted with learners to provide them with an opportunity to justify their chosen responses (Ndemo & Ndemo, 2023). We further recommend that, given the benefits of using a two-tier diagnostic test in identifying the goal of the research lesson in the current study, the process of determining the goal should be made explicit in the LS cycles globally, tailored to their local contexts. In other words, any other effective goal identification process could be considered.

Disclosure

Conflict of interest

The authors declare no conflict of interest

Author contributions

All the authors actively participated in this article and have approved the final version for publication.

ORCID

M Radingwane: https://orcid.org/0009-0000-5973-9924
D Sekao: https://orcid.org/0000-0003-1849-120X
MR Sibiya: https://orcid.org/0000-0003-4327-0046
KC Moremi: https://orcid.org/0000-0002-0311-7509

References

Aliustaoğlu, F., Tuna, A. & Biber, A.Ç. (2018). The misconceptions of sixth grade secondary school students on fractions. *International Electronic Journal of Elementary Education*, *10*(5), 591-599. https://doi.org/10.26822/iejee.2018541308

Alkhateeb, M.A. (2019). Common Errors in Fractions and the Thinking Strategies That Accompany Them. *International Journal of Instruction*, *12*(2), 399-416. https://doi.org/10.29333/iji.2019.12226a

Alsaeed, M.S. (2022). Supporting collaborative inquiry skills through lesson study: Investigation of high school mathematics professionals: Investigation of high school mathematics professionals. *Cogent Education, 9*(1). https://doi.org/10.1080/2331186X.2022.2064406.

Arhin, A. K. (2024). Developing Distractors for Mathematics Multiple Choice Items: A Literature Review. *Acta Educationis Generalis, 14*(3), 103-120. http://dx.doi.org/10.2478/atd-2024-0022.

Azzopardi Meli, M. (2023). Overcoming the limitations of a centralized educational system: a portfolio study examining a Jesuit school in Malta. [Master's Dissertation, Tampere University of Applied Sciences].

Boser, U. (2024). Confronting the Math Misconception Spiral. *Forbes*. https://www.forbes.com/sites/ulrichboser/2024/03/15/confronting-themathmisconception-spiral/?sh=2061019e6b7a

Botes, W., Moreeng, B., & Mosia, M. (2022). Pre-service teachers' experiences of a lesson study approach as a form of student support. *Issues in Educational Research*, *32*(1), 57-70.

Braithwaite, D.W. & Siegler, R.S. (2021). Putting fractions together. *Journal of Educational Psychology*, *113*(3). https://psycnet.apa.org/doi/10.1037/edu0000477

Braun, V. & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative research in psychology, 3*(2), 77-101. https://doi.org/10.1191/1478088706qp063oa

Chamundeswari, S. (2014). Conceptual errors encountered in mathematical operations in algebra among students at the secondary level. *International Journal of Innovative Science, Engineering and Technology*, *1*(8), 24-38.

Delastri, L. & Lolang, E. (2023). Students' Conceptual Error and Procedural Error in Solving Algebraic Problems. *Multicultural Education*, *9*(1). https://doi.org/10.5281/zenodo.7508092

Department of Basic Education. (2011). *Curriculum and Assessment Policy Statement Grades 4 -6.* Mathematics, Pretoria, South Africa: Government Printer.

Dudley, P. (2014). Lesson study: A handbook.

Elliott, J. (2019). What is LS? *European Journal of Education*, *54*(2), 175-188. https://doi.org/10.1111/ejed.12339

Faizah, S., Nusantara, T. & Rahardi, R. (2022). Constructing Students' Thinking Process through Assimilation and Accommodation Framework. *Online Submission*, *14*(1), 253-269.

Fluminhan, C. S. L., Schlünzen, E. T. M., & SHCLÜNZEN JUNIOR, K. L. A. U. S. (2024). Insights towards cultural aspects and related impacts of Lesson Study practices: a systematic review. *Educação em Revista*, *40*, e34104. http://dx.doi.org/10.1590/0102-469834104-T

Foster, C., & Trundley, R. (2024). Adapting lesson study for early-years practitioners. *International Journal for Lesson & Learning Studies*, *13*(3), 225-231. https://doi.org/10.1108/IJLLS-07-2024-184

Fujii, T. (2014). Implementing Japanese lesson study in foreign countries: Misconceptions revealed. *Mathematics teacher education and development*, *16*(1), n1.

Fujii, T. (2016). Designing and adapting tasks in lesson planning: a critical process of Lesson Study. *ZDM Mathematics Education, 48*, 411 – 423. http://dx.doi.org/10.1007/s11858-016-0770-3.

Fujii, T. (2019). Designing and adapting tasks in lesson planning: A critical process of Lesson Study. *Theory and Practice of LS in Mathematics: An International Perspective*, 681-704. https://doi.org/10.1007/s11858-016-0770-3

Fujii, T. (2020). Misconceptions and Alternative Conceptions in Mathematics Education. In: Lerman, S. (eds) *Encyclopedia of Mathematics Education*. Springer, Cham. https://doi.org/10.1007/978-3-030-15789-0_114.

Fujii, T. (2015). The critical role of task design in Lesson Study. *Task design in mathematics education: An ICMI Study 22*, pp.273-286. https://doi.org/10.1007/978-3-319-09629-2_9

García, F. J., Lendínez, E. M., Lerma, A. M., & Abril, A. M. (2024). Mechanisms and evidence of prospective teachers' learning through enquiry-oriented practices: the case of a lesson study intervention. *ZDM–Mathematics Education*, *56*(6), 1183-1196. https://doi.org/10.1007/s11858-024-01620-9.

Gardee, A. & Brodie, K. (2022). Relationships between teachers' interactions with learner errors and learners' mathematical identities. *International Journal of Science and Mathematics Education, 20*(1), 193-214. https://doi.org/10.1007/s10763-020-10142-1

Hansen, N., Jordan, N.C. & Rodrigues, J. (2017). Identifying learning difficulties with fractions: A longitudinal study of student growth from third through sixth grade. *Contemporary Educational Psychology*, *50*, 45-59. https://doi.org/10.1016/j.cedpsych.2015.11.002

Huang, R., & Shimizu, Y. (2016). Improving teaching, developing teachers and teacher educators, and linking theory and practice through LS in mathematics: an international perspective. *ZDM Mathematics Education, 48*, 93 – 409. https://doi.org/10.1007/s11858-016-0795-7

Hurrell, D. (2021). Conceptual knowledge or procedural knowledge or conceptual knowledge and procedural knowledge: Why the conjunction is important to teachers. *Australian Journal of Teacher Education (Online)*, *46*(2), 57-71. https://doi.org/10.14221/ajte.2021v46n2.4.

Ingram, J., Baldry, F. & Pitt, A. (2013). The influence of how teachers interactionally manage mathematical mistakes on the mathematics that students experience. *Conference proceedings*. 8th Congress of the European Society of Research in Mathematics Education. Ankara, European Society of Research in Mathematics Education. pp. 1487-1495.

Kanellopoulou, E. M., & Darra, M. (2018). The Planning of Teaching in the Context of Lesson Study: Research Findings. *International Education Studies*, *11*(2), 67-82. https://doi.org/10.5539/ies.v11n2p67.

Ketterlin-Geller, L.R., & Yovanoff, P. (2009). Diagnostic Assessments in Mathematics to Support Instructional Decision Making. *Practical Assessment, Research & Evaluation, 14*(16).

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding It Up: Helping Children Learn Mathematics.*

Kshetree, M. P., Acharya, B. R., Khanal, B., Panthi, R. K., & Belbase, S. (2021). Eighth Grade Students' Misconceptions and Errors in Mathematics Learning in Nepal. *European Journal of Educational Research*, *10*(3), 1101-1121. https://doi.org/10.12973/eu-jer.10.3.1101

Lai, C.F. (2012). Error Analysis in Mathematics. Technical Report# 1012. *Behavioral Research and Teaching*.

Lenz, K., Reinhold, F., & Wittmann, G. (2024). Topic specificity of students' conceptual and procedural fraction knowledge and its impact on errors. *Research in Mathematics Education*, *26*(1), 45-69. https://doi.org/10.1080/14794802.2022.2135132.

Lewis, C.C., Perry, R.R. & Hurd, J. (2009). Improving mathematics instruction through LS: A theoretical model and North American case. *Journal of mathematics teacher education*, *12*, 285-304. https://doi.org/10.1007/s10833008-9069-7

Mabena, N., Mokgosi, P.M., & Ramapela, S.S. (2021). Factors contributing to poor learner performance in mathematics: a case of selected schools in Mpumalanga province, South Africa. *Problems of Education in the 21st Century, 79*(3), 451 – 466. https://doi.org/10.33225/pec/21.79.451.

Machaba, F., & Moloto, M. (2021). Grade 6 teachers's mathematical Knowledge for teaching: The concept of fractions. *Journal for the Education of Gifted Young Scientists*, *9*(4), 283-297. http://dx.doi.org/10.17478/jegys.1000495.

Makinae, N. (2010). The origin of lesson study in Japan. In *The 5th East Asia Regional Conference on Mathematics Education: In Search of Excellence in Mathematics Education, Tokyo* (Vol. 2, pp. 140-47).

Manandhar, N. K., Pant, B. P., & Dawadi, S. D. (2022). Conceptual and procedural knowledge of students of Nepal in algebra: A mixed method study. *Contemporary Mathematics and Science Education*, *3*(1), 1-10. https://doi.org/10.30935/conmaths/11723.

Maslow, A. H. (1966). *The psychology of science*. Harper & Row, New York.

Mirza, H., Mirza, C. & Bellalem, F. (2023). Ethical considerations in qualitative research: Summary guidelines for novice social science researchers. *Social Studies and Research Journal*, *11*(1), 441-449.

Mohyuddin, R.G. & Khalil, U. (2016). Misconceptions of Students in Learning Mathematics at Primary Level. *Bulletin of Education and Research*, *38*(1), 133-162.

Moru, E.K. & Mathunya, M. (2022). A Constructivist Analysis of Grade 8 Learners' Errors and Misconceptions in Simplifying Mathematical Algebraic Expressions. *Journal of Research and Advances in Mathematics Education*, *7*(3), 130-144. https://doi.org/10.23917/jramathedu.v7i3.16784

Moyo, M. & Machaba, F.M. (2021). Grade 9 learners' understanding of fraction concepts: Equality of fractions, numerator and denominator. *Pythagoras*, *42*(1), 602. https://doi.org/10.4102/pythagoras.v42i1.602

Mtumtum, C.F. (2020). *The implications of teachers' understanding of learner errors in mathematics.* [Master's Dissertation, University of Pretoria].

Muthukrishnan, P., Kee, M. S. & Sidhu, G. K. (2019). Addition Error Patterns among the Preschool Children. *International Journal of Instruction*, *12*(2), 115-132. https://doi.org/10.29333/iji.2019.1228a

Nahdi, D. S., & Jatisunda, M. G. (2020). Conceptual understanding and procedural knowledge: A case study on learning mathematics of fractional material in elementary school. In *Journal of Physics: Conference Series* (Vol. 1477, No. 4, p. 042037). IOP Publishing. https://doi.org/10.1088/1742-6596/1477/4/042037.

Ndemo, Z., & Ndemo, O. (2023). Mitigating errors and misconceptions among Grade 11 learners in algebra through error analysis. *African Journal of Teacher Education and Development*, *2*(1), 11. https://doi.org/10.4102/ajoted.v2i1.11

Nikolopou, K. (2023). What is Purposive Sampling? Definition and Examples. *Scriber.* https://www.scribbr.com/methodology/purposive-sampling/

Nti, D., Appiah-Twumasi, E., Ameyaw, F., Agyemang, C., Asare, A.H.Y. & Issah, B. (2023). Analyzing Gender Differences in Misconception in Linear Momentum Using Two-tier Diagnostic Test Instrument. *reason*, *14*(1). https://doi.org/10.7176/JEP/14-1-10

Ojo, T. A., & Mathabathe, R. (2021). An investigation into the effectiveness of the curriculum and assessment policy statement (CAPS) in South African schools. *International Journal on Integrating Technology in Education*, *10*(2), 23-38. https://doi.org/10.5121/ijite.2021.10203.

Ojose, B. (2015). Students' Misconceptions in Mathematics: Analysis of Remedies and What Research Says. *Ohio Journal of School Mathematics,* 72.

Owusu, A., Bornaa, C., Kwakye, D., & Iddrisu, A. (2023). Pre-service mathematics teachers' errors and misconceptions of integer operations. *Journal of Statistics and Mathematical Concepts*, *1*(2), 1-19. https://doi.org/10.58425/jsmc.v1i2.127

Reinhold, F., Hoch, S., Werner, B., Richter-Gebert, J. & Reiss, K. (2020). Learning fractions with and without educational technology: What matters for high-achieving and low-achieving students? *Learning and Instruction*, 65, 101264. https://doi.org/10.1016/j.learninstruc.2019.101264

Richit, A., da Ponte, J.P., & Tomkelski, M.L. (2024). Professional Collaboration among Elementary School Teachers in Lesson Study. *Journal of Research in Mathematics Education, 13*(2), 111 – 131. http://dx.doi.org/10.17583/redimat.1.

Robinson, N., & Leikin, R. (2011). One teacher, two lessons: The lesson study process. *International Journal of Science and Mathematics Education*, *10*, 139-161. http://dx.doi.org/10.1007/s10763-011-9282-3.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education, 8*, 255–281. https://doi.org/10.1007/s10857-005-0853-5.

Sehole, L., Sekao, D., & Mokotjo, L. (2023). Mathematics conceptual errors in the learning of a linear function-a case of a Technical and Vocational Education and Training college in South Africa. *The Independent Journal of Teaching and Learning*, *18*(1), 81-97. https://doi.org/10.17159/ijtl.v18i1.17173

Sekao, D. (2023). *Teacher development through Lesson Study: A glocalised model for pre-service and in-service teachers in South Africa.* Van Schaik.

Sekao, D., & Engelbrecht, J. (2022). South African Primary Mathematics Teachers' Experiences and Perspectives About Lesson Study. *International Journal of Science and Mathematics Education.* https://doi.org/10.1007/s10763-02110214-w

Setiawan, Y. E. & Surahmat, S. (2021). Error analysis of prospective mathematics teachers in solving of applying radian measurement problem in trigonometry courses. *Al-Jabar: Jurnal Pendidikan Matematika*, *12*(2), 343-357. https://doi.org/10.24042/ajpm.v12i2.9874

Shimizu, Y., & Kang, H. (2022). Discussing students' thinking and perspectives for improving teaching: An analysis of teachers' reflection in post-lesson discussions in lesson study cycles. *ZDM–Mathematics Education*, *54*(2), 419-431. https://doi.org/10.1007/s11858-022-01371-5.

Siegler, R.S., Duncan, G.J., Davis-Kean, P.E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M.I. & Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological science*, 23(7), 691-697. https://doi.org/10.1177/0956797612440101

Simon, M.A., Placa, N., Avitzur, A. & Kara, M. (2018). Promoting a concept of fraction-as-measure: A study of the Learning Through Activity research program. *The Journal of Mathematical Behavior*, *52*, 122-133. https://doi.org/10.1016/j.jmathb.2018.03.004

Stigler, J. W., & Hiebert, J. (2016). Lesson study, improvement, and the importing of cultural routines. *ZDM*, *48*, 581-587. https://doi.org/10.1007/s11858-016-0787-7

Stockero, S.L. and Van Zoest, L.R., (2013). Characterizing pivotal teaching moments in beginning mathematics teachers' practice. *Journal of Mathematics Teacher Education*, *16*, 125-147. https://doi.org/10.1007/s10857-012-9222-3

Takahashi, A. & McDougal, T. (2018). Collaborative lesson research (CLR). *Mathematics LS around the world: Theoretical and methodological issues*, 143-152. https://doi.org/10.1007/978-3-319 75696-7_8

Tasci, G. (2024). Development of a two-tier diagnostic test to assess misconceptions about biology concepts at primary school. *Science Insights Education Frontiers, 25*(2), 4099-4122. https://doi.org/10.15354/sief.24.or652

Van de Walle, J.A., Karp, K.S., and Bay-Williams, J.M. (2016). *Elementary and middle school mathematics: Teaching developmentally*. Ninth edition, global edition. Upper Saddle River: Pearson.

Verma, P., & Choudhuri, R. (2025). Identification of Students' Misconceptions in Biology through Two-tier Diagnostic Test. *National Journal of Education, 23*(1).

Vetter, A. (2022). *Mathematics teachers' diagnostic assessment practices in the implementation of Lesson Study.* [Master's Dissertation, University of Pretoria].

Vitoria, L., Fauzi, F. & Ananda, N. (2017). Students' performance in solving problems involving fractions. *Proceedings of AICS-Social Sciences*, *7*, 707-714.

Wu, M., Tian, P., Sun, D., Feng, D., & Luo, M. (2025). Evaluating Students' Conceptual Understanding of Isomers Based on a Four-Tier Diagnostic Tool in Upper Secondary Schools. *International Journal of Science and Mathematics Education, 23*, 907 – 947. https://doi.org/10.1007/s10763-024-10494-y.

Yang, T.C., Chen, S.Y. & Hwang, G.J. (2015). The influences of a two-tier test strategy on student learning: A lag sequential analysis approach. *Computers and education, 82,* 366-377. https://doi.org/10.1016/j.compedu.2014.11.021

Yıldızhan, B. & Ertekin, E. (2022). General Understanding Regarding to the Fraction and Rational Number Concepts. *Research on Education and Psychology*, *6*(1), 46-67. https://doi.org/10.54535/rep.1115808

Yin, R. K. (2018). *Case study research and applications: design and methods*. SAGE Publications.